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BOOK 1

MODERN ALGEBRA

Structure and Method

HOUGHTON MIFFLIN · MODERN MATHEMATICS SERIES ·

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1965 Impression

HOUGHTON MIFFLIN · MODERN MATHEMATICS SERIES ·

Features of the Textbook

"Modern mathematics" has been on everybody's lips for the past decade, but few basic textbooks on the secondary school level have been published in this field. Houghton Mifflin Company now proudly offers the first book in a new series that is the culmination of years of evaluation of just what mathematics teachers are looking for in a "modern mathematics" program. *MODERN ALGEBRA*, *Structure and Method* not only clarifies, simplifies, unifies, and broadens old ideas in mathematics but also introduces new concepts. Its authors are nationally recognized leaders in the field; its editor, Dean Albert E. Meder, Jr., has been at the heart of the modern mathematics movement since its inception.

The authors of *MODERN ALGEBRA* have built the textbook on their conviction that a subject becomes significant to students when they utilize their newly gained knowledge. Consequently, the idea of *using* algebra permeates this book.

Basic features of the text are these:

1. Emphasis on the structure of algebra is combined with systematic instruction in the techniques of algebra as a reflection of its structure.
2. Precisely-worded presentation encourages reasoning and discovery and deliberately develops a familiarity with and understanding of algebraic proof; definitions are precise.
3. Flexible organization starts with the concrete and moves to the abstract; major concepts are given in depth and sequential review is provided.
4. Chapters open with appropriate motivating illustrations and caption, giving an idea of the subject of the chapter and what students should learn from it.
5. Centered heads focus students' attention on the major topics of each chapter.

6. Numbered side heads divide centered heads into sections, each conforming to a general lesson plan: explanation (text), try-out (oral exercises), practice (written exercises), and use (problems).

7. Sets are introduced in the first chapter and used throughout the text.

8. Abundant, interesting examples clarify explanations.

9. A multitude of carefully selected exercises and problems are graded A, B, or C, in order of increasing difficulty, to provide for individual student differences.

10. Meaningful photographs and drawings supply motivation.

11. Imaginative design employs visual aids as an integral part of the teaching process; Trans-Vision and the functional use of color and typography focus attention on significant ideas and operation.

12. Chapter Summary, Chapter Test, Chapter Review, and Cumulative Review serve as a built-in evaluation program.

13. Excellent historical, vocational, and recreational sections stimulate student interest.

14. Teaching aids include a complete solution key and a test booklet; student editions are available with or without answers.

Basic Philosophy

MODERN ALGEBRA, Structure and Method helps the student to:

1. understand some of the basic structure of algebra (the real number system);
2. recognize the techniques of algebra as reflections of this structure;
3. acquire facility in applying algebraic concepts and skills;
4. perceive the role of deductive reasoning in algebra;
5. appreciate the need for precision of language.

Because the understanding of concepts and the acquisition of skills are equally essential in a first course in algebra, this textbook guides the student in discovering mathematical principles and furnishes him with a wealth of exercise material to strengthen his comprehension of these principles. By providing many verbal problems the authors give the student constant opportunity to apply his understanding and skills to varied and realistic problem situations.

In preparing this text, the authors have considered the recommendations of many groups (such as the Commission on Mathematics of the College Entrance Examination Board and the School Mathematics Study Group) which have sought to improve the mathematics programs in secondary schools. Moreover the presentation incorporates modern knowledge about the teaching and learning of algebra. Consequently, the authors believe that their book constitutes a mathematically and pedagogically sound modern introduction to algebra.

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Order of Development

PART I — CHAPTERS 1–8

To be successful, a course must take into account the intellectual maturity of the students in it. Generally, when a student begins a first course in algebra, he knows how to operate with the whole numbers and common fractions, and he may even have some idea of such irrational numbers as π and $\sqrt{2}$. His experience with losses, debits, temperatures below zero, and so on, may have given him some notion of negative numbers. Nonetheless, he generally has only vague ideas about the structure of arithmetic and possibly a shallow intuition about the real number system.

Therefore, it would be most inappropriate to attempt to introduce algebra as an abstract, formal system, beginning with undefined terms and a set of axioms, and developing by logical inference into a complex structure of defined terms and proved theorems. Accordingly, the approach used in this book is not to construct the real number system but to assume that the real number system exists and to conduct a systematic investigation of its properties. Thus, the approach of *MODERN ALGEBRA* is mathematically correct, but informal and intuitive rather than axiomatic. Deductive reasoning is, however, woven into the presentation and the student is gradually led to think in terms of algebraic proof, to recognize the need for proof, and eventually to devise algebraic proofs himself.

Since the beginning student of algebra is familiar with the nonnegative real numbers, the so-called numbers of arithmetic, the first part of the book leads to a recognition of the general properties of these numbers under the arithmetic operations and under the relation of order. Exercise material serves the dual purpose of highlighting the number properties and providing an unobtrusive review of arithmetic.

Some of the more basic concepts and devices introduced are:

Number line — The device of picturing numbers on the number line is introduced at the outset and used thereafter to illustrate and motivate many concepts.

Set — This notion appears early and is employed throughout the book to clarify algebraic concepts.

Variables — These are introduced as symbols to represent any member of a specified set.

Number-numeral — The distinction between a number and its numerals is made in a careful agreement about the use of the equals sign.

Precision of language — Precise, explicit definitions are used throughout the development.

Open sentences — Both equations and inequalities are introduced along with their solution sets and the graphs of their solution sets. The student then begins his study of verbal problems by making up problems to fit given open sentences. This practice in translating from algebraic symbols to verbal expressions prepares the student for handling verbal-to-algebraic transformations with greater facility. In addition, he learns to state the properties of arithmetic numbers formally by means of open sentences.

Negative numbers — The negative numbers are next introduced as partners of points on the number line, and the concepts of order, opposites, and absolute value are defined. To devise rules for adding and multiplying in the extended number system, including negative numbers as well as the familiar numbers of arithmetic, the student is guided by three requirements:

1. the rules must be consistent with the rules for operating with arithmetic numbers;
2. addition and multiplication must continue to have the properties identified in the system of arithmetic numbers;
3. the rules must agree with intuitive notions about operations with numbers.

Through the use of Trans-Vision — transparent acetate overlays — the extended number system and its properties are reviewed and brought into focus (page 180).

Addition — Basic to addition is the assumption that the sum of a number and its opposite is zero, the identity element for addition. From this assumption, and the associative and commutative properties, the rules of addition follow.

Multiplication — Basic to multiplication is the property, proved in *MODERN ALGEBRA*, that multiplying a number by negative one (-1) produces the opposite of the number. From this and the other number properties, follow the rules of multiplication.

Subtraction and division — These are introduced as inverses of the operations of addition and multiplication. Throughout, reasoning and discovery are encouraged, and the role of proof is emphasized.

Equivalent equations and equivalent inequalities — They are developed and applied in the formal solution of open sentences and in verbal problems.

Polynomials — In discovering how to operate with polynomials, the student sees clearly the use of the associative, commutative, and distributive properties.

Factoring — In factoring, he learns the need for specifying the set of possible factors.

Algebraic fractions — Although work with fractions occurs throughout the book, a chapter devoted to the properties and applications of algebraic fractions has been included in view of the authors' conviction that particular attention must be paid to this frequently misunderstood area of elementary algebra. The logical development of the principles of fractions can help the student to recognize the rules for operating with fractions as simply reflections of the general properties of numbers. Experience suggests that this recognition usually does not develop from a casual or incidental treatment of rational expressions, but requires an energetic and full-scale presentation.

PART II — CHAPTERS 9–15

Coordinates in the plane — The student is led to this concept by a consideration of the solution sets of open sentences in two variables.

Graphing equations and inequalities — The basic principles of graphing equations and inequalities are developed and used to motivate the solution of systems of open sentences. Trans-Vision is again utilized to clarify these principles (page 356).

Having gained considerable intuition about numbers, the student is now ready to take a somewhat formal view of number systems, beginning with the set of natural numbers and extending through the set of positive fractions and integers and on to the rational numbers.

Irrational numbers — In the search for roots of positive numbers, irrational numbers are encountered.

Density and completeness — With the recognition of the existence of irrational numbers, the properties of density and completeness of the number system are identified and the system is now called the *real number system.* Through the use of Trans-Vision, the extended number system and its properties are reviewed and brought into focus (page 420).

Radicals — The properties of radicals are developed from the general properties of numbers.

Relation and function — Familiarity with sets of ordered pairs of real numbers leads to the concepts of relation and function.

Variation — This idea appears as a special type of function.

Quadratic equations and inequalities — A careful study of these concepts provides an extension of the basic course in elementary algebra.

Axioms in geometry, numerical trigonometry, vectors — A separate chapter on these areas furnishes additional enrichment, showing how algebraic and geometric ideas complement each other.

This sequence of topics provides a logical development of elementary algebra that is tailored to the mathematical maturity of the student. Since the treatment uses the spiral method, nearly every concept appears repeatedly, first in simple terms and later with broader or deeper significance. As a result, the student not only learns the usual skills of algebra, but also grows in his understanding of the nature of mathematics and in his ability to grasp mathematical ideas.

Over-all Organization

The book is divided into two parts: Part I comprises the first eight chapters, Part II, the remaining seven chapters. Appearing after Part I is a comprehensive cumulative review that covers all concepts and skills learned so far in the course. Additional cumulative reviews follow Chapters 3, 6, and 11.

The following time schedule suggests three possible levels of achievement. These may be varied according to the needs of students, the time available, and the requirements of the course of study. (See Assignment Guide, pages 47–64.)

180-Day Time Schedule

Chapter	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Minimum Course	8	10	14	15	10	15	18	19	17	17	17	10	0	0	10
Average Course	7	8	11	13	9	13	16	19	16	16	16	10	16	0	10
Maximum Course	7	8	10	12	8	12	15	18	15	15	15	10	15	10	10

It is important that your students be encouraged to read this textbook carefully. It is good to assign all students to read the sections to be discussed at the next class. Encourage questions and class discussions. Discourage students from accepting vague and imprecise statements. By emphazing the importance of self-reliance, a questioning attitude, and verbal precision, you will develop your students' mathematical maturity, while you teach the basic structure and skills of algebra.

Chapter 1

1–1 The number line, introduced here informally, will illustrate and motivate many concepts throughout the book. Students readily accept the assumption that a number can be assigned to every point to the right of 0 on the line. The question of whether the assigned number is rational or irrational is best avoided, however; if it is introduced by student discussion, it should be treated only casually. In Chapter 11, when the students have acquired the knowledge and maturity to appreciate the discussion, this question will be treated fully. The number line implies the existence of negative numbers, since it can be extended to the left of 0, as well as to the right. However, because the initial phase of the course considers the properties of the nonnegative real numbers (the numbers of arithmetic), no more than a passing recognition of the existence of negative numbers is appropriate.

Students should understand clearly that a *coordinate* is a number and that the *graph of a number* is a point. The phrases *partner of*, *paired with*, *corresponding to*, and the adjective *corresponding* should become familiar language to the class.

It is important to emphasize that the number line, by introducing the relations *is less than* and *is greater than,* orders the numbers of arithmetic. By tying these relations to their interpretations *is to the left of* and *is to the right of* on the number line, students acquire an intuitive basis for later formal definitions.

Each new mathematical term appears in red in the sentence defining it, while terms explained or illustrated, but not precisely defined, appear first in italics.

1–2 It is important to distinguish between a number and its name and to emphasize that the statement $a = b$ means that a and b are names for the same number. Notice that under this definition $1 + 7$ is a numeral for 8 rather than a command to add 1 and 7.

The term *numerical expression,* essentially defined as any numeral or combination of numerals representing a number, is to be compared with the term *variable expression,* introduced in Section 2–1. A variable expression is a variable or any combination of indicated sums, differences, products, or quotients which contain numerals and variables.

You will notice that the exercises hereafter in Chapters 1 and 2 provide a built-in review of arithmetic and suggest number properties that will be identified explicitly later. If perceptive students observe and announce some of these properties, commend their insight, but do not take time to develop the properties here.

You should point out that in characterizing a sentence as *true* or *false,* students *talk about* the sentence, but do not use it. Not all sentences talked about need be true, but sentences *used* in arithmetic discussions must be true. In writing "$(1 + 7) + 4 = 8 + 4$," students *use* the true statement "$1 + 7 = 8$." In writing " '$1 + 7 = 9$' is a false sentence," the student talks about "$1 + 7 = 9$." Since the statement is false, he may *discuss* it, but not *use* it.

1–3 You should emphasize that the size or complexity of a numeral does not indicate the size of the number represented. Thus, although the numeral 2 is larger than the numeral 7, you write $2 < 7$ because you are comparing *numbers,* not *numerals.*

1–4 Throughout the book, the idea of set clarifies many concepts, including properties of numbers, factoring, the solution of equations and inequalities, graphs, relations, and functions. Furthermore, the basic ideas of set are learned easily and can arouse considerable student interest. Therefore, the concept of set is a useful motivating topic at the beginning of the course.

The principal concept of set theory, that of *belonging,* is the main idea of this section. It should be emphasized that a set is determined by its elements, that sets are equal if, and only if, they have the same elements. Therefore, students should be careful that any rule proposed for determining a set can be applied universally to test any object for membership in the set.

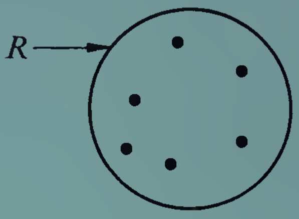
1–5 The student should understand clearly that in a one-to-one correspondence each element of one set is paired with an element of the other set, so that no element in either set is left out or used more than once.

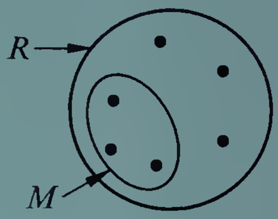
Notice that we avoid saying that an infinite set contains *an infinite number of elements,* or *an infinity of elements,* so that students will not imagine a very large number called *infinity.* You may say that an infinite set contains *infinitely many elements.*

In addition to confusing the empty set $\emptyset$ with $\{0\}$, students often confuse $\emptyset$ with $\{\emptyset\}$. Notice that $\emptyset$ contains no elements, whereas $\{\emptyset\}$ contains the single element, $\emptyset$.

To do the exercises some students may have to be reminded that an even number is a whole number that is a multiple of 2 and an odd number is a whole number that is not a multiple of 2.

1–7 An alternative definition of subset is: Set M is a subset of set R, if M contains no element that is not in R. You may find it helpful to use Venn diagrams to picture a set and one of its subsets. For example, to show that the points in the adjoining figure represent members of one set, you might draw a closed curve around them. If this diagram represents set R, then to show that M is a subset of R, you might draw the next figure.





The concept of subset will be helpful in discussing the solution sets of equations and inequalities.

Notice that we introduce only those set concepts that are very useful in elementary algebra. Thus, we have not used such set notation as $\subset$, $\supset$, $\cup$, $\cap$, or the set-builder symbols, since there is scant justification for employing them in the main development of this course. If you wish to use them in your class, you should, of course, do so. Intersection and union, and Venn diagrams illustrating these concepts, are treated in the Extra for Experts of Chapters 1 and 2.

1–8 Students then will find the necessary drill exercises more palatable, than if they consider them to be manipulations assigned for their own sake. Practice in using symbols of inclusion when writing expressions is given in Section 2–5.

1–9 The convention introduced here is universal in elementary mathematics and is presented to simplify notation. Because students frequently forget the rules of this convention, many advanced mathematics texts do not use it, but rely on symbols of inclusion to show the order of operations.

Chapter 2

This chapter introduces the concepts of variable, open expression, and open sentence including both equations and inequalities. In order to stress the role of the domain or replacement set of the variable in solving open sentences, formal methods of solution are not discussed. Rather, students use substitution, or even "educated guessing," to determine solution sets of simple equations and inequalities.

This chapter also aims to develop some ability in solving verbal problems. The successful device of having the student first interpret simple algebraic expressions verbally gives him an idea of what to look for when he later translates verbal phrases into algebra. Having gained facility in writing algebraic expressions for verbal phrases, students can more readily proceed to writing open sentences for verbal problems. In this chapter the emphasis should be on writing the sentence, rather than on finding the answer.

2–1 A variable is introduced here as a symbol standing for any element of a given set. Consequently, students should understand that in using a variable they must specify its replacement set. Discourage them from thinking of a variable as "something that varies."

Notice that the idea of a variable as an "unknown" is avoided in this section because that connotation, while useful, is definitely subsidiary to the more important one of place holder showing a pattern in a set of numbers.

In abstract algebra variables also appear as "indeterminates," as in the formal development of the theory of polynomials. This aspect of the concept of variable is not introduced in this course.

2–2 (1) Here the idea of *factor* is introduced very simply to provide a means of distinguishing between coefficients and exponents. Later the idea of factor will appear in the more specific language of *factoring over a set.*

(2) If a student says that an exponent shows the number of times a base is multiplied by itself, have him make the following computation:

Multiply 15 by itself once: $15 \times 15 = 225 = 15^2$
Multiply 15 by itself twice: $15 \times 15 \times 15 = 225 \times 15 = 3375 = 15^3$

Thus, to compute 15^2 he performs only *one* multiplication; to compute 15^3 he performs *two* multiplications.

2–3 (3) In this section the idea of a variable as an "unknown" is suggested, although it is wise to avoid this word, because of its misleading and imprecise connotations.

Notice that an open sentence is solved with respect to the domain of the variable. For example, in the sentence, "He is a member of the U. S. Senate," the solution set of the variable *he* depends on whether its domain is the set of all U. S. citizens or the set of all residents of a particular state. Similarly, the equation $3x + 1 = 2$ has no root if the domain of x is the set of whole numbers, but has one root if the domain is the set of common fractions.

(4) Students should expect to encounter open sentences whose solution sets coincide with the domains of the variables, as $y + 2 = 2 + y$ and $z \geq z$, as well as open sentences whose solution sets are proper (and even empty) subsets of the domain. Although an equation whose solution set coincides with the domain of its variables is often called *an identity* over that domain, in contrast to *conditional equations* which are not satisfied by all values of the variables, it is not essential to introduce this terminology here. Occasionally, however, you may wish to use these terms.

(5) To determine the solution of an equation like $2x + 5 = 17$ where $x \in \{\text{numbers of arithmetic}\}$, one cannot substitute for x all elements of the domain. But one might reason thus: "Since $2x + 5$ must equal 17, x cannot be too big. Replacing x by 5 gives $2 \cdot 5 + 5 = 15$; since $15 < 17$, try 6 as the replacement: $2 \cdot 6 + 5 = 17$. Thus, 6 is a root. Now, any number bigger than 6 will make $2x + 5$ bigger than 17, and a number smaller than 6 makes $2x + 5$ smaller than 17. So 6 is the only root." Of course, we do not make this argument precise, because we still must develop the properties of order on which it is based.

Some students may devise semiformal methods of solving open sentences, as: "If $2x + 5$ is to equal 17, $2x$ must equal 12; so, x has to be 6." Congratulate such students on their perception, but avoid making an issue of formal techniques at this time.

2–4 (6) By devising several interpretations for an algebraic expression, students develop a feeling for contexts in which the expression can arise. Later when they translate verbal phrases into algebraic expressions, they know what to look for in the verbal text.

In translating **+** into words, students should choose among: *greater than, the sum of, longer than, increased by,* etc. Various translations of **—** are: *less than, the difference of, shorter than,* and *decreased by.* Since $a - b \neq b - a$, students must be warned to be careful of order in translating $-$. Thus, $a - b$ means *the difference of a and b* or *the number that is b less than a.*

2–5 When the variable is introduced students should make clear what *number* is represented. Eventually, for ease of expression you may allow the somewhat loose language of a statement like "let n be the length in feet" rather than "let n be the number of feet in the length." However, students must understand that a variable represents a *number*, not a physical quantity.

2–6 The next step in problem solving is to write open sentences to fit problems. Since the open sentence is a statement relating two expressions by means of one of the symbols $=$, $<$, $>$, $\leq$, $\geq$, the student's experience in translating verbal phrases into algebraic expressions should enable them to proceed easily.

(7) If a class is not yet ready for this, however, you may wish to begin, as in Section 2–4 with the reverse process. For example, you may have the class make up a story to fit: $2w - 3 = 15$. One possible translation comes from adding one extra condition to the sentence given in Section 2–4: "The length of a room *15 feet long* is 3 feet less than twice the width."

The sample exercise on page 50 suggests a story for $\frac{v}{2} - 5 > 12$. To the second interpretation given in that sample, add the information: "Bill's club has more than 12 members."

Similarly, you may have your students use expressions in the exercises previously assigned on page 50 as the basis of open sentences. In each case, by adding one piece of information to their original interpretation of an expression, the pupils can create a story to fit the open sentences.

After three or four of these "algebra-to-English" translations, most classes will be ready for the word problems.

Chapter 3

(1) In this chapter we begin the study of the structure of algebra by calling attention to some of the axioms, or postulates, for numbers. Since many exercises in preceding chapters suggest these properties, most students will find the ideas familiar. Although some students may not be greatly impressed by formal statements of what may seem obvious facts, you can help them see the value of

featuring the postulates by (a) showing them how to use the properties to simplify computations, and (b) explaining that the whole algebra course will develop from such simple assumptions.

3–1 The axioms, or postulates, are statements accepted without proof. They are not self-evident truths. Rather, they are the arbitrary "ground rules" on which the development is based.

(2) In stating the properties of numbers in this chapter we omit the restrictive phrase *number of arithmetic* because: (a) No other numbers have been formally introduced up to this point. (b) The properties are, in fact, properties of all real numbers, so there is no need to make the statements seem formidable by adding a restriction.

(3) You notice that statements of properties appear in concise mathematical language as, "For any number a, $a = a$." To state this property in the language of sets, variables, and open sentences, you may write any of these:

a. If the domain of the variable a is the set of numbers of arithmetic, then any replacement of a converts the open sentence $a = a$ into a true statement.
b. If the domain of a is the set of numbers of arithmetic, the solution set of $a = a$ coincides with (equals) the domain.
c. Over the set of numbers of arithmetic, the equation $a = a$ is an identity.

The concise statement is given in the text in order to accustom students to conventional mathematical usage. It is good to bring in one of the longer forms, like *a* or *b* above, if any student seems unsure of the meaning of the concise statement.

(4) Several English idioms are logically equivalent to *for any number a*. You may use: *for each number a*, *for every number a*, *for all numbers a*, or even *for each* (*every*)(*any*) *a*. These phrases are used interchangeably in this book. To be sure that students recognize the meaning of these phrases (they are translations of the *universal* logical quantifier $\forall a$), ask the class to compare: *For each a*, $a = a$ and *for some a*, $a + 1 = 3$.

3–2 Other ways of saying *the sum* $a + b$ *is a unique number* are:

1. There is *exactly one* number, $a + b$, called the sum of a and b.
2. There is *one* number, *and only one* number, that is the sum, $a + b$, of a and b.

When we say "there is or there exists one number," we mean that there is *at least* one, but there may be more than one, number. When we say, "there is only one number," we mean that there is no more than one, but there could be none. By the statements *there is exactly one number* or *there is a unique number*, we mean that there is neither more than nor less than but precisely one number.

3–3 On the number line, $3 + 5$ is determined by starting at 3 and moving 5 units to the right, but $5 + 3$ is pictured by starting at 5 and counting 3 to the right. Each procedure ends at 8; thus, $3 + 5 = 5 + 3$. Students should understand that they do not start at 0.

3–4 **(5)** In the set of numbers of arithmetic not all differences exist. For example, $7 - 3$ is a number of arithmetic, but $3 - 7$ is not. Hence, in stating the distributive property over subtraction in this chapter, it is necessary to include the provision that $b - c$ is a number. Of course, by defining $b - c$ to be the unique number d (if it exists), such that $b = c + d$, you can prove this property. But we prefer to assume it, rather than attempt a proof which inexperienced students cannot appreciate.

(6) Do not allow expressions like *combining similar terms* to obscure the fact that you are using the distributive property to write an algebraic expression in another form. In particular, do not permit pupils to say, "You can't add 9 and $7b$, but you can add $5x$ and $3x$!"

(7) In a very good class you may wish to give this proof.

To prove $a(b + c + d) = ab + ac + ad$:

Proof:

$$\begin{aligned} a(b + c + d) &= a[b + (c + d)] && \text{Associative property} \\ &= ab + a(c + d) && \text{Distributive property} \\ &= ab + (ac + ad) && \text{Distributive property} \\ &= ab + ac + ad && \text{Associative property} \end{aligned}$$

3–5 (8) The proofs of the addition and subtraction properties of equality are very simple arguments every pupil will follow. Better students probably will discover the following proof of the subtraction property, although they may neglect to include the first step.

$$\begin{aligned} &a - c \text{ is a number} && \text{Given} \\ &a - c = a - c && \text{Reflexive property of equality} \\ &a = b && \text{Given} \\ &a - c = b - c && \text{Substitution principle} \end{aligned}$$

On the basis of the addition and subtraction properties of equality, you can prove that transforming an equation by adding (or subtracting) the same number in each member of the equation always produces an equivalent equation. You can present the idea with an informal argument for a specific equation like $n - 6 = 19$. Begin by saying that if a number n exists for which $n - 6 = 19$, the addition property of equality enables you to say that $n - 6 + 6 = 19 + 6$ or $n = 25$. On the other hand, if $n = 25$ is a true statement, the subtraction property of equality implies that $n - 6 = 25 - 6$ or $n - 6 = 19$. Hence, every number that satisfies one of the equations, $n - 6 = 19$ and $n = 25$, must satisfy the other, and the equations are equivalent.

(9) The transformations performed on the equation assure you that the final equation must be equivalent to the given one, *if no numerical errors occurred in the computation.* Because the check guards against such errors, students should always include it in the solution, even though it is not essential.

3–6 By an argument analogous to that for transformation by addition or subtraction, you can show that transforming an equation by multiplying (or dividing) each member by the same nonzero number produces an equivalent equation. It is good to run through the argument for a given equation.

3–7, 3–8 See text annotations.

Chapter 4

Chapter 4 extends the number concept to include a new kind of number. The guide in the investigation of the four fundamental operations in the extended number system is a *principle of permanence:* the rules for calculation in the larger system should preserve, as far as possible, the basic properties of the original number system. Thus, in applying the number properties discovered in Chapter 3 to derive rules of operation, students gain real appreciation of the significance of the properties.

4–1 **(1)** Although the negative numbers can be introduced abstractly, we believe that the more concrete approach of inventing number partners for the points on the left of the number line is appropriate for immature pupils. The informal experience of all students with negative numbers, suggested by the opening picture and the second paragraph of this chapter, provides a fairly natural introduction to negative numbers.

(2) You observe that we do not distinguish between the numbers of arithmetic and the nonnegative real numbers. If you make this distinction, then you must establish an isomorphism between nonnegative numbers and arithmetic numbers. Because a discussion of this isomorphism either confuses immature pupils or leaves them unconvinced of the logical necessity for the discussion, we omit what most students regard as an empty distinction.

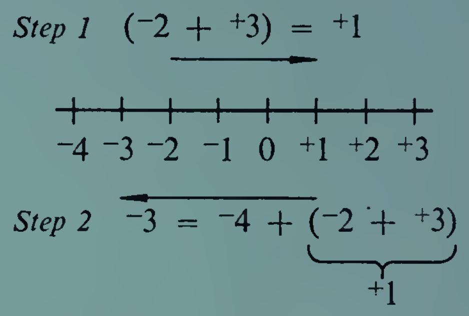
(3) The terms *signed numbers* and *directed numbers* are used mainly because students will meet them in mathematical literature. These terms, however, are misleading, since you can refer meaningfully to the sign as part of a numeral, but not as part of a number. Be sure that your students understand that $^{-}2$ and $^{-}3$ name *new* numbers, and not some operation performed on 2 and 3. If you wish, you need not use signed numerals like $^{+}4$ and $^{+}5$ for positive numbers, but from the outset simply use the numerals 4 and 5. Beginning with Section 4–4 the raised plus signs are omitted, and the raised minus signs are lowered. Using the raised signs in the introductory work does make the presentation clearer for most pupils, however.

For 0 and the positive and negative numbers, we use the term *the system of directed numbers* rather than *the system of real numbers* in order to postpone the discussion of irrational vs. rational numbers that might come from introducing the label, *the real numbers.* If you wish to use *real numbers,* do not let students give the word *real* any everyday non-mathematical meaning.

4–2 Note that $a < b$ in the system of directed numbers means that on the number line the point a is to the left of the point b. After addition is discussed, the order relation can be defined more formally.

4–3 **(4)** Since we have used the number line to introduce negative numbers, we now use it to lead informally to the addition of directed numbers. The preceding exercises have prepared the pupil to associate positive numbers with displacements to the right and negative numbers with displacements to the left. Thus, the addition of a positive number to a given number *a* is pictured by a displacement to the right of *a*, while the addition of a negative number is pictured by a displacement to the left.

This section prepares for a formal treatment of addition in Section 4–6. Do not ask students to give formal rules for adding numbers here, although your better students probably will devise rules for themselves. (5) The method of verifying the commutative property in some cases appears in the text. As an example of verifying the associative property in a special case, consider this diagram:


Step 1 $({}^{-}2 + {}^{+}3) = {}^{+}1$
$^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 $^{+}1$ $^{+}2$ $^{+}3$
Step 2 $^{-}3 = {}^{-}4 + \underbrace{({}^{-}2 + {}^{+}3)}_{{}^{+}1}$

4–4 (6) Introducing the concept of opposites prepares for the formal statement and derivation of the rules for operating with positive and negative numbers. Notice that every number is the opposite of exactly one number. Thus, $-a$ (the opposite of *a*) has meaning whether *a* be positive ($-3 = {}^{-}3$), negative ($-{}^{-}3 = 3$), or 0 ($-0 = 0$). Since -3 and $^{-}3$ name the same number, we displace the second symbol by the universal lowered minus sign, used as part of the numeral for *negative 3* or as part of the symbol for *the opposite of 3*.

The symbol for a positive number like $^{+}3$ hereafter is 3. From this point, the plus sign represents only the operation of addition.

Occasionally, students may ask why you do not use the lowered minus sign from the very beginning. We do this in order to distinguish between the opposite of a number -7 and a negative number $^{-}7$. If -7 is used for both, some students may receive the impression that the opposite of any number is negative, which is true only for positive numbers. In our development we never read the variable expression $-a$ as *negative a* but only as *the opposite of a*.

4–5 The concept of absolute value is important in advanced mathematics, as well as in elementary algebra. The customary definition in advanced texts,

$$|y| = \begin{cases} y \text{ if } y \geq 0 \\ -y \text{ if } y < 0, \end{cases}$$

is troublesome for most students; hence, we give a definition easily pictured on the number line. The more difficult definition is suggested by some exercises, in particular, Written Exercises 13 and 14 on page 124.

4–6 (7) The preceding sections have used the number line to suggest how to add directed numbers. Students now can see that the postulates enable them to derive rules for addition. The first four postulates on page 125 are familiar ones from the set of arithmetic numbers. The last two were introduced in Section 4–4. (8) No new rules are needed for adding nonnegative numbers, since these are the familiar numbers of arithmetic. Postulate 6, as shown in Example 1, enables students to compute the sum of two negative numbers by finding a sum of positive numbers. Examples 2 and 3 show how Postulates 4, 5, and 6 enable students to compute the sum of a positive and a negative number, if they can express a number appropriately as a sum. Give several examples like those in the book, especially like Examples 2 and 3.

The first step in each of these is the hard part, for the rest of the derivation follows easily. However, most students soon see what happens, even if they cannot verbalize it: they should express the number with the larger absolute value as a sum of two terms, one of which is the opposite of the number with the smaller absolute value. Thus, in $18 + (-5)$, $18 = 13 + 5$ and $5 = -(-5)$, while in $-9 + 2$, $-9 = (-7) + (-2)$ and $-2 = -(2)$.

(9) Unless you have an unusually able class, you should not attempt to derive the rules for addition on pages 125–126. Rather, have your students devise the rules on the basis of illustrative examples. Some teachers prefer not to state the rules formally, but prefer that their pupils "work out" sums by using the number line or the postulates, until experience finally suggests short cuts.

(10) You may point out that the property of the opposite of a sum is not independent of the other postulates on page 125, but can be proved. Students who successfully complete Exercise 26 on page 127 will prove it, in essence. The proof consists of showing that $(-a) + (-c)$ is the number whose sum with $a + c$ is 0. But the associative and commutative properties enable you to write

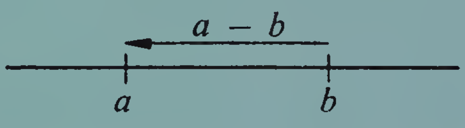
$$\begin{aligned}(a + c) + [(-a) + (-c)] &= [a + (-a)] + [c + (-c)] \\ &= 0 + 0 \qquad \text{Property of opposites} \\ &= 0 \qquad\quad\;\; \text{Additive property of zero}\end{aligned}$$

Similarly, $$[(-a) + (-c)] + (a + c) = 0.$$

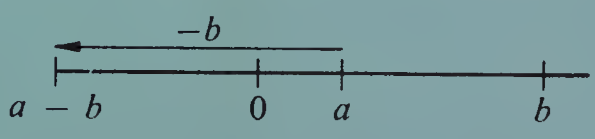
Since $(a + c)$ has only one additive inverse, and since $[(-a) + (-c)]$ has the property of the inverse, it follows that it *is* the inverse: $-(a + c) = (-a) + (-c)$.

In a very able class you may wish to develop this proof of the property of the opposite of a sum. In most classes it seems wiser to accept the property as a postulate.

4–7 (11) You must build on pupils' familiarity with subtraction in the set of arithmetic numbers in defining subtraction in the set of directed numbers. For this reason, $a - b$ is defined here as a directed number whose sum with b is a. This yields an immediate geometric interpretation: the *directed* distance *from b to a;* that is, $a - b$ is the number telling students how far and in what direction to move in going from b to a on the number line. It is pictured by the arrow in the figure.



(12) This interpretation of subtraction must be supplemented promptly by the theorem $a - b = a + (-b)$. Several examples suggest this, and it is anticipated in the sample on page 127. It gives a second geometric interpretation of $a - b$, the coordinate of the point reached by starting at a and moving the directed distance represented by $-b$. The distance from 0 to the point $(a - b)$ in this figure should equal the directed distance $(a - b)$ represented by the arrow in

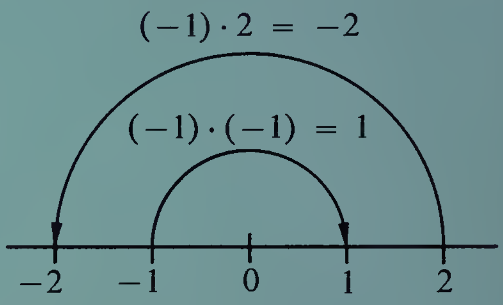


the preceding figure. Have your students compare the two geometric presentations of $a - b$ in several specific cases like $3 - 5$, and $-2 - (-4)$.

(13) Now that $a - b$ is defined for all a and b, you may have your students consider whether subtraction has such properties as commutativity, associativity, and existence of an identity. Examples show the answer in each case to be *no*. If pupils claim that 0 is the identity for subtraction because $a - 0 = a + (-0) = a + 0 = a$, ask them to consider $0 - a$. Does it equal a? It does not, because $0 - a = 0 + (-a) = -a$, and $-a \neq a$, unless a is 0.

4–8 (14) The postulates on page 134, together with the theorem $(-1)a = -a$, enable students to compute products of any directed numbers. By stressing the role of the properties in deriving calculating rules you help pupils to see the significance of the postulates and to appreciate the growing structure of the number system.

To be sure that students understand the result $(-1)a = a(-1) = -a$, illustrate it graphically, as shown in the figure. Thus, the effect of multiplying by -1 is a rotation through 180°; it produces the opposite of the number with which you start.



(15) It is wise to run through the steps in the following simplification.

$6x - 4y - 5x + 8y + 7x - 9y$ $= 6x + (-4y) + (-5x) + 8y + 7x + (-9y)$	Convention on order of operations and meaning of subtraction
$= [6x + (-5x) + 7x] + [(-4y) + 8y + (-9y)]$	Commutative and associative properties
$= [6x + (-5)x + 7x] + [(-4)y + 8y + (-9)y]$	$-ab = (-a)b$
$= [6 + (-5) + 7]\,x + [(-4) + 8 + (-9)]y$	Distributive property
$= 8x + (-5)y$	Addition of directed numbers
$= 8x + (-5y)$	$(-a)b = -ab$
$= 8x - 5y$	Meaning of subtraction

Naturally, although pupils should be able to go through these explicit steps, you will not expect them to write the details of the procedure each time an expression is simplified. Most of the steps will be done mentally.

4–9 (16) Just as subtraction is defined in terms of addition, so division is defined in terms of multiplication. Thus, addition and multiplication are the basic operations. Subtraction and division are defined, essentially, only for convenience.

This section parallels Section 4–7. The definition, "$\frac{a}{b}$ is the directed number you multiply by b to obtain a," is a direct extension of the meaning of division in the set of arithmetic numbers. The convenient result, $\frac{a}{b} = a \cdot \frac{1}{b}$, if $b \neq 0$, gives a second interpretation of division. It also makes it unnecessary to restate "rules of signs" like those stated for products on page 135.

(17) Be sure that students see why, if $b > 0$, then $\frac{1}{b} > 0$; and if $b < 0$, then $\frac{1}{b} < 0$. This involves an indirect argument like this: If b were positive and $\frac{1}{b}$ were negative, then $b \cdot \frac{1}{b}$ would be the product of a positive and a negative number, and would therefore be negative. But $b \cdot \frac{1}{b} = 1$, and 1 is positive. Therefore, either both b and $\frac{1}{b}$ must be positive or both must be negative. Of course, neither b nor $\frac{1}{b}$ can be 0, because $b \cdot \frac{1}{b} = 1 \neq 0$.

(18) The proof of $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$ consists in showing that $\frac{1}{a} \cdot \frac{1}{b}$ acts as the unique reciprocal of ab; that is, the product of ab and $\frac{1}{a} \cdot \frac{1}{b}$ should be 1. Your better students probably will devise this proof. By the commutative and associative properties:

$$\begin{aligned} (ab) \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) &= \left(a \cdot \frac{1}{a}\right)\left(b \cdot \frac{1}{b}\right) & \\ &= 1 \cdot 1 & \text{Property of reciprocals} \\ &= 1 & \text{Multiplicative property of 1} \end{aligned}$$

Encourage your students to study the text and diagrams in the accompanying insert. It provides a brief development of the number system, starting with the natural numbers and following the steps the students have taken in extending their concept of number. The transparency crystallizes much of the algebraic structure taught in this course, and also suggests additional concepts to be met later. The developing diamond not only depicts the growing structure of the number system, but also emphasizes the diamond symbol used throughout the book to signify statements of algebraic structure. A later transparency extends the number concept to include irrational numbers, so that together these transparencies provide a succinct discussion of the real number system.

After completing this chapter and having developed some feeling for structure and proof, exceptionally perceptive students may ask whether there can be more than one 0 or 1. You may even wish to pose this question to your best students. Of course, the answer is in each case *no*. And the reasoning is similar for both cases: Suppose 0 and $0'$ were both zeros (additive identities); *i.e.* for every a, $a + 0 = 0 + a = a$, and $a + 0' = 0' + a = a$. Then, in particular

$$\begin{aligned} & 0 + 0' = 0 && \text{because } 0' \text{ is an additive identity.} \\ \text{and } & 0 + 0' = 0' && \text{because } 0 \text{ is an additive identity.} \\ \therefore\ & 0 = 0' && \text{by the symmetric and transitive properties of equality.} \end{aligned}$$

Replacing 0 by 1, $0'$ by $1'$ and $+$ by $\times$, you analogously show that if 1 and $1'$ are multiplicative identities, $1 = 1'$.

4–10 See text annotations.

Chapter 5

This chapter strengthens pupils' skill in solving equations; it develops with them some of the order properties of the set of real numbers; and it extends their ability to solve word problems.

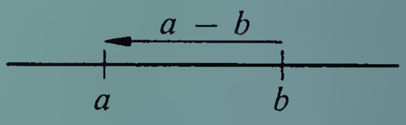
5–1 Remind your students of the processes for transforming an equation into an equivalent equation:

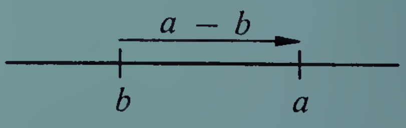
(a) Substituting for any expression in the equation an equivalent expression. This is Step 1 in Example 1 on page 158.
(b) Adding to (or subtracting from) each member of the equation the same number, or the same product of the variable and a number. (At this point students add and subtract monomials only.) In Steps 2 and 3 of the example this process is used.
(c) Multiplying (or dividing) each member by the same nonzero number. This is Step 4 of the example.

Be sure that your pupils understand how the properties of equality insure that these processes lead to equivalent equations. Process (a) depends on the substitution principle; (b), on the addition and subtraction properties of equality; and (c), on the multiplication and division properties of equality. Do not permit students to regard these processes only as manipulations and to describe (b) and (c) as "doing the same thing to both sides of the equation." It is good to have the class explain why equations like $-4x = 16$ and $x = -4$ are equivalent, in this manner: If x is a root of $-4x = 16$, then by the division property of equality $\frac{-4x}{-4} = \frac{16}{-4}$ or $x = -4$. On the other hand, if x is a number such that $x = -4$, then by the multiplication property of equality, $-4 \cdot x = -4(-4)$ or $-4x = 16$. Therefore, every root of $-4x = 16$ is a root of $x = -4$, and every root of $x = -4$ is a root of $-4x = 16$. Similar explanations should be given for the equivalence of $\frac{1}{2}x + 5 = 9$, $\frac{1}{2}x = 4$, and $x = 8$, and of $3x - 6 = x$, $-6 = -2x$ and $3 = x$.

5–2 In this section we make four formal assumptions about the relation of order in the set of directed numbers (real numbers). We assume that any two numbers can be compared and that the order relation is transitive. We also assume properties of addition and multiplication.

On the basis of these assumptions, you can prove other properties that some of your students may have discovered already. For example: if $a < b$, then on the number line a lies to the left of b. But this means that $a - b$ must be negative: $a - b < 0$. On the other hand: if $a > b$, a is to the right of b and $a - b$ is positive: $a - b > 0$. This suggests the following result.





Theorem A. Let a and b be directed numbers.

1. $a < b$ if, and only if, $a - b < 0$ *2.* $a > b$ if, and only if, $a - b > 0$

Proof of Part 1: (a) If $a < b$, then $a - b < 0$

$a < b$	Given
$a + (-b) < b + (-b)$	Addition property of inequality
$a + (-b) < 0$	Property of opposites
$\therefore a - b < 0$	Meaning of subtraction

(b) If $a - b < 0$, then $a < b$

$a - b < 0$	Given
$a + (-b) < 0$	Meaning of subtraction
$[a + (-b)] + b < 0 + b$	Addition property of inequality
$a + [(-b) + b] < 0 + b$	Associative property of addition
$a + 0 < 0 + b$	Property of opposites
$\therefore a < b$	Additive property of zero

Part 2 is proved analogously.

Although you may not wish to have all pupils prove this theorem, your better students may profit by trying to devise the proof. It also offers you an opportunity to discuss the significance of *if and only if.* Notice also that if you let $a = 0$, the theorem states:

1. $0 < b$ if, and only if, $-b < 0$ *2.* $0 > b$ if, and only if, $-b > 0$

An exceedingly worthwhile project on which your better students can work is the discovery and proof of other properties of the order relation like this simple one.

Theorem B. If $a + c < b + c$, then $a < b$

Proof:

$a + c < b + c$	Given
$(a + c) + (-c) < (b + c) + (-c)$	Addition property of inequality
$a + [c + (-c)] < b + (c + -c)$	Associative property
$a + 0 < b + 0$	Property of opposites
$\therefore a < b$	Additive property of zero

The multiplication property of inequality is not an independent postulate, but can be proved.

Theorem C. *1.* If $a < b$ and $c > 0$, then $ac < bc$.
2. If $a < b$ and $c = 0$, then $ac = bc$.
3. If $a < b$ and $c < 0$, then $ac > bc$.

Proof of Part 1:

$a < b$	Given
$c > 0$	Given
$a - b < 0$	*Theorem A–1*
$(a - b)c < 0$	Product of a negative and a positive number is negative
$ac - bc < 0$	Distributive property
$\therefore ac < bc$	*Theorem A–1*

Proof of Part 2:

$c = 0$	Given
$ac = 0$ and $bc = 0$	Multiplicative property of zero
$\therefore ac = bc$	Symmetric and transitive properties of equality

Proof of Part 3:

$a < b$	Given
$c < 0$	Given
$a - b < 0$	*Theorem A–1*
$(a - b)c > 0$	Product of two negative numbers is a positive number
$ac - bc > 0$	Distributive property
$\therefore ac > bc$	*Theorem A–2*

(1) *Parts 4* and *5* of the multiplicative property are implied by *Parts 1* and *3*, and also by the fact that $\frac{a}{c} = a \cdot \frac{1}{c}$ and $\frac{b}{c} = b \cdot \frac{1}{c}$, where c and $\frac{1}{c}$ are either both positive or both negative numbers.

Other results easily proved are these:

Theorem D. *1.* If $0 < a$ and $a < b$ $(0 < a < b)$, then $\frac{1}{a} > \frac{1}{b}$.

2. If $a < b < 0$, then $\frac{1}{a} > \frac{1}{b}$.

Proof of Part 1 (*Part 2* is proved similarly.):

$0 < a$ and $a < b$	Given
$0 < b$	Transitive property of inequality
$\frac{1}{a}$ and $\frac{1}{b}$ are numbers	Every nonzero number has a reciprocal.
$0 < \frac{1}{a}$ and $0 < \frac{1}{b}$	A number and its reciprocal are both positive or both negative
$\therefore 0 < \frac{1}{a} \cdot \frac{1}{b}$	The product of positive numbers is positive.
$\left(\frac{1}{a} \cdot \frac{1}{b}\right) a < \left(\frac{1}{a} \cdot \frac{1}{b}\right) b$	Multiplication property of inequality
$\frac{1}{b} < \frac{1}{a}$	Associative property of multiplication, property of reciprocals, and multiplicative property of 1.

Specific examples show that without the restriction that a and b are both positive or both negative, the statement of *Theorem D* is false. For example, $-2 < 3$, but $\frac{1}{-2} \not> \frac{1}{3}$.

Theorem E. If $a < b$ and $c < d$, then $a + c < b + d$.

Proof:	$a < b$ and $c < d$	Given
	$a + c < b + c$ and $b + c < b + d$	Addition property of inequality
	$\therefore a + c < b + d$	Transitive property of inequality

Theorem F. If $0 < a < b$ and $0 < c < d$, then $ac < bd$.

Proof:	$a < b$ and $0 < c$	Given
	$ac < bc$	Multiplication property of inequality
	$0 < a$ and $a < b$	Given
	$0 < b$	Transitive property of inequality
	$c < d$	Given
	$bc < bd$	Multiplication property of inequality
	$\therefore ac < bd$	Transitive property of inequality

A special case of *Theorem F* occurs for $a = c$ and $b = d$: If $0 < a < b$, then $a^2 < b^2$.

Theorem G. If $a < b < 0$ and $c < d < 0$, then $ac > bd$

Proof:	$c < d$ and $d < 0$	Given
	$c < 0$	Transitive property of inequality
	$a < b$	Given
	$ac > bc$	Multiplication property of inequality
	$c < d$ and $b < 0$	Given
	$bc > bd$	Multiplication property of inequality
	$\therefore ac > bd$	Transitive property of inequality

In a special case, if $a = c$ and $b = d$, then *Theorem G* becomes: If $a < b < 0$, then $a^2 > b^2$.

By encouraging your very able students to discover and prove theorems like those above, you will develop in them an appreciation of the essential character of mathematics, conjecture based on intuition guided by experience, and proved by deductive reasoning.

(2) Processes that transform an inequality into an equivalent inequality are:

(a) Substituting for any expression in the inequality an equivalent expression.

(b) Adding to (or subtracting from) each member of the inequality the same number, polynomial, or other expression having a definite value for all directed numbers.

(c) Multiplying (or dividing) each member by the same positive number.

(d) Multiplying (or dividing) each member by the same negative number and reversing the order of the inequality.

5–3 This optional section provides the better student with insight into the mathematical usage of the words *and,* and *or.* It also provides a good opportunity to employ the concepts of intersection (page 30) and union (page 60) of sets.

5–4 Like this chapter's presentation of the solving of open sentences, the treatment of problem solving is a reinforcement of earlier teaching. More emphasis is given to deliberate analysis, to the thinking by which students can arrive at the formation of the necessary open sentence, and to the general step-by-step scheme into which particular problems may be fitted.

Reading comprehension is essential. Confusion is more likely to result from careless reading than from ignorance of the words in a problem. Speed is unimportant at this point. Thinking, rather than manipulations, is all-important.

You may wish to have students analyze some problems orally in class to help them gain skill in forming open sentences. Some leading questions may be asked, such as, "What numbers are described in the problem?" "Does the problem suggest an equation by describing the same number in two ways?" "Is an inequality relation suggested?" "Can you form an expression for each number described?" After the preliminary reading, have pupils translate each significant word, phrase, or clause into its algebraic equivalent.

You must decide for each class how many problems to assign in this and later sections of the chapter. You may wish to assign one or two problems from this chapter along with assignments later in the course.

5–5 Problems about consecutive integers are admittedly somewhat academic, yet they have the special value of affording practice in applying a general problem-solving method to very simple specific cases. Furthermore, some ideas suggested here are used in more challenging problems later.

5–6 See text annotations.

5–7 Despite their feeling of familiarity with the formula $d = rt$, pupils often confuse the variables. A short discussion regarding the units of d, r, and t should reduce the number of these errors. Show that the units must correspond: if r is given as 4 miles per hour, d is expressed in miles, and t in hours.

Motion problems lend themselves readily to dramatization. It is not too childish to have pupils act out each of the three types presented in this section, nor even to use props such as dime-store toys in visualizing the situations.

(3) Do not permit pupils to use the chart as a crutch. It should encourage an organization of data to show relationships, not the blind adherence to a pattern. The intelligent charting of data can be a valuable technique.

Chapter 6

(1) This chapter presents the formal techniques of calculating with polynomials. In advanced courses, polynomials may be regarded simply as *forms* for which suitable operations of addition and multiplication are defined. From this point of view, the letter x in $5 + 4x + 3x^2$ does not denote an element of some set of numbers, and the + signs do not indicate addition. Rather, the + signs, as well as x and x^2, function as "bookkeeping devices" to keep track of the coeffi-

cients 5, 4, and 3. This is your meaning when you say that a polynomial is determined by listing its coefficients in order.

On the other hand, if you think of x as denoting an element of a set of numbers, then the polynomial $5 + 4x + 3x^2$ represents some number for each value of x. For example, if you replace x by 2, then $5 + 4x + 3x^2$ represents the *sum* of 5, 8, and 12, or 25. In this case, you regard polynomials as *functions*. With this interpretation, the commutative, associative, and distributive properties of operations with numbers determine operations with polynomials.

Both interpretations appear in elementary algebra, although the concept of a polynomial as a *form*, while implicit in manipulations like factoring and simplifying, is usually not discussed explicitly with immature pupils. The distinction should be clear to you, but pupils at this level are easily confused by the idea of x as an indeterminate. Furthermore, for polynomials over the set of real numbers (or over any infinite set) the two concepts are abstractly identical in the sense that the set of polynomial forms and the set of polynomial functions are isomorphic.

Consequently, in this chapter we regard symbols like x and y as variables in the sense of the definition in Chapter 2, and we justify operations with polynomials on the basis of the properties of the set of real numbers.

6–1 Pupils should master the language introduced in this section by repeatedly using it correctly in appropriate situations.

(2) When adding polynomials in a horizontal arrangement, some pupils find it helpful to check (√) each term as it is used. In the vertical arrangement it is most helpful to align similar terms underneath each other. Students should realize that the commutative and associative properties are used in this alignment.

6–2 We do not define subtraction of polynomials in terms of addition formally, since we are simply applying our postulates. Basically, we are using the fact that if A and B are polynomials, then $A - B = A + (-B)$. The property of the opposite of a sum enables us to express $-(2x^2 + x - 3)$, in Example 1, as a polynomial in customary form $-2x^2 - x + 3$ or $(-2)x^2 + (-1)x + 3$.

6–3 **(3)** Like addition, multiplication of numbers is a binary operation defined for pairs of numbers. When we write $a \cdot b \cdot c$, we may mean either $a(bc)$ or $(ab)c$. The associative property makes these definitions equivalent. Furthermore, the associative property permits us to insert parentheses freely in the product $\underbrace{b \cdot b \cdot \ldots b}_{m + n \text{ factors}} = b^{m+n}$. In particular, then, $b^{m+n} = \underbrace{(b \ldots b)}_{m \text{ factors}}\ \underbrace{(b \ldots b)}_{n \text{ factors}}$.

(4) It is good to compare the numbers $2^3 \cdot 2^4$, 2^{3+4}, and $2^{3 \cdot 4}$ (128, 128, 4096) to help students see the difference between b^{m+n} and $b^{m \cdot n}$. Also point out that $m + n$ counts the number of dots in *two* boxes, one containing m dots, the other, n dots. But $m \cdot n$ is the number in m boxes, each containing n dots.

$$\boxed{\cdot\ \cdot\ \cdot} + \boxed{\cdot\ \cdot\ \cdot\ \cdot} \qquad 3 + 4 = 7$$

$$\boxed{\cdot\ \cdot\ \cdot\ \cdot} \quad \boxed{\cdot\ \cdot\ \cdot\ \cdot} \quad \boxed{\cdot\ \cdot\ \cdot\ \cdot} \qquad 3 \cdot 4 = 12$$

6–4 (5) You may wish to say that the operation of raising to a power is distributive over multiplication: $(ab)^m = a^m \cdot b^m$. As we demonstrate shortly, it is not distributive over addition. We have not mentioned this usage of *distributive*, because we wish *the distributive property* to make students think of *the distributive property of multiplication over addition*. Nevertheless, at some point in the course, perhaps here or perhaps as part of a review, you may wish to consider, in general, whether or not one operation is distributive over another. Is addition distributive over multiplication; does $a + bc = (a + b)(a + c)$ for every replacement of a, b, and c by numbers? The consideration of such questions can be a useful exercise, particularly for better students.

(6) Have students who compute $(-5x^4)^2$ incorrectly, write the expression in full:

$$\begin{aligned}(-5x^4)^2 &= (-5x^4)(-5x^4)\\ &= (-5)(-5)x^4 \cdot x^4\\ &= 25 \cdot x^{4+4}\\ &= 25x^8. \quad \text{Hence, } (-5x^4)^2 = (-5)^2(x^4)^2 = 25x^8.\end{aligned}$$

You will have to emphasize again the distinction between b^{m+n} and $b^{m \cdot n}$.

6–5 See text annotations.

6–6 It is useful to point out that the computation of products of integers is an abbreviated form of the process of multiplying polynomials. Consider what happens in the text illustration if x is replaced by 10: $(2 \cdot 10 + 3)(4 \cdot 10 + 5)$ or 23×45. Now, $(2 \cdot 10 + 3)(4 \cdot 10 + 5) = 23 \times 45$.

$$\begin{array}{lclcrcrcl}
4 \cdot 10 + 5 & & & & & & 45 & & \\
\underline{2 \cdot 10 + 3} & & & & & & \underline{23} & & \\
8 \cdot 10^2 + 10 \cdot 10 & = & 800 + 100 & = & 900 & \nwarrow\;\nearrow & 135 & = & 3(4 \cdot 10 + 5) \\
\underline{\qquad\quad 12 \cdot 10 + 15} & = & \underline{\qquad 120 + 15} & = & \underline{135} & \swarrow\;\searrow & \underline{900} & = & 2 \cdot 10(4 \cdot 10 + 5) \\
8 \cdot 10^2 + 22 \cdot 10 + 15 & = & 800 + 220 + 15 & = & 1035 & \longleftrightarrow & 1035 & = & (2 \cdot 10 + 3)(4 \cdot 10 + 5)
\end{array}$$

By multiplying polynomials from right to left as in the arithmetic example, you can make the correspondence even clearer. As a special project, some students may work out a comparison of algebraic and arithmetic multiplication in which one factor is a trinomial.

6–7 See text annotations.

6–8 Although at this stage most students will not attempt sight multiplication, some may give the answer to Exercise 1 as $a^2 + b^2$. This amounts to assuming that the operation of raising to a power is distributive over addition and subtraction. You may illustrate the error in this assumption arithmetically ($[4 + 3]^2 \neq 16 + 9$) and geometrically (See Section 7–5).

6–9 Pupils should be familiar with these rules, but familiarity must be based on understanding. Thus, if a student simplifies $\dfrac{-8a^2b^7}{-56a^{12}b^3}$ incorrectly, he should reason thus: $\dfrac{-8a^2b^7}{-56a^{12}b^3} = \dfrac{-8}{-56} \cdot \dfrac{a^2}{a^{12}} \cdot \dfrac{b^7}{b^3} = \dfrac{1}{7} \cdot \dfrac{a^2}{a^{10} \cdot a^2} \cdot \dfrac{b^4 \cdot b^3}{b^3} = \dfrac{1}{7} \cdot \dfrac{1}{a^{10}} \cdot b^4,$

since $\frac{a^2}{a^2} = 1$ and $\frac{b^3}{b^3} = 1$. Therefore, the quotient is $\frac{b^4}{7a^{10}}$.

If a student uses the word *cancel*, ask him to explain his meaning, as above.

6–10 The concepts of zero and negative exponents could be developed here, but they are not essential in this course. The Extra for Experts in this chapter, however, does introduce negative exponents, showing their use in scientific notation.

6–11 See text annotations.

6–12 The principal idea in this and Section 6–11 is: if A and B ($B \neq 0$) are polynomials in a given variable, then there exist polynomials Q and R such that $A = QB + R$, where $R = 0$ or the degree of R is less than the degree of B. This fact, as well as the technique for determining Q and R, should be made clear to students.

It is important to stress that repeated subtraction of multiples of the divisor from the dividend is the basis of the division algorithm. As suggested in the text, you can underline this by presenting some examples of the division of integers.

Students are very interested that some large-scale electronic computers do division in a manner quite analogous to the process for polynomials. If a computer has two registers, an accumulator register (AC) and a quotient register (QR), initially, the dividend is put into the AC and 0 into the QR. Let $d \neq 0$ be the divisor. Then, from the AC the computer subtracts the largest value of $d \times 10^n$ that gives a positive result and adds 10^n to the QR. After repeating this subtraction-addition pattern as many times as possible, the QR eventually contains the quotient and the AC, the remainder. Here are the steps of such a computer in computing 273 ÷ 12.

	AC	QR	
Put dividend in AC.	273	0	Put 0 in QR.
Subtract $d \times 10$ from AC.	$\underline{120}$	$\underline{10}$	Add 10 to QR.
	153	10	
Subtract $d \times 10$.	$\underline{120}$	$\underline{10}$	Add 10.
	33	20	
Subtract $d \times 10^0 = d \times 1$	$\underline{12}$	$\underline{1}$	Add 1.
	21	21	
Subtract $d \times 1$.	$\underline{12}$	$\underline{1}$	Add 1.
Remainder =	9	22	= Quotient

Chapter 7

This chapter presents the basic ideas about factors and factoring and thereby lays a foundation for work with algebraic fractions. The factoring of polynomials is introduced in analogy with the factoring of integers and is applied in solving equations and problems.

Some pupils like to think of factoring as finding out "how polynomials got that way"; and they enjoy learning "how to take them apart," as they would take apart an alarm clock to see its works. If at some point in the chapter the pupils ask, "Why are we learning to factor?" the answer may be given in terms of problem solving. Some problems lead to equations that cannot be solved by the techniques learned to date. To illustrate this reason, the sample problem on page 267 may be put on the board without the solution of the open sentence.

7–1 The fact that concepts in factoring depend on the set over which the factoring occurs cannot be overemphasized. For example, over the set of integers, 5 is a factor of 10 ($10 = 5 \cdot 2$), but 4 is not. Over the set of rational numbers, however, 4 *is* a factor of 10, since $10 = 4 \cdot \frac{5}{2}$. Similarly, x is a factor of $x^2 + x$ over the set of polynomials with integral coefficients, but x^2 is not. If, however, the factoring takes place over the set of algebraic fractions, then x^2 is a factor of $x^2 + x$, because $x^2 + x = x^2\left(1 + \frac{1}{x}\right)$. Of course, the idea of a factor becomes interesting only when the nature of possible factors is restricted. In this chapter, integers are factored over a specified subset of the set of integers, and polynomials with integral coefficients are factored over a subset of the set of polynomials with integral coefficients.

(1) It can be proved that the factoring of a positive integer over the set of prime numbers is unique, except for the order in which the factors are written. Although the proof of this theorem is above the level of this course, you can bring the result to the attention of your students by asking them to factor an integer like 392 into prime factors in as many ways as they can.

7–2 Sometimes pupils do not see the significance of a sentence like $4ab + 6a = 2a(2b + 3)$. Point out that because of the distributive property this sentence is true for *every* replacement of a and b. Thus, the sentence resulting from a correct factoring of a polynomial is an identity — an equation true for all values of the variables. From time to time this idea should be reiterated by asking pupils such questions as, "For what value of x is $x^2 - 9 = (x + 3)(x - 3)$ true?"

7–3 See text annotations.

7–4 Reference is made to Table 3, page 541. This table should be used rather than the table of square roots (Table 4, page 542), because it gives only perfect squares. It may be necessary to explain how the table is to be used.

In a factoring involving more than one step a pupil is less likely to "lose" factors if his work is arranged as follows:

$$\begin{aligned} 2a^4 - 32 &= 2(a^4 - 16) \\ &= 2(a^2 + 4)(a^2 - 4) \\ &= 2(a^2 + 4)(a + 2)(a - 2) \end{aligned}$$

At this time certain students may feel that the sum of two squares should be factorable. If a pupil claims that the factoring of $a^2 + 4$ should be $(a + 2)(a + 2)$, ask him to expand this product and so discover his mistake. Other students may insist that with time they can find the right factors, in which case you may wish to anticipate the discussion on page 256 and have the class *prove* that $a^2 + 4$ cannot be factored over the set of polynomials (in a) with integral coefficients.

7–5 The geometric illustration makes the squaring process concrete. It can be reproduced profitably by using cutout squares and rectangles. Making its counterpart for $(a - b)^2$ (Problem 2, page 250) will be especially valuable, for it will firmly fix the fact that the middle term of the product is subtracted from the *sum* of the other two terms.

7–6 See text annotations.

7–7 To introduce this material, the teacher may send a number of pupils to the blackboard with instructions to write out the multiplication of any two binomials. From the examples before them, the class can then try to formulate generalizations regarding the three terms in the product.

If the binomials are written next to each other horizontally, the second step in the procedure (page 253) can be given as, "Multiply the inner pair of terms; then multiply the outer pair of terms; add the products."

To anticipate the next three sections, have the class discuss the relationship among coefficients of terms in the product and coefficients of terms in the binomial factors.

In this section and throughout the chapter you must maintain a proper balance of emphasis on technique and on the ideas behind it. Students who manipulate symbols without understanding do not learn algebra. Pupils who understand the basic ideas, but who are sluggish or slipshod in applying them are also unsatisfactory. Thus, pupils must have the triple goal of learning "*what* you do," "*how* you do it," and "*why* you do it." You must use judgment in devising lessons and assignments to foster the development of technical facility *and* the appreciation of the logical deduction of results from postulates.

7–8 **(2)** Sometimes pupils suspect that you know a "secret weapon" capable of revealing the factors directly. Assure them that the teacher's only advantage is experience, which they will soon acquire. Tell them that with experience the procedure, which may seem long to them in print, will take less than a minute.

(3) A student may ask whether $x^2 + 10x + 8 = (-1)(-x^2 - 10x - 8)$ is a factoring of the given polynomial over the set of polynomials with integral coefficients. Such a factoring is uninteresting because (-1) is a factor of every polynomial, and the second factor $(-x^2 - 10x - 8)$ is not of lower degree than the given polynomial. Therefore, $x^2 + 10x + 8$ is prime *over the set of polynomials with integral coefficients.* Do not permit students to omit the italicized phrase. Later they will discover that this polynomial can be factored *over the set of polynomials with real coefficients:* $x^2 + 10x + 8 = (x + 5 + \sqrt{17})(x + 5 - \sqrt{17})$.

7–9 See text annotations.

7–10 This section may be considered optional. It is not so marked because the principles are the same as in Sections 7–8 and 7–9, and the practice is worthwhile. The greater number of possible combinations challenges pupil ingenuity.

(4) To find integers a, b, r, and s such that $(ax + r)(bx + s) = abx^2 + asx + rbx + rs$, note that in the right member of the identity the product of the quadratic and constant terms, $abx^2(rs)$, equals the product of the linear terms, $asx(rbx)$. So to factor $6x^2 - 25x + 14$, express $6x^2(14)$ or $84x^2$ as a product of linear factors whose sum is $-25x$. Since $84 > 0$ while $-25 < 0$, the desired pair of linear factors of $84x^2$ must have negative coefficients: $-x(-84x)$, $-2x(-42x)$, $-21x(-4x)$. The last pair gives $-21x + (-4x) = -25x$. Thus, $6x^2 - 25x + 14 = 6x^2 - 21x - 4x + 14 = 3x(2x - 7) - 2(2x - 7) = (3x - 2)(2x - 7)$.

7–11 If you have omitted Section 7–10, you should replace Written Exercises 5, 6, 23, 24, 25, 30, 31, 38–43, 46, 47, and 56–59 with unused exercises from each of the other sections on factoring. The text summation should not be bypassed.

7–12 The multiplicative property of zero implies that if at least one factor of a product is zero, the product must be zero. The main point of this section is that the converse of this proposition is also true.

To introduce this converse, you may ask, "Can you name two numbers neither of which is zero, such that their product is zero?" Although pupils soon admit that they cannot, point out that it is not enough to *guess*, but that they must *prove* that this is impossible. The proof consists in showing that if one factor of ab is not zero, the other factor must be zero, if ab itself is zero.

(5) You should discuss the words *if and only if* with the class. Emphasize that *two* statements are combined in the sentence on page 263. More extensively you would say: 1. If at least one factor of a product is zero, then the product is zero.
2. If a product is zero, then at least one of its factors is zero.

The multiplicative property of zero shows that the first statement is true. The proof in the text shows that the second statement is true.

(6) Some pupils attempt to extend the theorem to products equal to numbers other than zero. Show that neither part is true in such cases. Suppose that 0 is replaced by 1: 1. If at least one factor is 1, then the product is 1? But $1 \cdot 6 \neq 1$.
2. If a product is 1, then at least one of its factors is 1? But, $\frac{1}{7} \cdot 7 = 1$ and neither $\frac{1}{7}$ nor 7 is 1.

7–13 Not all problems can be solved by linear equations. Thus, introduce the quadratic equation as a necessary tool in problem solving.

Notice that in this section students are using the *factor theorem:* if $x - c$ is a factor of the polynomial A, then c is a root of the equation $A = 0$.

The converse of the factor theorem is the basis of Exercises 43–48.

7–14 Students may be surprised that a root of the equation they set up fails to satisfy the problem. Point out that problems frequently involve conditions not expressed in the equation. Thus, in Example 2, t must satisfy not only the equation $16t^2 + 160t - 9600 = 0$, but also the inequality $t > 0$. Encourage pupils to be alert for such conditions implicitly stated in problems.

This discussion gives an opening for comparing a conditional statement with its converse. Thus, the truth of the statement, *if p then q* does not imply or deny the truth of *if q, then p.*

For example, "If a man is hit by an atom bomb, then the man dies." True. "If a man dies, then he was hit by an atom bomb." False.

If two angles of a triangle are equal, then the sides opposite these angles are equal. True. If two sides of a triangle are equal, then the angles opposite are equal. True.

Chapter 8

The aim of this chapter is to extend the fundamental operations of algebra to fractions, by building on the work of Chapters 4 and 7. Much student knowledge about fractions is limited to the "how" of doing things. This chapter analyzes previous methods to show the "why" also. It gives students a new sense of power over a kind of number about which they may have felt unsure. They should continue to rely less and less on intuitive number relations and more and more on general algebraic laws.

8–1 (1) Notice that we have not excluded expressions like $\frac{\sqrt{x^2 + 1}}{3}$ from the class of algebraic fractions. If we were to restrict the discussion to indicated quotients of polynomials, we would be dealing with a subset of the set of algebraic fractions (the set of rational expressions). Although the examples and exercises are drawn from that subset, the discussion applies to the whole set of algebraic fractions.

Pupils must understand that for all algebraic fractions we assume that the variables have no values for which the denominator is zero. At this stage the resulting restrictions are explicitly stated; later they will not be stated in each case.

With such algebraic fractions as $\frac{x + 3}{x^2}$, as with polynomials, we think of the form, or of the number represented for each value of x. Thus, the *fraction* refers to the symbol $\frac{2}{3}$ or to the number $\frac{2}{3}$. Although this usage does not strictly distinguish between number and numeral, form and function, we use it for simplicity, as long as meaning is clear.

8–2 (2) The general proof is easy:

$$\begin{aligned}
\frac{ac}{bc} &= \frac{a}{b}\cdot\frac{c}{c} && \text{Property of quotients}\\
1 \cdot c &= c && \text{Multiplicative property of 1}\\
\frac{c}{c} &= 1 && \text{1 is the root of } x \cdot c = c\\
\therefore\ \frac{ac}{bc} &= \frac{a}{b}\cdot 1 && \text{Substitution principle}\\
&= \frac{a}{b} && \text{Multiplicative property of 1}
\end{aligned}$$

(3) Some teachers prefer to have pupils repeat the steps of this argument each time they reduce a fraction, rather than having them use the multiplicative property of fractions. In either case, pupils should understand the underlying ideas. Illustrating the process with arithmetic fractions helps understanding.

8–3 See text annotations.

8–4 Here are problems involving per cents, given from a new point of view. The repeated reference to per cent in the textbook is made in an attempt to develop clear understanding of the concept and its application to daily living.

8–5 Here, and earlier in the reduction of fractions, pupils may suggest that they have "canceled" such factors as $x - 1$. The word *cancellation* emphasizes the mechanics, rather than the ideas behind the process. For this reason, we also avoid "slash marks" as in $\frac{\overset{1}{\cancel{(x+1)}}\overset{1}{\cancel{(x-1)}}(x-2)}{\cancel{(x-1)}\cancel{(x+1)}(x+2)} = \frac{x-2}{x+2}$. Although slash marks can be a useful "bookkeeping" device, so many algebraic crimes are committed in the name of cancellation or with slash marks that we urge extreme caution in their use.

8–6 Having the class occasionally run through a deductive argument like the following will remind them that procedures depend on properties.

$3 - t = (-1)(-[3 - t])$	Multiplicative property of -1
$= (-1)(-[3 + (-t)])$	Meaning of subtraction
$= (-1)(-3 + t)$	Property of the opposite of a sum, and $-(-t) = t$
$= (-1)(t + (-3))$	Commutative property of addition
$= (-1)(t - 3)$	Meaning of subtraction

8–7 See text annotations.

8–8 Stress that just as the common factor x is used in combining $3x$ and $4x$, so, too, in combining $\frac{3}{5}$ and $\frac{4}{5}$, the common denominator 5 (the common factor $\frac{1}{5}$) should be retained. This emphasis helps pupils avoid the common errors: $\frac{1}{3} + \frac{4}{5} = \frac{5}{8}$, or $\frac{1}{3} + \frac{4}{5} = \frac{4}{15}$. If such errors appear, have students try to justify their computations on the basis of number properties. Whether or not a calculation conforms with these properties determines its validity.

8–9 (4) The sum of two pennies and four nickels does not equal six of any monetary unit. Before such quantities can be combined, some common unit must be found; here both quantities may be expressed in pennies. So, to combine fractions, find a common denominator. Pupils should note that they are doing the reverse operation to reduction of fractions when they transform a fraction into another having a specified denominator.

(5) The determination of the L.C.D. of a pair of common fractions is a good application of the prime factoring of integers. In Section 7–1 it was used to find the *greatest common factor* of two integers; here, it is used to find their *least common multiple.*

By emphasizing number properties as the basis of technique, you will help students avoid confusion between addition and multiplication of fractions, which leads to errors like: $\frac{3}{5} + \frac{2}{3} = \frac{6}{15} = \frac{2}{5}$.

8–10 See text annotations.

8–11 Encourage pupils to use discretion in choosing the method to simplify a complex fraction. Often, when there is just one fraction in the numerator and in the denominator, Method II is simpler. But when there is a sum of fractions in both numerator and denominator, Method I is usually more effective. Thus, in Written Exercises 1–10, 13, and 14, pupils may be expected (not forced) to choose Method II. In the other exercises Method I seems advisable.

8–12 In each solution, students transform the equation by multiplication by a nonzero number (20). In Solution 2 the transformation is preceded by the addition of the fractions in the left member, so that the multiplier shows up explicitly as the L.C.D. of these fractions. In Solution 1, the pupil still must find the L.C.D., though he does not actually write each fraction with 20 as denominator.

8–13 This interest formula can be thought of as a special case of the percentage formula, $P = B \cdot R$, by identifying P (percentage) with i (interest), B (base) with p (principal), and R (rate or per cent) with rt (equivalent yearly interest rate). By *equivalent yearly interest rate* we mean the annual rate that would produce the same return in one year that $r\%$ per year produces in t years. This approach permits investment problems to be seen as one application of the general concept of other per cent problems. It also provides a fresh approach for those pupils who were unsuccessful with investment problems in arithmetic classes, or who solved them merely by rote. When students have gained an understanding of percent as rate, the formula $P = BR$ may be considered in terms of parallels for *percentage* (annual interest, income, yield, return, dividend) and for *base* (investment, capital, principal, amount deposited).

8–14 See text annotations.

8–15 This is a good place to review the processes that transform an equation into an equivalent one. Emphasize particularly that transformation by multiplication does not permit multiplication by zero (page 84).

Some pupil may suggest the following solution in which the terms in the left member are combined initially: $\dfrac{z^2 + 24}{z^2 - 36} = \dfrac{5}{z - 6}$.

But $\dfrac{5}{z - 6} = \dfrac{5(z + 6)}{z^2 - 36}$; therefore, $\dfrac{z^2 + 24}{z^2 - 36} = \dfrac{5(z + 6)}{z^2 - 36}$

The student reasons: *The denominators of these fractions are equal, so the numerators must also be equal.* Thus, $z^2 + 24 = 5(z + 6)$, $z^2 - 5z - 6 = 0$, and the roots are 6 and -1.

This pupil will claim that he did not multiply by zero; but his error, of course, lies in the italicized sentence above. The sentence is true if the denominators are not zero; but, for $z = 6$ each denominator is zero!

8–16 The pupils here have a chance to use fractions to solve a kind of problem they find intriguing. They should realize that the time in which several people together do a job (if they work at the same rate as when alone) must be less than the time in which one of them does it alone. Consequently, it is not the periods of time that are added, but the fractional parts of the whole job.

8–17 These motion problems represent a variation of those studied earlier. The binomial denominators will cause little difficulty, but the section may be considered optional, if you wish.

Chapter 9

This chapter extends work with graphs from the line to the plane by associating ordered pairs of numbers with points of the plane. The graphs of solution sets of open sentences in two variables, particularly linear equations and inequalities, are drawn. Parabolic graphs are introduced, and the use of graphs in statistics is reviewed.

9–1 This section uses a simple problem to introduce open sentences in two variables and thereby provides motivation for discussing *ordered* pairs of numbers. In the examples and written exercises, solution sets are finite because the replacement set of at least one of the variables is a finite set.

9–2 With the introduction of coordinate axes in the plane you can interpret an ordered pair of numbers geometrically and thus see the geometric significance of the order in which the numbers appear in the pair. In plotting the graph of an ordered pair, stress that the first coordinate is associated with the horizontal number line, and the second, with the vertical number line. You may wish to have pupils compare the graphs of (3, −2) and (−2, 3), or (5, 1) and (1, 5), etc.

(1) Labeling the horizontal and vertical axes x and y, respectively, is a convenience, dictated by mathematical custom. Other letters would serve equally well. What matters is that the first coordinate (abscissa) is associated with the horizontal axis and the second coordinate (ordinate), with the vertical axis.

(2) Some pupils forget that points on the axes, like all other points in the plane, have two coordinates. Oral Exercises 21–24, 29–32 are designed to emphasize that giving one of its coordinates does not locate a point.

9–3 (3) Students should understand that we have not proved that all points whose coordinates satisfy $y = 1 - 2x$ lie on one straight line, nor that every point on this line has coordinates satisfying this equation. Based on experience with a few of the ordered pairs in the solution set of this equation, we *assume* that this is so. An alternative approach would be to define a straight line to be any set of points whose coordinates satisfy a linear equation in two variables. In this case, you would define a linear equation as an equation of the form $ax + by = c$, where at least a or b is not zero.

Notice that the terms *line* and *straight line* are used interchangeably. We never speak of a *curved line*, as we never refer to a *positive negative number*.

(4) You should point out that you draw only an incomplete "picture" of the graph of a linear equation. It is a picture because points and lines are conceptual, not physical objects. The picture is incomplete, because a line extends indefinitely in both directions.

9–4 A pupil who draws several straight lines and several curves, moving his pencil slowly, will quickly gain an intuitive feeling for the slope of a line. By asking, "What makes a straight line straight?" you can lead into a formal discussion of slope. References to the grade of a hill, the pitch of a roof, or the *rise/run* of a stairway or ramp help illustrate the definition of slope. Stress that *rate* of change, not amount, determines the slope.

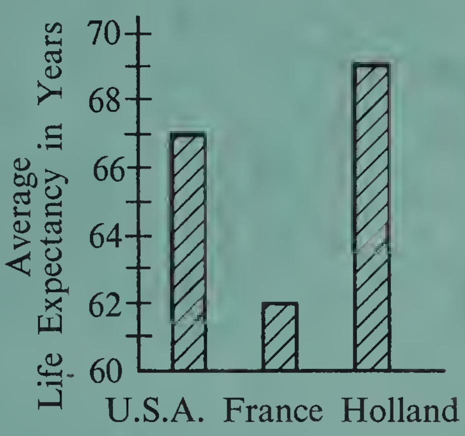
9–5 To introduce this material, you may have several pupils at the blackboard complete tables and draw the graphs of such equations as $y = -2x$, $y = 5x$, $y = 6x + 3$ and $y = x - 2$. From the examples before them, the class can try to relate the slope of the line and the ordinate of its point of intersection with the y-axis to the equation of the line.

9–6 To introduce this section, you may send several pupils to the board to draw, in the same coordinate system, different lines which all have the same slope, say $\frac{5}{2}$. Judicious questioning will bring out that the lines are parallel and that each has an equation of the form $y = \frac{5}{2}x + b$. To find the equation of a particular line, some pupil may insist that he also must know the y-intercept. By showing him that this means knowing a particular point of the line, he can be led to see that knowing any point of the line enables him to determine b.

9–7 If pupils compare the ordinates of several points on a vertical line through $(x, 3)$ for any value of x, say $x = 1$, with the ordinate of $(x, 3)$, they will see that above the line of $y = 3$, $y > 3$, and below, $y < 3$. Apply the same reasoning to the line $y = 2x + 1$. Start at any point on the line, and compare ordinates of points on the vertical line through the starting point with the ordinate of the starting point. The comparison leads to the conclusion: above the line of $y = 2x + 1$, $y > 2x + 1$; below the line, $y < 2x + 1$.

9–8 In connecting the isolated points in Figure 9–15 by a smooth curve, you assume (as when you join the points associated with roots of a linear equation) that the graph has no jumps or other irregularities. By plotting additional points of the graph you can approximate the true shape of the curve more closely. But, since you cannot plot every such point, your curve can be only an approximation. Pupils should realize that the graphs shown are incomplete, since there are points on the curve corresponding to every value of x however large in absolute value.

9–9 Failure to start the scale of a bar graph at zero produces a misleading picture, as indicated by the adjoining graph. A casual look might suggest that life expectancy in the U. S. A. is $3\frac{1}{2}$ times as great as in France, while the difference shown is only 5 years of life.



9–10 See text annotations.

Chapter 10

This chapter presents the usual methods of solving linear systems in two variables. The graphical solution is a convenient bridge from the previous chapter and introduces the concept of equivalent systems. Algebraic skills developed here are applied in solving various problems.

10–1 In discussing a pair of independent equations it is important that pupils understand that any equivalent system will also contain two equations. Furthermore, all methods of solving the system aim to determine the special equivalent system consisting of the equations of the horizontal and vertical lines through the point of intersection.

10–2 (1) If two systems are equivalent, any ordered pair satisfying one system will also satisfy the other. (Your proof of this depends on the properties of equality developed in Chapter 3 and extended in the Sample and Exercises 37–42 on page 85.) If (x, y) is a common root of equations of the first system, then both of these sentences are true: $x + 4y = 27$ and $x + 2y = 21$. Hence, the subtraction property of equality implies that $(x + 4y) - (x + 2y) = 27 - 21$ or $2y = 6$ is true. Then the division property of equality implies that $y = 3$ is true. Now, the substitution principle permits you to replace y by 3 in either of the original *true* sentences and so obtain a true statement; for example,

$x + 4 \cdot 3 = 27$ or $x + 12 = 27$. The subtraction property thus gives the true sentence $x = 15$. Consequently, any common root of the given system also satisfies the system $x = 15$, $y = 3$.

On the other hand, if $x = 15$ and $y = 3$ are true sentences, then $4y = 12$ and $2y = 6$ are also true by the multiplication property of equality. Then, by the addition property $x + 4y = 15 + 12 = 27$ and $x + 2y = 15 + 6 = 21$. Thus, (15, 3), the common solution of $x = 15$ and $y = 3$ must satisfy the given system. This proves that the systems are equivalent. Unless the class is uniformly poor, such a discussion is very worthwhile in striking home the meaning of equivalent systems. Poorer pupils may have difficulty following the argument, but better students should be able to justify the steps in the discussion.

(2) To keep the purpose of the addition or subtraction always before the class, you may use the Written Exercises for oral work. Consider them in connection with questions such as, "When should you add?" and "Which variable can you eliminate by subtraction?"

10–3 The now familiar problem-solving procedure is adapted here to situations involving two unknowns. Emphasize the necessity of forming two equations, each representing a separate relationship. Although the Oral Exercises are very simple, they afford needed practice in setting up pairs of equations. Similar practice is provided by the problems that follow these exercises.

10–4 The use of a negative multiplier leads some students to errors. Such pupils may be advised to consider only the absolute values of the coefficients when determining the multiplier and then to add or subtract, as necessary. Indicating the multiplier will help your pupils.

10–5 (3) Careful questioning will lead the class to the statement that on the graph of $x + 4y = 3$, the abscissa of a point equals the difference: 3 minus four times the ordinate ($x = 3 - 4y$). This relationship holds true, in particular, at the point (if any) where this graph intersects the graph of $2x - 3y = 17$. Hence, at that point the open sentence obtained by replacing x, in $2x - 3y = 17$, by $3 - 4y$ must also be true: $2(3 - 4y) - 3y = 17$.

But this equation is equivalent to $y = -1$. Thus, the ordinate of any common point can be nothing other than -1. But, if the ordinate is -1, the fact that the relationship $x = 3 - 4y$ holds at the point of intersection implies that $x = 3 - 4(-1) = 7$ must be true, and the abscissa must be 7. Consequently, if the two graphs intersect, $(7, -1)$ must be the coordinates of the point of intersection. Substituting 7 for x and -1 for y in both equations shows that the equations are satisfied. Hence, the only possible root $(7, -1)$ actually is a root of the system.

(4) Notice the difference in approach between the elimination and substitution methods. In the elimination scheme, the system is replaced by a sequence of equivalent systems. The final system consists of a pair of simple equations whose common solution set is evident. And, since the initial and final systems are equivalent, that solution set is also the solution set of the given system. In the substitution method, we argue that *if* a common root of the equations exists, then for that pair of numbers (x, y) the expression for x in terms of y given by the first

equation can be substituted in the second equation. From this, we obtain the only possible pair of numbers that can be a common root. There remains the need to prove by actual substitution that the only possible root does satisfy the equations. Thus, in the elimination method the check is included just to guard against numerical errors; in the substitution method, the check is an essential part of the solution.

10–6 This section may be considered optional. However, it provides an easy extension of the graphical technique of solving linear systems and a good review of the graphing of inequalities.

10–7 The problems thus far in this chapter can be solved about as easily with one variable as with two. But this section has problems which are not solved easily with one variable. These provide pupils with new insight into the operation of the number system. The chief difficulty will come from the tendency to confuse *sum of the digits* with *number*, and may be avoided by giving numerical examples of numbers and the sum of the digits of their numerals.

10–8 Many consider the problems in Section 10–8 to be optional in an elementary course; however, they are especially appropriate at this point because their solution calls for the use of two variables. Stress the fact that the motion of the plane depends not only on the rate of the plane in still air, but also on the direction and speed of the wind. Otherwise, the problem is like previous ones. An extra caution regarding change of units is needed.

10–9 Of course, these problems can also be done using one variable. Some classes will find it profitable to compare the two methods of solution.

Chapter 11

This chapter completes the discussion of the number system by introducing irrational numbers. Particular attention is paid to operations with radicals and to an iterative method of determining decimal approximations of square roots.

11–1 This section briefly reviews the kinds of numbers pupils have used and introduces two properties of the set of rational numbers, not yet mentioned.

(1) Questions such as these will point up closure properties of the rational number system: Why is the sum of $\frac{1}{3}$ and $\frac{4}{5}$ a rational number? Why is their product rational? Is their quotient rational? Written Exercises such as 25–31 can initiate worthwhile discussions of comparisons between the properties of integers and of rational numbers.

11–2 The first part of this section may be a review for many. The important idea to transmit is that the set of numbers expressible by terminating or repeating decimals coincides with the set of rational numbers.

(2) Converting a repeating decimal to a common fraction will not be familiar to many students. Notice that the conversion makes the unstated assumption that a repeating decimal does represent some number, N. An indication of how that assumption can be proved is given in the third transparency. The proof, itself, requires the use of the limit theorems of analysis.

11–3 Students must understand that $\sqrt{a}$ for $a > 0$ represents a *positive* number.

(3) Thus, $\sqrt{x^2} = x$ if $x > 0$; but, $\sqrt{x^2} = -x$ if x is negative; $\sqrt{5^2} = 5$, but $\sqrt{(-5)^2} = 5$, not -5. In general, for every x, $\sqrt{x^2} = |x|$. The answers to Oral Exercises 17, 18, 21, and 22 reflect this, as do Written Exercises 23, 24, 25, and 26.
(4) Avoid asserting that, "negative numbers do not have square roots," without adding the phrase, "in the set of directed numbers." Students may be led to the proper formulation by asking them to solve equations like $x^2 = -4$, $y^2 = -1$, $z^2 = -25$, given that the replacement set of each variable is the set of directed numbers.

The reasons for the steps in the proofs on page 404 are:

(5)
1. Given
2. Meaning of principal square root
3. Product of nonnegative numbers is nonnegative.

1. Meaning of a square
2. Commutative and associative properties
3. Meaning of $\sqrt{a}$ and $\sqrt{b}$
4. Substitution principle

11–4 (6) The operations of addition, multiplication, subtraction, and division (except by zero) can be performed freely in the system of rational numbers. The system is also closed under involution. The extraction of roots, however, is not always possible in the set of rational numbers; square roots of negative numbers, for example, cannot be found in this set. But can you always find the square roots of every positive rational number in this system? The purpose of this section is to show that you cannot.

To introduce this section, you may ask pupils to solve the following equations in the set of positive rational numbers: $x^2 = 1$, $y^2 = 2$, $z^2 = 4$. The problem of whether or not $\sqrt{2}$ represents a rational number arises immediately. The argument on page 407 indicates the main ideas in a proof of the following theorem: If the positive square root of a positive integer is a rational number, then the square root must be an integer.

But $1 < \sqrt{2} < 2$; hence, $\sqrt{2}$ cannot be an integer and therefore cannot be rational. Similarly, $1 < \sqrt{3} < 2$ and $2 < \sqrt{5} < 3$, so $\sqrt{3}$ and $\sqrt{5}$ cannot be rational.
(7) We prefer the iterative procedure for determining square roots, over the algorithm frequently taught in arithmetic, for these reasons: First, it focuses the pupils' attention on the fact that $\sqrt{r}$ is the positive number whose square is r. Second, it is a self-correcting procedure. Even though a numerical error may produce a relatively poor estimate for $\sqrt{r}$, the effect of the error is dissipated after one or two iterations. Third, in the absence of numerical errors, the procedure gives a rapid calculation of successive digits. In general, each iteration produces an approximation for $\sqrt{r}$, correct to about twice as many digits as the preceding estimate. Furthermore, the iterative technique is part of a general method of determining roots of any index and, in fact, of approximating real roots of any equation.
(8) The steps outlined in the text are based on the fact that, in general, an iteration beginning with an estimate correct to n digits ends with an approximation correct to $2n - 1$ digits, which has either no error or an error of only 1 or 2 in

the 2*n*th digit. Since a two-decimal-place approximation is usually sufficient for elementary work, two iterations usually give the desired approximation for any number in the range 1–100. By using the product and quotient properties of square roots, every positive number can be replaced by a number in that range before the iteration begins. For example: $\sqrt{1764} = 10\sqrt{17.64}$, $\sqrt{.279} = \frac{\sqrt{27.9}}{10}$, and $\sqrt{24336} = 100\sqrt{2.4336}$.

11–5 (9) Although pupils must have been exposed to this theorem previously, you should not assume that they know how to use it. You should go through the explanatory material and illustrations with them carefully. Pupils should learn to identify the hypotenuse by identifying the right angle in the triangle. Point out that the discovery of this theorem proved that there exist distances like the length of the diagonal in Figure 11–2 that cannot be measured by rational numbers. Figure 11–1 suggests a geometrical argument which may have led the early Greek mathematicians to this theorem.

(10) In connection with the Pythagorean theorem, you may anticipate the common student error: If $c^2 = a^2 + b^2$, then $c = a + b$. Have students making this error discover their mistake by squaring each member of the equation $c = a + b$ and by measuring c in Figure 11–2. Measurement shows that c in that figure does not equal $a + b$ ($\sqrt{2} \neq 1 + 1$).

11–6 This section provides drill in the use of the product and quotient properties of square roots: $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ and $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$. Practice in basic algebraic manipulations is also included. Note, in particular, the frequent use of the fraction property: $\frac{a}{b} = \frac{ac}{bc}$. In rationalizing a numerical denominator, pupils can use their skill in factoring an integer into a product of prime factors.

11–7, 11–8 See text annotations.

11–9 Radical equations are sometimes called *irrational equations*. Careful questioning should lead students to see that the equation in Step 2 of the Example would have resulted if Step 1 had been either $3 - x = 2\sqrt{x}$ or $-(3 - x) = 2\sqrt{x}$. Hence, the equation in Step 2 has as roots the roots of *both* of these equations. Also, because $\sqrt{x}$ (by definition of the symbol $\sqrt{\ }$) represents a nonnegative number, $3 - x = 2\sqrt{x}$ can be satisfied only if $3 - x$ also represents a nonnegative number. Thus, only those roots of $(3 - x)^2 = (2\sqrt{x})^2$ that satisfy the additional provision $3 - x \geq 0$ can satisfy the given equation. The second tentative root, 9, fails to meet that provision because $3 - 9 < 0$. The principle to remember is: If $a^2 = b^2$ then $a = b$, provided that both a and b are positive *or* that neither a nor b is positive.

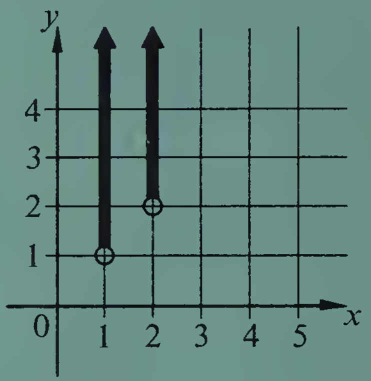
Chapter 12

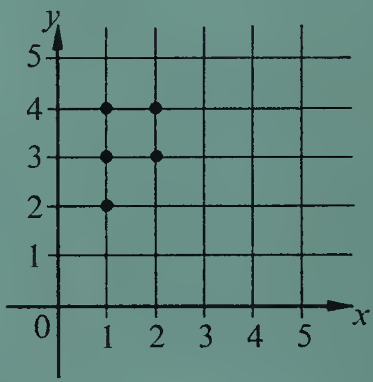
This chapter provides a basic introduction to one of the most important concepts of mathematics: function. Because the ideas of set and of ordered pairs are now thoroughly familiar to students, the development begins with the concept of a relation as a set of ordered pairs. Although a function is introduced as a special kind of relation, the idea of a function as an association or correspondence from one set to another set is brought in quickly. Since the special functions involved in direct and inverse variation are important in general and specific science courses, as well as in mathematics courses, we emphasize them. The variety of practical applications of the function concept is evident in the examples, exercises, and problems drawn from many fields.

12–1 (1) The idea of a relation as a set of ordered pairs of numbers is fundamental. The several methods of specifying a set (roster, rule, graph) should be employed to describe various relations. The words *domain* and *range* should be understood clearly and even described formally in such terms as, "the domain is the set of all first coordinates of a relation," and "the range is the set of all second coordinates."

(2) In introducing open sentences as rules for relations, you should make clear that the relation coincides with the solution set of the sentence, not with the sentence itself. Thus, $y = 2x$, with the domain or set of values of x being $\{1, 2, 3\}$, defines the relation $\{(1, 2), (2, 4), (3, 6)\}$. The same equation, with $x \in \{0, -3\}$, defines a different relation $\{(0, 0), (-3, -6)\}$.

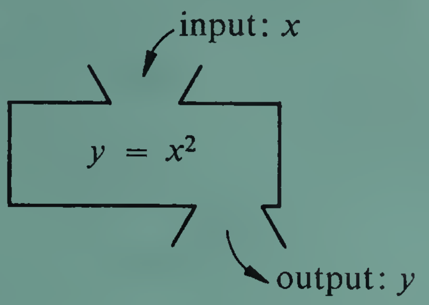
(3) This example also illustrates the importance of specifying the domain of a relation defined by an open sentence. Unless we state otherwise, the range of such relations is the set of numbers obtained by substituting in the open sentence values from the domain. Additional restrictions may be placed on the range. For example, the open sentence $y > x$ with $x \in \{1, 2\}$ defines the relation graphed in the adjoining figure. If y is restricted to $\{2, 3, 4\}$, however, you have the relation graphed in the next figure. This relation is a very small subset of the relation shown in the first figure. With a good class, exercises of this type may be based on Written Exercises 1–18 on pages 437–438 by suitably restricting the domain (values of x) and the range (values of y).

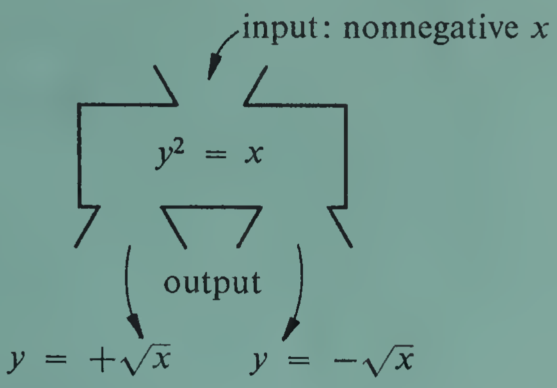




12–2 The function concept is of prime importance in mathematics. We introduce function as a special type of set of ordered pairs because of students' familiarity with the idea of ordered pairs and of the graph of a set of ordered pairs. However, in many situations the idea of a function as an association, a correspondence, or mapping from one set (the domain) to another set (the range) is more useful. With a good class, you may wish to pursue this aspect of the function concept more thoroughly. In this case, you may have students describe each function in the Written Exercises on pages 441–442 in terms of association; then Exercise 18 would read, "the function which assigns to each integer between -2 and 2, inclusive, the negative (or opposite) of its cube."

Notice that the term *function* is reserved for what some textbooks call a "singlevalued function." "Multiple-valued functions" in our development are called *relations.* The fact that a function pairs with each number in the domain, a unique number in the range, can be emphasized by having students think of a machine (*below left*) which delivers one item of output for each item of input. On the other hand, the machine (*below right*) for a relation produces one or more units of output for each x.





The function concept is illustrated by many familiar relationships like those between the weight of a letter and the postage required to post it, a man's age and his life insurance premium, and the speed of a car and its braking distance. Such illustrations emphasize the practical significance of the idea of a function and show that not all functions can be defined as solution sets of equations.

12–3 (4) The setup in the Figure on page 443 may be readily reproduced as a class demonstration, and k may be calculated for the spring used. Such concrete treatment will clarify the nature of a constant and give meaning to the term *direct variation.* It will also demonstrate that the value of the constant is not as immediately important in showing the relationship as the fact that it *is* a constant.

(5) Additional properties of proportions which students may profitably be asked to prove are: If $\frac{y_1}{x_1} = \frac{y_2}{x_2}$, then

1. $\frac{x_1}{y_1} = \frac{x_2}{y_2}$ if $y_1 \neq 0; y_2 \neq 0$
2. $\frac{y_1 + x_1}{x_1} = \frac{y_2 + x_2}{x_2}$
3. $\frac{y_1 - x_1}{x_1} = \frac{y_2 - x_2}{x_2}$

Proof of Part 2: $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ Given

$\frac{y_1}{x_1} + 1 = \frac{y_2}{x_2} + 1$ Addition property of equality

$\frac{y_1}{x_1} + \frac{x_1}{x_1} = \frac{y_2}{x_2} + \frac{x_2}{x_2}$ Substitution principle

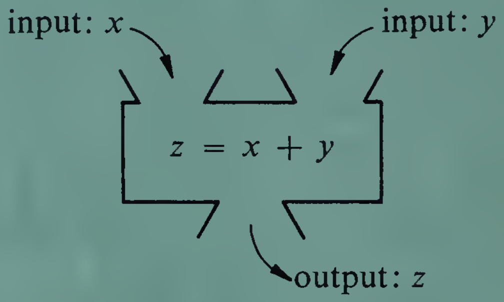
$\frac{y_1 + x_1}{x_1} = \frac{y_2 + x_2}{x_2}$ Addition of fractions

(6) The variety of uses for scale drawings in the home, on the farm, on the job, and so on, should be mentioned to the students. Road maps or wall maps may be studied for scale relationships. In reading a scale, the first number in the ratio

refers to the figure or drawing under inspection; the second number, to the object represented by the figure or drawing.

12–4 In a minimum course, inverse variation may be considered an optional topic, but omitting it leaves students with a very limited concept of variation. Many junior-high-school general-science teachers successfully develop inverse variation in connection with the study of simple machines. The topic is not too difficult and lends itself well to classroom demonstration. To illustrate the lever, various small weights can be hung on a yardstick or meter stick, and calculations may be based on actual observations. For example, these values might be obtained: $d_1 = 6$ cm, $W_1 = 16$ g, $d_2 = 8$ cm, $W_2 = 12$ g. Here k is 96.

12–5 In introducing this section you may wish to have a brief discussion of the idea of a function of two or more variables. A function of two variables assigns to each member of a set of *ordered pairs* of numbers a unique number. Thus, the domain of such a function is a set of ordered pairs, while the range is a set of numbers. You may picture this kind of function by a machine requiring two input items for each item of output. To graph such functions requires a three-dimensional coordinate system; since the graph of the domain is a set of points in the x, y plane. Introducing the z coordinate locates a point in space.



The extension to functions of more than two variables proceeds similarly, except that a physical visualization of the graph of these functions is not easily available.

The Extra for Experts here briefly introduces an important practical application of the solution of linear systems — linear programing. The basic theorem of linear programing is discussed on pages K and L of the second transparency insert in the text.

Chapter 13

In this chapter the solution of the quadratic equation in one variable is extended to the general case, an extension made possible by the algebraic background established in Chapters 11 and 12. The chapter begins with the solution of equations of the forms, $x^2 = s^2$ and $(x - k)^2 = s^2$. This leads to the method of completing the square, in solving quadratic equations, and thus to the quadratic formula. The use of graphs of quadratic functions to determine the solution sets of quadratic open sentences strengthens pupils' graphing techniques and deepens their understanding of the function concept.

13–1 Class discussion should produce a proof of this property along these lines: The addition and subtraction properties of equality imply that the equations $r^2 = s^2$ and $r^2 - s^2 = 0$ are equivalent. But $r^2 - s^2 = (r - s)(r + s)$, so the given equation is equivalent to $(r - s)(r + s) = 0$. A product of two real numbers is zero, if, and only if, *at least one* factor is zero, so $r^2 = s^2$ if, and only if, either $r - s = 0$ or $r + s = 0$; that is, $r = s$ or $r = -s$.

In applying the property just proved, to solve $y^2 = 12$, you also use the fact that every positive real number has two real square roots. As suggested in Chapter 11, this follows from the completeness property of the real number system.

13–2 **(1)** You may introduce this section by having several students go to the board to expand the left members of such equations as $(x - 2)(x - 3) = 0$, $(x + 1)(x + 4) = 0$, $(x - \sqrt{2})(x + \sqrt{2}) = 0$, and $[x - (-3 + \sqrt{10})][x - (-3 - \sqrt{10})] = 0$. From the examples before them, the class may then try to discover the relations between the roots and coefficients of each equation.

(2) To emphasize that the relations as stated depend on having the equation in standard form, $ax^2 + bx + c = 0$ with $a = 1$, you may have the class suggest the relations in case $a \neq 1$. In general, $r + s = -\frac{b}{a}$ and $rs = \frac{c}{a}$.

13–3 The ability to complete a trinomial square is useful in many situations in algebra, analytic geometry, and analysis. Therefore, pupils should understand and be able to use this technique, even though they may not often employ it to solve quadratic equations.

A series of oral exercises like the following, in which pupils supply the missing terms, can introduce the technique.

$$\begin{array}{llll}
x^2 + 6x + \underline{\quad} = (x + 3)^2 & \qquad & z^2 + 2az + a^2 = (z + \underline{\quad})^2 \\
x^2 - 8x + \underline{\quad} = (x - 4)^2 & & z^2 - 2bz + b^2 = (z - \underline{\quad})^2 \\
y^2 \underline{\qquad} + 25 = (y + 5)^2 & & t^2 - 2t + \underline{\quad} = (t - \underline{\quad})^2 \\
y^2 \underline{\qquad} + 36 = (y - 6)^2 & & t^2 + 4t + \underline{\quad} = (t + \underline{\quad})^2
\end{array}$$

13–4 **(3)** All students should understand and be able to use the quadratic formula. Better students should be able to produce its derivation.

In working with the formula, pupils should write the formula each time they use it. Remembering it will come naturally for most students. The first step should be one of substitution only, since the chances for error are greatly increased when too much is attempted at once.

(4) Better students can be encouraged to use the quadratic formula to factor polynomials, as follows:

$$\begin{aligned}
ax^2 + bx + c &= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] \\
&= a\left[x - \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\right]\left[x - \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\right] \\
&= a\left[\left(x + \frac{b}{2a}\right) - \frac{\sqrt{b^2 - 4ac}}{2a}\right]\left[\left(x + \frac{b}{2a}\right) + \frac{\sqrt{b^2 - 4ac}}{2a}\right]
\end{aligned}$$

Using the discriminant, $b^2 - 4ac$, pupils can determine whether or not the polynomial is factorable over the set of real numbers and can also find the factors. Thus, for $a = -2$, $b = 4$, $c = -1$, they have:

$$\begin{aligned} -2x^2 + 4x - 1 &= -2[x^2 - 2x + \tfrac{1}{2}] \\ &= -2\left[x - \frac{-4 + \sqrt{8}}{-4}\right]\left[x - \frac{-4 - \sqrt{8}}{-4}\right] \\ &= -2\left[(x - 1) + \frac{\sqrt{2}}{2}\right]\left[(x - 1) - \frac{\sqrt{2}}{2}\right] \end{aligned}$$

13–5 Be sure that pupils understand the difference between the quadratic equation in *one* variable $ax^2 + bx + c = 0$ and the quadratic function determined by the quadratic equation $y = ax^2 + bx + c$ in *two* variables. The quadratic function consists of *all* the ordered pairs (x, y) in the solution set of $y = ax^2 + bx + c$. The solution set of $ax^2 + bx + c = 0$ consists of the first coordinates of those ordered pairs in which the second coordinate is zero.

13–6 To assert that a is a positive number means that $a > 0$; while $a < 0$ means that a is negative. Thus, the laws governing the multiplication of real numbers stated on page 135 in Chapter 4 can be expressed

$$\begin{aligned} ab > 0 \quad &\text{if, and only if, } a > 0 \text{ and } b > 0, \text{ or} \\ &\phantom{\text{if, and only if, }} a < 0 \text{ and } b < 0. \\ ab < 0 \quad &\text{if, and only if, } a > 0 \text{ and } b < 0, \text{ or} \\ &\phantom{\text{if, and only if, }} a < 0 \text{ and } b > 0. \end{aligned}$$

Pupils should understand clearly the significance of the fact that the product in each case is compared with zero, not any other number.

Emphasize that this multiplicative property can be used in solving a quadratic inequality only when one member of the inequality is zero. For example, $ab > 1$ does not imply either that $a > 1$ or that $b > 1$; thus, $27 \cdot \frac{1}{9} > 1$, but $\frac{1}{9}$ is not greater than 1. If the property is clearly understood to be a restatement of the "law of signs" for real numbers, such false extensions are less likely to occur.

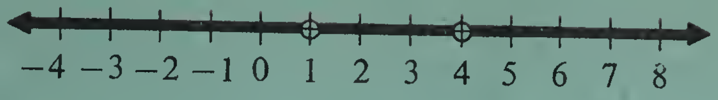
The method of completing the square can also be used for quadratic inequalities.

Example: $x^2 - 2x > 3$ Solution: $x^2 - 2x + 1 > 3 + 1$

$$\therefore \ (x - 1)^2 > 4$$

This inequality will be satisfied if, and only if, $|x - 1| > 2$. But this means that on the number line x must be more than 2 units from 1 ($x < -1$ or $x > 3$).

13–7 The essence of the graphic solution is the discovery of the real roots of the equation $ax^2 + bx + c = 0$. Since there are at most two such roots, they determine at most three intervals on the number line for x in which to examine the nature of $ax^2 + bx + c$. As the graph of $y = ax^2 + bx + c$ shows, the expression $ax^2 + bx + c$ does not change sign *within* any of these intervals. Hence, it is sufficient to determine the nature of the expression for some convenient value of x in each interval to know the nature of the expression throughout the interval. Consider $x^2 - 5x + 4 > 0$. The roots of $x^2 - 5x + 4 = 0$ are 1 and 4. The three intervals on the number line are: $x < 1$, $1 < x < 4$, and $x > 4$.



Now, determine whether $x^2 - 5x + 4$ is positive or negative in each interval by testing this expression for some value within each interval:

Interval	x	Value of $x^2 - 5x + 4$	
$x < 1$	0	4	positive
$1 < x < 4$	2	-2	negative
$x > 4$	5	4	positive

Thus, $x^2 - 5x + 4 > 0$ if $x < 1$ *or* $x > 4$.

Chapter 14

The aim of this chapter is to give students a taste of some important mathematical concepts of later high-school courses. The unit on numerical trigonometry also provides practice in algebraic skills, review of the function concept, and interesting applications. Furthermore, the chapter ties together several threads that have been running through this course: deduction of theorems from postulates, ratio and proportion, measurements and approximations, geometric concepts, and the idea of function.

14–1 This course is planned so that students appreciate that the whole structure of algebra is built on the foundation of a few assumptions about sets of numbers. In this and the following two sections students meet assumptions about geometrical figures (sets of points) and receive a rudimentary idea of how theorems in geometry follow from geometric axioms. These sections also review the geometrical facts needed for the unit on trigonometry.

14–2 In discussing geometric assumptions with a superior class, you may wish to emphasize that geometry deals with abstract concepts, not concrete objects. Therefore, the axioms are not "evident truths" dictated by experience, but are assumptions accepted as the basis for the geometry to be developed. To the extent that the points and lines of this geometry are suitable idealizations of physical objects, the geometry is applicable in physical situations. None the less, the significance of a geometry lies in its being a logically consistent system, not in its applicability to the physical world.

14–3 (1) Note that an angle (sometimes called a *geometric angle*) is simply a set of points. When a rotation is associated with an angle, the resulting object is called a *directed angle* (page 173). For the purposes of numerical trigonometry, the simple notion of angle is sufficient.

14–4 The ideas included in this brief section are of fundamental importance for the sections on trigonometry. An intuitive understanding without proof is quite sufficient, but absolutely essential. Make sure your pupils have a thorough grasp of *right triangle*, the *Pythagorean theorem*, and the *angle-sum theorem.*

14–5 The emphasis, as the text notes, is not on measuring an angle by the use of a protractor, but on assigning a measure to each angle.

14–6 (2) In this section the tangent function is defined by actually setting up the ordered pairs (angle, number) that constitute the function. Do not allude to the tangent as the opposite side over the adjacent side of a right triangle at this point; that will come in good time.

The general plan of development is as follows. The pupils already know the meaning of slope of a line. The tangent of an angle is defined in terms of slope. Then the "line-value" of the tangent is introduced by using a particular point on the terminal ray of the angle. This enables the pupil to visualize the new function.

The corresponding "line-values" for the sine and the cosine are introduced, the sine and cosine defined in terms of them, and the basic properties easily derived.

Finally, it is shown that $\sin A = \frac{y}{r}$, $\cos A = \frac{x}{r}$, and $\tan A = \frac{y}{x}$ when $P(x, y)$ is any point on the terminal ray of an angle in standard position and $r = OP$. The right triangle definitions come in last of all as a convenience in handling applied problems.

14–7 (3) With some classes it may be necessary to stress the fact that the point of the proof is that it enables us to use *any* point (not merely a special, particular point) on the terminal ray to find sine, cosine, or tangent.

14–8 Because the table of trigonometric functions looks more complicated than the tables pupils have hitherto used, it should be carefully examined in class. Use it not only for finding required function values; use it also to develop familiarity with the way in which values vary with change in angle. Students should note that only the tangent can exceed 1 in value, and they should learn how to estimate the measure of an angle from its function value. With rapid learners, you may show interpolation. If time permits, you may also have pupils draw graphs of the trigonometric functions.

14–9 In general, these questions should be asked in solving a problem in numerical trigonometry: 1. What is the relation of each side (given or required) with respect to the angle? (opposite, adjacent, or hypotenuse?) 2. What function of the angle involves two such sides?

(4) The meaning of *angle of elevation* or *angle of depression* may be clarified by having pupils sight light bulbs, floor plugs, and the like. Emphasize that such angles are measured from the horizontal to the line of sight.

At the beginning of each period, for several days from now on, you may draw various right triangles on the blackboard, indicate the given and required parts, and have the pupils do the exercises. These skeletal diagrams should simulate problem situations, but the numbers should be small. Such drill will strengthen understanding.

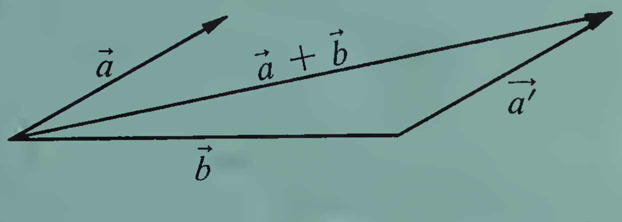
To keep the material simple, we have not defined the cotangent, secant, and cosecant functions, but you may introduce these, if you wish. In particular, by defining the cotangent function, you may discuss the relation between the functions of complementary angles. You may then show that long division can be avoided by using the complementary angle whenever an angle and the opposite side are given and the side adjacent to the angle is required.

Pupils get a great deal of satisfaction from the construction and use of instruments for indirect measurement. Suggestions are given in the Eighteenth Yearbook of the National Council of Teachers of Mathematics, *Multisensory Aids in the Teaching of Mathematics.* See also the Nineteenth Yearbook, *Surveying Instruments, Their History and Classroom Use.*

14–10 (5) The theorem that the sides of similar triangles are proportional is based on the definition of the sine. The exercises are selected to show that while frequently the solution of triangles by numerical trigonometry and by similar triangles are virtually identical, there are cases where the latter is genuinely easier. This may appeal to scouts in the class who may have used such methods in field work. Incidentally, they might be helpful as leaders in outdoor work to find the height of the school, a flagpole, or a nearby monument.

14–11 Only an informal understanding of angles greater than 180° is required. Make sure the ideas are clear but do not press for precise definitions.

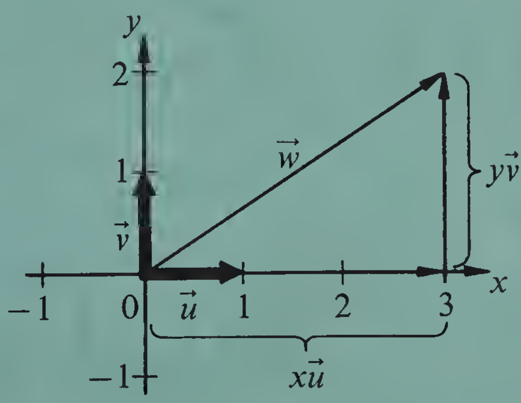
(6) This section gives pupils opportunity to extend their previous learning in connection with motion problems, to use their knowledge of scale drawings, and to apply their newly gained skill with trigonometric functions. Since pupils' knowledge is limited to the trigonometry of right triangles, they now can compute resultants only of vectors acting at right angles to each other. The resultants of other vectors can, of course, be estimated from scale-drawings and the use of the triangle law of addition: $\vec{a}$ and $\vec{a'}$ are equivalent vectors.



14–12 (7) The fundamental idea in this section is that every vector can be expressed as the sum of vectors in two arbitrarily chosen directions. With a superior class you may wish to formalize this idea as follows: Let $\vec{u}$ and $\vec{v}$ be the unit vectors (length 1) in the given coordinate system. Then, every vector $\vec{w}$ can be expressed in the form

$$\vec{w} = x\vec{u} + y\vec{v},$$

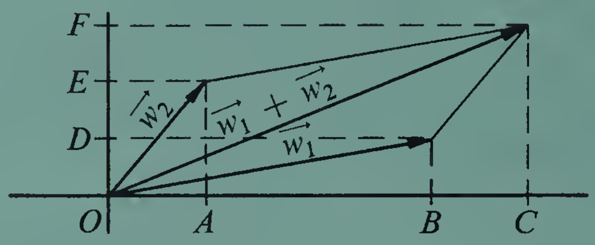
where $x\vec{u}$ is a vector of length $|x|$ and direction $\vec{u}$ (if $x \geq 0$) or $-\vec{u}$ (if $x < 0$), and $y\vec{v}$ has length $|y|$ and direction $\vec{v}$ ($y \geq 0$) or $-\vec{v}$ ($y < 0$).



(8) When two vectors have been resolved into their horizontal and vertical components, they can be combined by adding corresponding components. Thus, if $\vec{w}_1 = x_1\vec{u} + y_1\vec{v}$ and $\vec{w}_2 = x_2\vec{u} + y_2\vec{v}$ then

$$\vec{w}_1 + \vec{w}_2 = (x_1 + x_2)\vec{u} + (y_1 + y_2)\vec{v}.$$

The proof of this requires more knowledge of geometry than your students possess, but it can be made plausible, at least, by considering such figures as this:

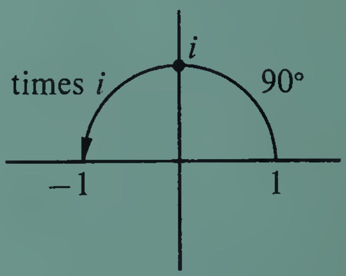


$OA = BC; \therefore OC = OB + OA$
$OD = EF; \therefore OF = OD + OE$

(9) In extending the concept of number, the idea of numbers as partners for points has been exploited throughout the course. Thus, we paired the numbers of arithmetic with the points on a ray, then, by considering the points on the opposite ray, we introduced negative numbers. We suggested the existence of irrational numbers by the need to have a number for every point on the line.

Therefore, it seems natural to motivate the further extension of the number system by seeking number partners for every point in the plane. It is convenient to think of each such point as determining a vector from the origin, so that the extension involves pairing numbers with vectors. This approach to complex numbers has the advantage of removing the feeling that complex numbers are peculiar or "unreal." That they are conceptual objects is true, but so, too, are the arithmetic numbers. Tying the idea of complex number to that of vector also motivates the definition of addition.

To help pupils see why we define $i^2 = -1$, you may point out that with every complex number is associated an angle having as one side the corresponding vector and as the other side the positive half of the x-axis. With 1 is associated an angle of 0°; with -1, 180°; with i, 90°. When a number a is multiplied by 1, the angle of the product is the same as that of a. When -1 is the multiplier, a is turned counterclockwise through 180°. So, when i is the multiplier, a should be turned through 90°:



But then, $i \cdot i = -1$, since rotating i counterclockwise through 90° produces -1.

(10) Students may be surprised to learn that the system of complex numbers is complete in the sense that any polynomial equation over this system is completely solvable in this system. The proof of this fundamental theorem of algebra is beyond the scope of high-school mathematics courses, but pupils should know that the theorem is true.

Assignment Guide

This Assignment Guide will assist you in planning a minimum, average, or maximum course of study. The time schedule (see p. 6) and the daily assignments are suggestions, not prescriptions, and may be modified according to needs of students, requirements of the course of study, length of each classroom period, and number of days classes meet during the school year.

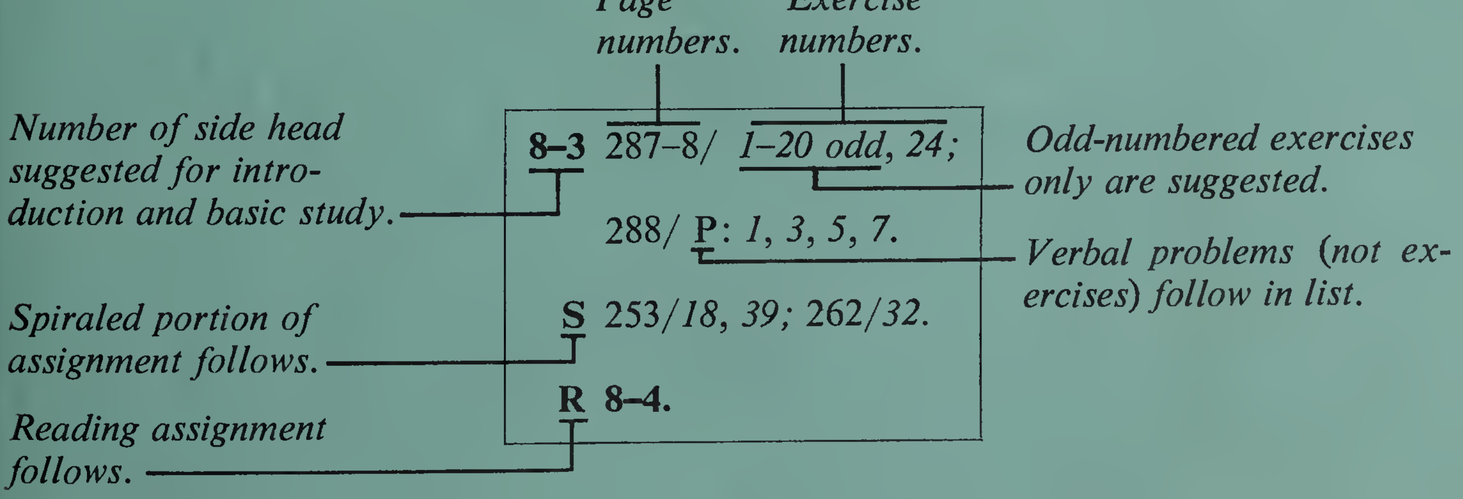
For each course, 170 specific daily assignments are suggested. The 10 days remaining in the time schedule are left for work with Chapter 15, Comprehensive Review and Tests, or for any other use you deem appropriate. Exercise and problem assignments are generally ample to provide necessary practice. You may wish to poll your students periodically to determine whether modifications in assignments are necessary to keep homework demands within the time span recommended by your school.

Spiraled assignments — an important feature of the daily lessons suggested in this guide — will contribute significantly to the student's understanding of concepts and to improvement or maintenance of skills. The use of spiraled assignments means that exercises or problems on a specific topic are assigned at spaced intervals following the initial day of study on this topic. A more complete discussion of spiraled assignments may be found in THE LEARNING OF MATHEMATICS, Its Theory and Practice, The Twenty-First Yearbook of The National Council of Teachers of Mathematics, pages 323–326.

Assignments may be enriched with selections from Extra for Experts, Just for Fun, Human Equation and Vocational Interest sections. Many students will profit from the use of PROGRAMMED PRACTICE FOR MODERN ALGEBRA, Structure and Method, Book One; this is a self-checking workbook to be used in connection with home assignments.

To further assist you, Progress Tests to Accompany MODERN ALGEBRA, Structure and Method, Book One, contain 47 tests: 42 cover small blocks of material; 4 are Cumulative Review Tests, and 1 is a year-end Comprehensive Test. Progress Tests are available in several forms.

Daily assignments are arranged as follows:


Page numbers.
Exercise numbers.
Number of side head suggested for introduction and basic study.
8–3 287–8/ 1–20 odd, 24;
288/ P: 1, 3, 5, 7.
S 253/18, 39; 262/32.
R 8–4.
Odd-numbered exercises only are suggested.
Verbal problems (not exercises) follow in list.
Spiraled portion of assignment follows.
Reading assignment follows.

Lesson	Minimum Course	Average Course	Maximum Course
1	**1–1** 3–5/1–25 odd.	**1–1** 3–5/1–27 odd.	**1–1** 3–5/1–29 odd. **R 1–1, 1–2, 1–3.**
2	**1–2** 6–7/1–10odd, 11, 15. **1–3** 9–10/1–20 odd, 25–36 odd. **S** 4/4.	**1–2** 6–7/1–15 odd. **1–3** 9–10/1–50 odd. **S** 4/4.	**1–2** 6–7/1–17 odd. **1–3** 9–10/13–54 odd. **S** 3–5/2, 4, 24.
3	**1–4** *Use orals in class.* **1–5** 14–5/1–10 odd, 11, 13. **S** 6/2; 10/25–36 even.	**1–4** *Use orals in class.* **1–5** 14–5/1–34 odd. **S** 6–7/6, 16; 9–10/20, 48.	**1–4** *Use orals in class.* **1–5** 14–15/1–34 odd. **S** 6–7/10, 16; 9–10/22, 42. **R 1–6, 1–7.**
4	**1–6** 17/1–16 odd. **S** 4–5/6, 26; 14–15/2, 12.	**1–6** 17/1–18 odd. **1–7** 19/1–14 odd. **S** 4–5/6, 26; 14–15/2, 12.	**1–6** 17/5–18 odd. **1–7** 19/1–14 odd. **S** 4–5/6, 28; 15/12, 20, 24. **R 1–8.**
5	**1–7** 19/1–14 odd. **S** 6–7/4, 12, 14; 9/2–10 even; 10/42, 44; 17/2, 4.	**1–8** 22/1–24 odd. **S** 6–7/8, 18; 9–10/4, 36; 17/2; 19/6.	**1–8** 22/7–11, 13–20 odd. **R 1–9.** 24/1, 3, 5. **S** 6–7/8, 18; 10/44; 17/10, 12; 19/2, 10.
6	**1–8** 22/1–12 odd, 14. **S** 14–5/4, 14; 19/2.	**1–9** 24/1–25 odd. **S** 14–5/4; 22/13–19 even. *Announce chapter test.*	**1–9** 24/7–32 odd. **S** 15/26, 28; 22/12, 16, 22. *Announce chapter test.*
7	**1–9** 24/1–16 odd. **S** 4/10; 17/6, 10; 22/1–12 even. *Announce chapter test.*	*Administer chapter test.* **2–1 R 2–1.** 37–8/5–16 odd. **S** 4/8; 17/12; 19/14; 24/18.	*Administer chapter test.*
8	*Administer chapter test.* **S** 6/10; 10/42; 19/4; 24/1–29 even.	*Discuss test. See p.* 27, T.E. **S** 37–8/1–19 even; 38–9/P: 2, 4; 6/10; 10/23; 22/20.	*Discuss test. Note annotation, p.* 27, T.E. *Students should study Extra for Experts “on their own.”*
9	*Discuss test. See p.* 27, T.E. **2–1 R 2–1.** 37–8/1–10 odd. **S** 15/8; 22/13, 22.	**2–2** 42–3/1–36 odd. **S** 15/14; 24/6, 30; 38/12; 39/P: 9.	**2–1** 37–9/1–15 odd, P: 1–12 odd. **S** 5/30; 10/54; 17/2. **R 2–1, 2–2.**
10	**2–2** 42/1–24 odd. **S** 17/12, 14; 24/17, 19, 21; 38/P: 1–8 odd.	**2–3** 47–8/1–30 odd. **S** 17/4; 19/12; 42/8, 20.	**2–2** 42–3/1–30 odd, P: 3, 5, 7. **S** 15/22; 39/13–20 odd. **R 2–3.**
11	**2–3** 47/1–10 even. **S** 5/27; 19/10; 42/15–30 even.	**2–4** 50/1–14 odd. **S** 4–5/29; 22/12; 38/14, P: 6; 47/4.	**2–3** 47–8/5, 7, 9, 21–40 odd. **S** 19/8; 43/47, 49, 51. **R 2–4.**

Lesson	Minimum Course	Average Course	Maximum Course
12	**S** 6/*8*; 10/*41, 43*; 22/*15, 17*; 38/*10, 18*; 38/P: *2, 6*; 47–8/*1–30 odd*.	**2–5** 54/*1–25 odd*; 54–5/P: *1, 5, 11*. **S** 6–7/*12*; 10/*42*; 24/*14*; 42/*18*; 43/P: *4*; 48/*31, 33*.	**2–4** 50–1/*1–10 odd, 17, 19*. **S** 15/*30*; 39/*18*; 48/*41, 43, 45*. **R 2–5.**
13	**2–4** 50/*1–10 odd*. **S** 15/*10*; 24/*23, 30*; 42–3/*25–32 odd*.	**2–6** 58–9/P: *1, 4, 7*. **S** 15/*20*; 47/*8, 41, 46*; 54/*11–20 even*.	**2–5** 54–5/*1–25 odd*, P: *5–12 odd*. **S** 17/*8*; 44/*9*; 51/*18*. **R 2–6.**
14	**2–5** 54/*1–20 odd*. **S** 17/*16*; 48/*12, 18, 41*; 50/*1–10 even*.	**S** 17/*16*; 19/*8*; 38/*8*; 55/P: *8, 12*; 58/P: *2*; 59/P: *5, 8, 10, 11, 16, 18*. *Announce chapter test.*	**2–6** 59/P: *4–18 odd*. **S** 15/*32, 34*; 55/*13, 14*. *Announce chapter test.*
15	**S** 19/*8*; 38/P: *4*; 54/*1–10 even*; 54–5/P: *1–12 odd*.	*Chapter test.* **R 3–1.** 70/*1–8, write answers.* **S** 22/*8*; 43/*32*; 43/P: *6*.	*Administer chapter test.* **S** 44/P: *11*; 59–60/P: *18–26 even*.
16	**2–6** 58–9/P: *1–10 even*. **S** 5/*27*; 22/*18, 24*; 43/*33, 35*; 50/*12*.	*Discuss test. See p.* 27, T.E. **3–2** 73/*1–12 even*. **S** 4/*18*; 24/*24*; 47/*6*; 58/*3*.	*Discuss test and assignment.* **S** 48/*47*; 55/P: *15, 16*; 58/P: *10, 12, 14*. **R 3–1, 3–2.**
17	**S** 7/*11, 13*; 10/*45, 54*; 24/*25, 31*; 54/*12, 18*; 54–5/P: *2, 4*; 58–9/P: *1–16 odd*.	**3–3** *Use odd orals in class.* 74–5/*1–26 even, write answers.* **S** 7/*14*; 10/*28, 38*.	**3–1** *Use orals in class.* **3–2** 73/*1–12 odd*. **S** 19/*12*; 51/*23*; 59–60/P: *18–25 odd*. **R 3–3.**
18	**S** 15/*17; 19*; 48/*14, 40*; 58–9/P: *11–20 even*. *Announce chapter test.*	**3–4** 79/*1–20 odd*. **S** 15/*23–34 even*; 37/*7*; 38/P: *3*; 54–5/*2 (girls)*, 4 (*boys*).	**3–3** *Use orals* 74/*1–20 in class.* 75/*21–24, write answers.* **S** 55/P: *17*; 60/P: *27*; 70/*7*; 73/*1–8 even*. **R 3–4.**
19	*Administer chapter test.* **3–1** **R 3–1.** 70/*1–8, write answers.*	**S** 17/*3, 8*; 19/*2, 10*; 42/*14*; 44/P: *10*; 59/*6*; 79/*1–30 even*.	**3–4** 79/*11–30 odd*; *35–42 odd*. **S** 48/*49*; 75/*25*. **R 3–5.**
20	*Discuss test. See p.* 27, T.E. **3–2** 73/*1–12 odd*. **S** 19/*14*; 43/P: *2–5*; 54/*23*.	**3–5** 83/*1–30 odd*. **S** 22/*15*; 48/*34, 35*.	**3–5** 83/*1–30 odd*. **S** 44/P: *15*; 73/*10*; 79/*22–32 even*. **R 3–6.**
21	**3–3** 75/*21, 23, 25, write answers.* **S** 22/*19*; 48/*37, 42*; 73/*2, 6, 10*.	**3–6** 84–5/*1–18 odd, 37*. **S** 24/*8*; 79/*21, 23*; 83/*1–10 even*.	**3–6** 84–5/*5–22 odd*. **S** 55/P: *18*; 79/*31*; 83/*16–22 even*. **R 3–7.**

Lesson	Minimum Course	Average Course	Maximum Course
22	**3–4** 79/*1–20 odd.* S 24/*27*; 59/*17*; 75/*22*.	**3–7** (*Omit example 2*) 88/*1–28 odd.* S 54/*1–10 even*, 84–5/*1–18 even*, *40*.	**3–7** 88–9/*15–32 odd*, P: *1–8 odd.* S 60/P: *28*; 85/*37*. R **3–8**
23	S 50/*14*; 73/*8*; 79/*1–28 even.*	**3–7** (*Example 2*) 89/P: *1, 3, 5, 7* (*boys*), *8* (*girls*). S 38/P: *5*; 59/P: *13, 14*; 83/*28*; 88/*20–30 even.*	**3–8** 93–4/*23–32 odd*, P: *2, 4, 8.* S 73/*12*; 85/*38*; 88–9/*44, 47, 49*; 89/P: *8*; 90/P: *26.*
24	**3–5** 83/*1–20 even.* S 38/*11*; 55/P: *6, 8*; 75/*24*; 79/*21.*	**3–8** (*Omit example 5*) 93/*1–18 odd.* S 43/P: *2, 5*; 85/*31, 36*; 89–90/P: *2, 6, 9, 26.*	S 75/*26*; 85/*39*; 89/*45, 46*; 90/P: *28*; 94/*35–44 odd*; 95/P: *9–16 odd.* *Announce chapter test.*
25	**3–6** 84/*1–18 even.* S 43/P: *6*; 79/*23*; 83/*1–25 odd.*	**3–8** (*Example 5*) 94/P: *1, 3, 4, 7.* S 47/*10*; 88–9/*1–10 even, 46, 47*; 93/*1–18 even.* *Announce test.*	*Administer chapter test.* R **4–1, 4–2.**
26	S 48/*16, 28, 44*; 59/P: *19*; 73/*4*; 83/*22, 24*; 84/*1–18 odd, 37.*	*Administer chapter test.* S 102–4/*1–30 odd.*	*Discuss test.* **4–1** 113/*1–12 odd;* **4–2** 114–5/*1–12 odd, 13, 15, 17.* S 79/*33*; 95/P: *20.*
27	**3–7** (*Omit example 2*) 88/*1–28 odd.* S 75/*26*; 83/*26*; 85/*39.*	*Discuss test and assignment questions with students.* S 104/*31–53 odd.*	**4–3** 118–9/*9, 11, 13, 29, 31, 33.* S 95/P: *16*; 113/*10*; 114/*10*; 115–6/*21, 23*, P: *3, 5, 7, 9.*
28	**3–7** (*Example 2*) 89/P: *1–16 odd.* S 79/*25*; 85/*19, 21.* 88/*1–10 even.*	**4–1** 113/*1–12.* S 59/P: *20*; 88/*33*; 94–5/P: *5, 9.*	**4–4** 122–3/*7, 8, 9*; 13–24 even. S 118/*12, 14*; 119/*35, 37*; 119–20/P: *4, 6, 10.* R **4–5**
29	**3–8** (*Omit example 5*) 93/*1–18 odd.* S 55/*10*; 88/*12, 14*; 89/P: *1–12 even.*	**4–2** 114–5/*1–26 odd*; 115–6/P: *1–10 even.* S 89/P: *4*; 112–3/*12–28*, *write answers to these orals.*	**4–5** 124/*1–16 odd.* S 60/P: *30*; 90/P: *30*; 120/P: *11*; 123/*21–27 odd.*
30	**3–8** (*Example 5*) 94–5/P: *1–16 odd.* S 83/*27*; 88/*16*; 93/*1–12 even.*	**4–3** 118–9/*1–32 even*, *46.* 119/P: *1, 3, 5.* S 83/*14*; 94/*33, 34*; 114/*6.*	**4–6** 127/*5–14 odd; 27–35 odd.* 128/P: *1–9 odd.* S 124/*6, 8.*

Lesson	Minimum Course	Average Course	Maximum Course
31	**S** 58–9/P: *21,* 85/*22, 34;* 90/P: *18–22;* 94/P: *1–8 even.* *Announce chapter test.*	**4–4** 122–3/*1–25 odd.* **S** 73/*5;* 85/*33;* 94/*8;* 119/*42, 45.*	**4–7** 131/*1–15 odd.* **S** 85/*41;* 127/*5–14 even, 19, 21, 28, 30.*
32	*Administer chapter test.* **S** 102–3/*1–26 even.*	**4–5** 124/*1–25 odd, 14.* **S** 88/*31;* 114–5/*8, 16, 32;* 123/*22, 28.*	**S** 95/P: *21;* 115/P: *4, 6.* 123/*26;* 124/*14, 18;* 127/*32, 34;* 131/*13–30 even.*
33	*Discuss test and assignment.* **S** 104/*27–53 even.*	**4–6** 127/*1–25 odd.* **S** 79/*25;* 90/*21, 24;* 119/P: *6;* 124/*2, 6.*	**S** 85/*40;* 131/*17—30 odd;* 132/*31–36 odd;* 132/P: *1–12 odd.* **R 4–8.**
34	**4–1** 113/*1–12.* **S** 83/*28;* 88/*30;* 90/*24.*	**S** 93/*24, 26;* 123/*26, 27;* 127/*1–34 even;* 128/P: *1–7 odd.*	**4–8** 137/*11–30 odd.* **4–9** 141/*14, 16, 18.* **S** 91/P: *32;* 124/*16;* 132/*34.*
35	**4–2** 114/*1–12 odd.* 115–6/P: *1–10 odd.* **S** 73/*12;* 85/*28;* 93/*19.*	**4–7** 131/*1–30 odd.* **S** 83/*18, 20;* 95/P: *13;* 114/*10;* 124/*8.*	**S** 132/*36;* 137/*16, 18, 20;* 137/*33–44 odd;* 141/*13–22 odd.* **R 4–10.**
36	**4–3** 118–9/*1–14 odd, 29, 33.* 119/P: *1, 3, 5.* **S** 95/P: *10, 12;* 115/P: *2, 4.*	**S** 85/*26, 41;* 120/P: *12;* 131/*1–30 even;* 132/P: *1–12 even.*	**4–10** 143/P: *1, 3.* **S** 85/*42;* 132/P: *6;* 137–8/*41–52 even,* 141/*13–24 even.* *Announce chapter test.*
37	**4–4** 122–3/*1–20 even.* **S** 79/*30;* 88/*20;* 120/P: *7, 9, 11.*	**4–8** 137/*1–40 odd.* **S** 88/*34, 36;* 123/*24, 34.*	*Administer chapter test.* **S** 103/*9–26 odd;* 144/P: *5, 7.* **R 5–1.**
38	**4–5** 124/*1–12 odd.* **S** 90/P: *23, 27;* 115/*13, 15, 17;* 123/*13, 15, 17.*	**4–9** 141/*1–30 odd.* **S** 90/P: *27;* 124/*16;* 132/*33, 34;* 137/*1–20 even.*	*Discuss test and assignment.* **5–1** 159/*7–28.* **S** 104/*45–53 odd;* 141/*27.*
39	**4–6** 127/*1–20 odd;* 128/P: *2, 4, 6.* **S** 83/*29;* 93/*21, 24;* 120/P: *8;* 124/*2, 13, 14.*	**S** 95/P: *18;* 115/*35;* 128/P: *9;* 137–8/*30–50 even;* 141/*1–20 even.* *Announce chapter test.*	**5–2** 163/*1–12 odd.* **S** 159/*31, 33, 41, 43, 45.* **R 5–3, 5–4.**
40	**S** 85/*31;* 95/P: *14;* 123/*19;* 127/*1–20 even;* 128/P: *1–9 odd.*	*Administer chapter test.* **5–1 R 5–1.** 159/*1–20 even.* **S** 60/P: *26, 28.*	**5–3** 165–6/*2–12 odd, 17.* **5–4** 168/P: *2–10 odd.* **S** 163/*17–28 odd.*

Lesson	Minimum Course	Average Course	Maximum Course
41	**4–7** 131/*1–20 odd;* 132/P: *1–12 odd.* S 115/*21;* 124/*15.*	*Discuss test and assignment.* S 123/*20;* 132/P: *1, 5 odd;* 142/*31;* 159/*1–30 odd.*	**5–5** 171–2/P: *5, 7, 10, 12.* **5–6** 175–7/*5–12 even, 17, 19,* P: *3, 5, 7.* S 166/*15;* 169/P: *14, 17, 19.*
42	S 88/*31;* 119/*30, 34;* 127/*27–30;* 128/P: *8;* 131/*1–25 even;* 132/*33, 34.* 132/P: *2, 4, 6.*	**5–2** 163/*1–20 odd.* S 124/*18;* 159/*26, 40, 42.*	**5–7** 181/P: *2, 3, 5, 9, 12.* S 159/*39, 44;* 176/*22;* 177/P: *8, 13.*
43	**4–8** 137/*1–30 even.* S 90/P: *26;* 123/*23–30 odd.*	S 127/*25–35 odd;* 137/*41, 43;* 159/*28, 44, 46;* 163/*6–25 even.*	**5–8** 183–4/P: *1–10 odd.* S 172/P: *16;* 181/P: *6, 8, 10;* 182/P: *15, 17.*
44	**4–9** 141/*1–20 even.* S 93/*25;* 124/*17;* 131/*26;* 137/*1–10 odd.*	**5–4** 168/P: *1, 3, 6, 9, 11.* S 115/*22;* 142/*32;* 159/*48;* 163/*26.*	S 159/*46;* 176/*23;* 177/P: *14;* 182/P: *14;* 183–4/P: *1–10 even;* 189/*8, 19.* **R** *Inserts pp.* A–E. *Announce chapter test.*
45	S 95/*19;* 116/P: *10;* 127/*31;* 137/*11–40 odd;* 141/*1–20 odd, 29.*	**5–5** 171–2/P: *1, 5, 9.* **5–6** 175–6/P: *1, 5, 10, 14, 20, 21, 25.* S 120/*11;* 132/*3, 9;* 163/*22;* 168/*5.*	*Administer chapter test.* **R 6–1, 6–2.** S 182/P: *16, 20;* 184/P: *11, 12.*
46	S 119/*35–40;* 127/*31–35;* 131–2/*28, 35;* 138/*49–54;* 141/*22–30 even.* *Announce chapter test.*	**5–7** 180–1/P: *1, 3, 5.* S 123/*22;* 132/P: *7;* 159/*5;* 172/P: *10;* 175/*6;* 177/P: *9.*	*Discuss test.* **6–1** 199–200/*10–20 odd, 23.* **6–2** 202/*4, 6, 9, 20, 22.* S 172/P: *17.*
47	*Administer chapter test.* **5–1 R 5–1.** 159/*1–20 even.* S 60/P: *26.*	**5–8** 183–4/P: *1, 3, 5.* S 124/*4;* 137/*24;* 159/*36;* 168/*7;* 181/P: *2, 11.*	**6–3** 204/*15, 17, 21, 23.* S 191/*55;* 200/*18, 20, 30;* 202/*3, 23, 30.* **R 6–4.**
48	*Discuss test and assignment.* **5–2** 163/*1–12 even;* S 124/*20, 21;* 159/*21–36 even.*	S 141/*22;* 163/*25;* 172/P: *15;* 176/*22;* 181/P: *4;* 184/P: *4, 6.* *Announce chapter test.*	**6–4** 206/*12–18 even.* **6–5** 207–8/*9, 11, 13, 19, 21.* S 204/*25, 27.* **R 6–6.**
49	S 127/*21;* 137/*36, 38, 40;* 159/*1–28 odd;* 163/*1–22 odd.*	*Administer chapter test.* **6–1 R 6–1.** 199–200/*1–22 even.* S 42/*28, 30.*	**6–6** 210/*6, 7, 9, 17, 21.* **6–7** 212/P: *2, 5.* S 191/*59;* 206/*17;* 208/*23, 25.* **R 6–8.**

Lesson	Minimum Course	Average Course	Maximum Course
50	**5–4** 168/P: *1–12 odd.* **S** 115/*33–35;* 141/*27;* 159/*40.*	*Discuss test and assignment.* **6–2** 202/*1–29 odd.* **S** 200/*15–29 odd.*	**6–8** 214/*1–10 odd.* **S** 203/*32;* 210/*10, 18, 22, 41;* 212–3/P: *3, 8, 9.*
51	**5–5** 171–2/P: *1–12 odd.* **S** 119/*41;* 131/*30;* 163/*14;* 168/P: *2, 4.*	**6–3** 204/*1–24 odd.* **6–4** 206/*1–21 odd.* **S** 172/P: *11;* 175/*2;* 202/*8, 10.*	**6–9** 218/*15–24 odd.* **S** 191/*56;* 206/*22;* 210/*43;* 213/P: *11;* 214/P: *1, 3, 5.*
52	**5–6** 175–6/*1–18 odd;* 177/P: *1–12 odd.* **S** 123/*24;* 171/P: *2, 4.*	**6–5** 207–8/*1–22 odd.* **S** 163/*21;* 182/P: *16;* 200/*24;* 204/*2, 4;* 206/*2, 4.*	**6–10** 219/*1–10 odd.* **S** 210/*27;* 213/P: *6;* 215/*6;* 218/*15–24 even.*
53	**5–7** 180–1/*1–10 odd.* **S** 124/*19;* 138/*49;* 159/*41;* 175/*1–12 even.*	**6–6** 210/*1–24 odd.* **S** 141/*24;* 163/*2;* 184/P: *7;* 202/*12;* 208/P: *1, 3.*	**6–11** 220/*9, 12, 20.* **S** 172/P: *18;* 212/P: *6.* **R 6–12.**
54	**5–8** 183–4/P: *1–8 odd.* **S** 127/*24;* 141/*25;* 163/*20;* 172/P: *10;* 181/*2.*	**6–7** 212–3/P: *1, 3, 5, 9.* **S** 168/P: *2;* 204/*25;* 206/*18;* 210/*25.*	**6–12** 223/*19, 24, 26, 28.* **S** 189/*13, 17;* 191/*60;* 210/*44;* 219/*6, 8.*
55	**S** 169/P: *18;* 172/P: *12;* 177/P: *2, 6;* 181/P: *4;* 183/P: *2.*	**6–8** 214/*1–13 odd;* 214–5/P: *1–9 odd.* **S** 159/*22;* 176/*20;* 208/P: *5;* 212/P: *2.*	**S** 191/*53;* 203/*33;* 204/*32;* 210/*28;* 220–1/*6, 25, 31;* 223/*11, 20, 29, 30.*
56	**S** 168/P: *8;* 172/P: *15;* 176/*19;* 182/P: *15;* 184/P: *4.* *Announce chapter test.*	**6–9** 218/*1–20 odd.* **S** 182/P: *19;* 202/*20;* 210/*26;* 214/*2.*	**S** 191/*52, 57;* 200/*31;* 204/*33;* 208/P: *8;* 209/P: *9;* 210/*29;* 213/P: *11, 12;* 215/P: *10;* 223/*27, 36.* *Announce chapter test.*
57	*Administer chapter test.* **6–1 R 6–1.** 199–200/*1–20 even.* **S** 43/*32.*	**6–11** 220/*1–24 odd.* **S** 163/*4;* 184/P: *8;* 204/*26;* 206/*10;* 218/*6, 8.*	*Administer chapter test.* **S** 190/*41;* 191/*61;* 204/*34;* 232/*45.* (*Optional:* **R** *Extra for Experts:* 233/*1–15 odd.*)
58	*Discuss test and assignment.* **6–2** 202/*1–23 odd.* **S** 43/*34;* 138/*56;* 163/*16;* 176/*20;* 199–200/*1–20 odd.*	**6–12** 223/*1–15 odd.* **S** 208/*18;* 215/P: *6;* 220/*1–12 even.*	*Discuss test and assignment.* **7–1** 240/*5, 15, 17, 21–24, 31, 33.* **S** 191/*54;* 213/P: *14;* 215/P: *9;* 221/*32.*
59	**6–3** 204/*1–20 odd.* **6–4** 206/*1–15 odd.* **S** 141/*23;* 181/P: *6;* 200/*21;* 202/*2, 10.*	**S** 168/P: *8;* 200/*28;* 210/*14;* 218/*12–24 even;* 223/*1–12 even.* *Announce chapter test.*	**7–2** 242–3/*13–27 odd.* **S** 190/*50;* 210/*30;* 219/*19;* 240/*6, 16, 25, 37.*

Lesson	Minimum Course	Average Course	Maximum Course
60	**6–5** 207–8/*1–18 odd.* S 168/P: *12;* 184/P: *8;* 200/*23;* 204/*21, 23;* 206/*17.*	*Administer chapter test.* S 230–1/*1–28 odd.*	**7–3** 246/*11, 13, 15.* **7–4** 247/*10–18 odd.* S 209/P: *10;* 223/*22;* 242–3/*16, 18, 20, 22, 30;* 244/P: *6.*
61	**6–6** 210/*1–20 odd.* S 171/P: *8;* 202/*4;* 207–8/*2, 6, 16,* P: *2.*	*Discuss test and assignment.* S 231–2/*29–54 odd.* *Announce test.*	**7–5** 250/P: *1, 3.* **7–6** 253/*11–20 odd.* S 213/P: *13;* 244/*7;* 247–8/*18–29 odd;* 248/*44.*
62	**6–7** 212/P: *1–8 odd.* S 159/*45;* 176/*21;* 204/*21;* 206/*16;* 210/*22.*	*Administer cumulative test.* *Progress Test* **20** *is recommended.*	**7–7** 254/*1–14 odd.* S 221/*33;* 240/*18;* 248/*33, 40;* 253/*21, 25, 27, 31, 37.*
63	**6–8** 214/*1–12 odd.* 214–5/P: *1–8 odd.* S 163/*18;* 181/P: *8;* 200/*30;* 207/*8;* 212/P: *2.*	*Discuss test.* **7–1** 240/*1–45 multiples of 4.* S 210/*6, 8;* 220/*16, 18.*	**7–8** 257/*7, 9, 11, 17, 19, 21, 25, 27, 31.* S 243/*26;* 253/*32, 38;* 254/*2, 4, 6, 8, 21.*
64	**6–9** 218/*1–15 odd.* S 184/P: *6;* 202/*8, 12.* 210/*24;* 214/*2,* P: *2.*	**7–2** 242–3/*1–33 odd.* S 212/P: *6;* 223/*23;* 240/*1–30 odd.*	**7–9** 258/*6, 7, 14–28 odd.* S 223/*25;* 248/*34;* 257/*16, 18, 23, 28, 33.*
65	**6–11** 220/*1–20 odd.* S 168/P: *10;* 204/*24;* 206/*2, 4;* 212/P: *4;* 218/*2, 4.*	**7–3** 246/*1–21 odd.* S 202/*22;* 215/*8;* 242–3/*1–20 even.*	**7–10** 260/*5–14 odd, 21.* S 233/*14;* 248/*35;* 253/*33;* 257/*8, 10;* 258/*12, 14, 26, 31.*
66	**6–12** 223/*1–10 even.* S 172/P: *13;* 207–8/*10, 12,* P: *1, 5;* 214/*4,* P: *4;* 220/*1–15 even.*	**7–4** 247–8/*1–27 odd.* S 204/*28;* 206/*16;* 218/*4;* 240/*18;* 246/*2.*	**7–11** 262/*13–31 odd.* S 243/*38;* 257/*12, 32, 34;* 260/*14, 22.*
67	S 176/*22;* 200/*22, 31.* 210/*1–15 even;* 218/*6, 8, 10;* 223/*1–15 odd.*	**7–5** 250/P: *1, 2.* S 207/*2;* 220/*20;* 242/*28;* 247/*2, 4, 6.*	**7–12** 264/*3, 5, 9, 13.* S 258/*20, 22, 23;* 262/*13–31 even, 38.*
68	S 181/P: *10;* 202/*14;* 212/P: *8;* 220/*16–24 even;* 223/*11–20 even.*	**7–6** 252–3/*1–30 odd.* S 210/*2;* 223/*14;* 246/*4;* 251/P: *7.*	**7–13** 266/*9–26 odd.* S 248/*37;* 253/*35;* 260/*16, 23;* 264/*8, 15, 17.*
69	S 184/P: *10;* 204/*4, 6;* 206/*6, 8;* 214/*6;* 215/P: *6;* 223/*16–24 odd.* *Announce chapter test.*	**7–7** 254/*1–14, 21.* S 212/P: *4;* 240/*22;* 247/*8;* 252/*2, 4.*	**7–14** 269–70/P: *3, 5, 13, 19.* S 221/*34;* 255/*24;* 262/*44;* 263/*58;* 266/*26–35 even.*

Lesson	Minimum Course	Average Course	Maximum Course
70	*Administer chapter test.* **S** 230–1/*1–37 odd.*	**7–8** 256–7/*1–24 odd.* **S** 214/*4, 6;* 242/*22, 24;* 250/P: *3;* 255/*23.*	**S** 212/P: *17;* 243/*31;* 255/*25;* 257/*36;* 258/*29;* 260/*18;* 262/*49;* 266/*27;* 267/*41, 43;* 269–70/P: *7, 12, 16, 21, 22.*
71	*Discuss test and assignment.* **S** 231–2/*38–54 odd.*	**7–9** 258/*1–24 odd.* **S** 218/*2;* 242/*26;* 246/*6;* 252/*6, 8;* 256/*2, 4, 6.*	**S** 223/*37;* 231/*43;* 253/*39;* 259/*34;* 262/*53;* 263/*57;* 266/*35;* 267/*42, 45;* 269/P: *10;* 270/P: *20.* *Announce chapter test.*
72	**S** 230–2/*1–54 multiples of 4.* *Announce test.*	**7–10** 260/*1–12 odd.* **S** 220/*22;* 243/*30, 32;* 247/*18;* 257/*8.*	*Administer chapter test.* **S** 230/*1, 4, 5, 15, 19.*
73	*Administer cumulative test.* *Progress Test* **20** *is recommended.*	**7–11** 262/*1–27 odd.* **S** 223/*16;* 240/*26;* 251/*5;* 257/*25;* 260/*2, 4.*	**8–1** 282/*5, 9, 11, 17, 21, 23, 26.* **8–2** 285/*11–24 odd.* **S** 230–1/*17, 31, 39, 42.*
74	*Discuss test.* **7–1 R 7–1.** 240/*1–45 multiples of 4.*	**7–12** 264/*1–12.* **S** 246/*12, 14;* 253/*14, 37;* 255/*25;* 260/*14, 21;* 262/*1–18 even.*	**8–3** 287–8/*3, 9, 14, 15, 19, 21;* 289/P: *7, 9, 12, 15.* **S** 282/*24, 27;* 285/*16, 33.*
75	**7–2** 242–3/*1–25 odd.* **S** 207/*4, 14;* 220/*21, 23.* 240/*2, 10, 18, 26, 34.*	**7–13** 266/*1–38 odd.* **S** 171/P: *3;* 258/*12, 14;* 264/*13, 15.*	**8–4** 290–1/*3, 11, 18, 21, 25;* 291/P: *6–16 odd.* **S** 288/*10, 22;* 289/P: *10, 16.*
76	**7–3** 246/*1–12 odd.* **S** 210/*16;* 223/*25.* 242–3/*1–25 even.*	**7–14** 269–70/P: *1, 3, 5, 7, 11, 19.* **S** 258–9/*25, 34;* 266/*1–10 even.* *Announce chapter test.*	**8–5** 294/*13, 15, 19, 27.* **S** 285/*24, 35;* 290/*12, 20;* 291/*27,* P: *14, 16.*
77	**7–4** 247–8/*1–24 odd.* **S** 213/P: *9;* 240/*6, 14;* 246/*2, 17.*	*Administer chapter test.*	**8–6** 295–6/*6, 13, 16, 21, 23, 24.* **S** 282/*10;* 294/*14, 26, 40.*
78	**7–5** 250/P: *1, 2.* **S** 215/P: *8;* 243/*26, 27;* 247–8/*1–10 even.*	*Discuss test.* **8–1** 282/*1–21 odd.* **S** 247/*10;* 258/*2;* 264/*17;* 269/P: *2, 6.*	**8–7** 297/*4, 10, 11, 16, 19.* **S** 289/P: *13;* 291/P: *17;* 296/*20, 22, 27;*
79	**7–6** 252–3/*1–20 odd.* **S** 218/*12;* 246/*4, 14.* 250/P: *3.*	**8–2** 284–5/*1–24 odd.* **S** 250/P: *4;* 260/*6;* 266/*22;* 282/*2, 4.*	**8–8** 298–9/*2, 4, 7, 9.* **8–9** 301/*5, 7, 12, 14, 17.* **S** 297/*15, 17.*

Lesson	Minimum Course	Average Course	Maximum Course
80	**7–7** 254/*1–14 odd.* **S** 221/*25*; 240/*22, 30*; 247–8/*12, 25*; 252/*1–10 even.*	**8–3** 287–8/*1–20 odd, 23.* 288–9/P: *1, 3, 5, 7.* **S** 253/*18, 39*; 262/*32*; 269/P: *4*; 285/*4, 12.*	**8–10** 303–4/*6, 8, 18, 24.* **S** 299/*11*; 301/*18, 19, 21, 23.* 302/*27, 31.*
81	**7–8** 256–7/*1–18 odd.* **S** 223/*26*; 243/*30*; 251/P: *7*; 254/*1–14 even.*	**8–4** 290–1/*1–28 odd.* 291/P: *1, 3, 5, 9.* **S** 254/*18*; 262/*34*; 282/*8*; 287/P: *2.*	**8–11** 305/*2, 6, 7, 11, 15, 17.* **S** 285/*36*; 294/*42*; 299/*13*; 303–4/*11, 20, 23.*
82	**7–9** 258/*1–16 odd, 25.* **S** 246/*6*; 252–3/*11–20 even*; 256–7/*1–12, even.*	**8–5** 293–4/*1–21 odd.* **S** 257/*33*; 264/*19*; 285/*34*; 291/P: *2.*	**8–12** 307/*15–24 odd.* **S** 291/P: *18*; 302/*30, 32*; 305/*8, 12, 16, 18, 26.*
83	**7–10** 260/*1–12 odd.* **S** 247–8/*20, 22*; 254/*21*; 258/*17–24 odd.*	**8–6** 295–6/*1–23 odd.* **S** 258/*28, 32*; 266/*24*; 288/*12*, P: *2*; 293/*2, 10.*	**S** 289/P: *18*; 297/*13*; 301/*22, 24*; 307–8/*15–30 even.* *Announce test* **(8–1—8–12).**
84	**S** 240/*1, 9*; 251/P: *8*; 257/*13–24 even*; 258/*1–16 even, 26*; 260/*1–6 even.*	**8–7** 296–7/*1–16 odd.* **S** 260/*22*; 269/P: *8*; 282/*18*; 291/P: *6*; 295/*2, 10.*	*Administer test.* **R 8–13.** 309/P: *7–16 odd.* **S** 285/*43*; 302/*26*; 305/*21*; 308/*29.*
85	**7–11** 262/*1–18 odd.* **S** 240/*17, 23, 33*; 243/*28*; 253/*21*; 258/*27, 31.*	**8–8** 298–9/*1–14 odd.* **S** 262/*36, 48*; 285/*16*; 294/*22*; 296/*2.*	*Discuss test and assignment.* **8–14** 311/P: *1–12 odd.* **S** 297/*14*; 309/P: *10, 14.*
86	**7–12** 264/*1–12 odd.* **S** 246/*8*; 254/*15, 22*; 260/*21*; 262/*38.*	**8–9** 301/*1–24 odd.* **S** 288/*16*; 289/P: *10*; 295/*8*; 298–9/*2, 4, 10.*	**8–15** 313/*7–16 odd.* **S** 311/P: *4, 10, 13, 14.*
87	**7–13** 266/*1–16 odd.* **S** 248/*27, 28*; 257/*25, 31, 34*; 264/*2, 13.*	**8–10** 303/*1–15 odd.* **S** 264/*16*; 297/*6*; 301–2/*12–30 even.*	**8–16** 315–6/P: *1–12 odd.* **S** 302/*28*; 313/*10, 16.* **R 8–17.**
88	**7–14** 269–70/P: *1–12 odd.* **S** 251/P: *6*; 258–9/*32, 34*; 262/*39*; 266/*1–10 even.*	**8–11** 305/*1–17 odd, 21.* **S** 266/*34*; 282/*24*; 294/*14*; 299/*6*; 303/*2, 14.*	**8–17** 318/P: *3–7.* **S** 311/P: *2*; 316/P: *6, 10, 12.*

Lesson	Minimum Course	Average Course	Maximum Course
89	**S** 243/*29, 30;* 253/*22;* 260/*8;* 264/*8, 14;* 266/*12, 16.* 269/P: *2.* *Announce chapter test.*	**8–12** 307/*1–10 odd.* **S** 270/P: *12;* 285/*36;* 295/*14;* 302/*27;* 305/*2.*	**S** 302/*37;* 305/*23;* 310/P: *17;* 311/P: *6;* 313/*14;* 318/P: *8–11;* 328/*40.* *Announce test* **(8–13—8–17).**
90	*Administer chapter test.*	**8–13** 309/P: *1–12 odd.* **S** 288/*18;* 297/*4;* 303/*4;* 307/*11, 13, 15, 17.*	*Administer 30-minute test.* **S** 327/*5, 7, 8, 11, 15, 22.*
91	*Discuss test.* **8–1** **R 8–1** 282/*1–12 odd.* **S** 246/*10;* 257/*28;* 262/*44;* 269/P: *6.*	**8–14** 311/P: *1–10 odd, 13.* **S** 299/*12;* 305/*10;* 309/P: *2.*	*Discuss tests and assignment.* **R 9–1.**
92	**8–2** 284–5/*1–17 odd.* **S** 248/*26;* 258/*33;* 264/*4;* 282/*2, 17.*	**8–15** 313/*1–14 odd.* **S** 294/*16;* 302/*25;* 307/*4, 6;* 311/P: *2, 11.*	**9–1** 336–7/*1, 3, 9, 11, 13, 27.* **S** 308/*27;* 311/P: *12;* 316/P: *8, 13;* 327/*5.*
93	**8–3** 287–8/*1—10 odd;* 288–9/P: *1–12 odd.* **S** 251/P: *5;* 260/*14;* 266/*24;* 285/*19, 21.*	**S** 282/*20;* 296/*16;* 303/*10;* 309/P: *15;* 311/P: *4, 12;* 313/*6–20 even.*	**9–2** 340/*1, 4, 8, 9, 13, 19, 20.* **S** 310/P: *18;* 316/P: *14;* 328/*41;* 336/*14, 17, 19.*
94	**8–4** 290–1/*1–28 multiples of 4;* 291/P: *1–12 odd.* **S** 253/*23;* 269/P: *8;* 282/*10.*	**8–16** 315–6/P: *1–10 odd.* **S** 285/*18, 37;* 297/*10;* 305/*12;* 311/P: *14;* 313/*4.*	**9–3** 342/*1, 6, 8, 9, 13, 18, 20.* **S** 328/*27;* 340/*21;* 343/*26.*
95	**8–5** 293–4/*1–18 odd.* **S** 255/*23;* 263/*56;* 284–5/*1–10 even;* 290/*2, 14, 22;* 291/P: *2.*	**8–17** 318/P: *1–8 odd.* **S** 288–9/P: *4, 11;* 297/*16;* 307/*10;* 313/*15;* 315/P: *2.* *Announce chapter test.*	**9–4** 345–6/*3–12 even.* **S** 328/*43;* 342/*11, 12, 14, 22.*
96	**8–6** 295–6/*1–18 odd.* **S** 257/*21;* 264/*15;* 287/*2;* 288/P: *2;* 293/*2, 10.*	*Administer chapter test.* **9–1** **R 9–1.** 336/*1–16 odd.*	**9–5** 348/*1, 3, 5, 11.* **S** 345–6/*5, 9, 11, 13.*
97	**8–7** 296–7/*1–10 odd.* **S** 258/*18;* 266/*14;* 282/*14;* 285/*12, 14, 16;* 295/*1–10 even.*	*Discuss test and assignment.* **9–2** **R 9–2.** 340/*1–14 odd.* **S** 293/*12;* 303/*8;* 311/P: *6;* 315/P: *4;* 336/*1–16 even.*	**9–6** 350/*1, 3, 10, 16.* **S** 348/*2, 6, 16.*

Lesson	Minimum Course	Average Course	Maximum Course
98	**8–8** 298–9/*1–10 odd.* **S** 260/P: *22;* 270/P: *12;* 285/*18, 20;* 296–7/*1–10 even.*	**9–3** 342/*1–14 odd.* **S** 295/*12;* 305/*4;* 313/*2;* 318/P: *2;* 340/*2, 14.*	**9–7** 352–3/*11, 13, 15, 19.* **S** 336/*21;* 348/*7, 10, 13.* **R** *Inserts* G–L *following p.* 356.
99	**8–9** 301/*1–20 odd.* **S** 287/*4;* 288/P: *4;* 295/*12;* 298–9/*1–10 even.*	**9–4** 345–6/*1–14 odd.* **S** 297/*12;* 307/*12;* 336/*17;* 342/*2.*	**S** 313/*19;* 337/*28;* 350/*5, 17, 23;* 352–3/*12, 14, 17.* **R** *Extra for Experts pp.* 461, 462.
100	**8–10** 303/*1–15 odd.* **S** 262/*20;* 282/*12;* 297/*11, 12;* 301/*1–10 even.*	**9–5** 348/*1–14 odd.* **S** 299/*14;* 309/P: *4;* 316/P: *6;* 340/*4, 16;* 345/*2.*	**S** 328/*42;* 342/*21;* 346/*14;* 350/*11, 21, 25, 27;* 462/*3.*
101	**S** 264/*6;* 282/*13;* 293–4/*6, 14;* 299/*11;* 301/*12; 20, 23;* 303/*2, 12.*	**S** 301/*2;* 311/P: *8;* 318/P: *4;* 342/*14;* 345/*4;* 346/*8;* 348/*2, 4.*	**9–8** 354/*2, 7, 13.* **S** 316/P: *15;* 328/*39;* 352/*10;* 353/*16.*
102	**8–12** 307/*1–10 odd.* **S** 266/*17;* 285/*23;* 295/*14;* 301/*14.*	**9–6** 350/*1, 5, 9, 11, 15, 17.* **S** 303/*12;* 313/*17;* 336/*18;* 345/*6.*	**9–9** 356–7/P: *2, 4, 6.* **S** 342/*19;* 354/*14.*
103	**8–13** 309/P: *1–10 odd.* **S** 269/P: *4;* 287/*6;* 289/P: *6;* 297/*13;* 303/*4;* 307/*2, 4.*	**S** 305/*14;* 340/*19;* 348/*6, 12, 14, 16.* 350/*2, 10, 19.*	**9–10** 357–8/*1, 3.* **S** 337/*26;* 354/*15.* 357/P: *5.*
104	**8–14** 311/P: *1, 3, 5, 7.* **S** 290/*13;* 291/P: *4;* 299/*12;* 309/P: *2, 4.*	**9–7** 352/*1–12 odd.* **S** 124/*20;* 307/*8;* 316/P: *11;* 342–3/*21, 23.*	**S** 307/*25;* 311/P: *8;* 316/P: *16;* 328/*45;* 354/*19;* 357/P: *7;* 358/*2.* *Announce chapter test.*
105	**8–15** 313/*1–10 odd.* **S** 294/*16, 18;* 301/*16, 24;* 307/*6, 8;* 311/P: *9.*	**S** 115/*34;* 309/P: *8;* 318/P: *6;* 343/*24;* 346/*14;* 350/*23, 29;* 352–3/*13, 19.*	*Administer chapter test.*
106	**8–16** 315–6/P: *1, 3, 5, 7.* **S** 282/*15, 16;* 296/*18;* 303/*6;* 309/P: *11;* 313/*2, 8, 12.*	**9–8** 354/*1, 3, 5.* **S** 311/P: *10;* 336/*21;* 348/*8.*	*Discuss test.* **S** 328/*44;* 343/*25;* 346/*17;* 345/*16.* **R** **10–1.**
107	**8–17** 318/P: *1, 3, 5.* **S** 285/*22, 24;* 297/*14;* 311/P: *10;* 315-6/P: *2, 9.*	**S** 313/*19;* 340/*18, 20;* 353/*21;* 354/*7, 9, 13.*	**10–1** 370/*3, 5, 7, 10, 13, 16.* **S** 358/*4.*

Lesson	Minimum Course	Average Course	Maximum Course
108	S 289/P: *8, 12;* 299/*13;* 307/*10, 11;* 313/*4, 13.* 318/P: *2, 7.* *Announce chapter test.*	**9–9** 356/*1, 3.* S 342–3/*20, 26;* 350/*20, 24.*	**10–2** 372/*13–22 odd.* S 354/*8;* 370/*6, 14.*
109	*Administer chapter test.* S 326–7/*1–18.*	**9–10** 357–8/*1, 3.* S 316/P: *8;* 346/*10;* 354/*14;* 356/*4.*	**10–3** 374/P: *2, 6, 8, 9.* S 354/*17;* 372/*10, 18, 22.*
110	*Discuss test.* S 327–8/*19–34 all, 40–45 odd.*	S 318/P: *10;* 348/*18;* 352/*14;* 358/*2.* *Announce chapter test.*	**10–4** 376/*5–14 odd.* S 374/P: *7.*
111	**9–1** 336/*1–16 odd.* S 296/*16;* 313/*6, 15.* 315/P: *4.*	*Administer chapter test.*	**10–5** 378/*6, 7, 14.* S 370/*11;* 376/*6, 8, 14.*
112	**9–2** 340/*1–14 odd.* S 297/*15;* 307/*12;* 316/P: *6;* 336/*2, 8.*	*Discuss test.* **10–1** 370/*1, 3, 5, 13.*	S 377/P: *5, 7, 11;* 378/*9;* 379/P: *4, 5.*
113	S 299/*13;* 309/P: *10;* 318/P: *4;* 340/*14.*	**10–2** 372/*1–12 odd.* S 342/*4, 8, 19;* 370/*2, 6, 17.*	**10–6** 380/*6, 9.* S 377/P: *10;* 378/*12, 18;* 379/P: *6, 8.*
114	**9–3** 342/*1, 5, 7, 11, 13.* S 301/*18;* 311/P: *2;* 336/*14.*	S 346/*12;* 352/*2;* 356/*2;* 370/*15;* 372/*13, 15, 17, 21.*	**10–7** 382–3/P: *2, 3, 7, 9, 11.* S 372/*23;* 376/*10.* 380/*7;* 381/*18.*
115	**9–4** 345–6/*1–12 odd.* S 303/*8;* 313/*10;* 340/*19;* 342/*2, 3, 14.*	**10–3** 374/P: *1–10 odd.* S 348/*17;* 358/*4;* 370/*4.*	**10–8** 384/P: *1, 3, 5, 7.* S 381/*19;* 383/P: *8, 10, 13;* 462/*5.*
116	**9–5** 348/*1, 5, 9, 13, 17.* S 316/P: *8;* 345–6/*2, 12, 13.*	**10–4** 376/*1–12 odd.* S 354/*8;* 372/*9;* 374/P: *2.*	**10–9** 385/P: *2, 4, 6.* S 370/*17;* 384–5/P: *4, 6, 8, 9;* 386/P: *9.*
117	S 307/*13, 15;* 318/P: *8;* 336/*4;* 342/*4;* 348/*2.*	S 350/*8, 25;* 376/*2, 4, 6, 13;* 377/P: *1–10 odd.*	**10–10** 387/P: *1, 3, 5, 7.* S 372/*25;* 380/*11;* 385/P: *1, 5.*
118	S 309/P: *8;* 340/*16;* 345–6/*4, 14;* 348/*11.*	**10–5** 378/*1–10 odd.* S 318/P: *9;* 370/*9;* 374/P: *4.*	S 376/*12;* 377/P: *1;* 379/P: *7, 9;* 382/P: *1;* 383/P: *12;* 387/P: *4, 6, 8.*

Lesson	Minimum Course	Average Course	Maximum Course
119	**9–7** 352/*1–12 odd.* **S** 311/P: *8;* 348/*7, 15.*	**10–5** 379/P: *1, 3, 5, 7.* **S** 352/*12;* 372/*12;* 376/*8;* 378/*2, 11.*	**S** 374/P: *10;* 377/*19,* P: *12;* 379/P: *10;* 382/P: *5;* 385/P: *3;* 387/P: *2, 11.* *Announce chapter test.*
120	**S** 313/*14;* 316/P: *10;* 342/*21;* 352/*2, 6, 13.*	**10–6** 380/*1, 3.* **S** 377/P: *12;* 378/*15;* 379/P: *8.*	*Administer chapter test.* **S** 429–30/*1–14 odd.*
121	**9–8** 354/*1, 3, 5, 7.* **S** 316/P: *11;* 336/*6;* 346/*8.*	**10–7** 382–3/P: *1–10 odd.* **S** 354/*17;* 374/P: *6;* 378/*6;* 380/*5, 7.*	*Discuss test and assignment.* **R** *Inserts* A–E *following p.* 180. **R 11–1.**
122	**S** 318/P: *6;* 340/*17;* 348/*8;* 352/*12.* 354/*8.*	**10–8** 384/P: *1, 3, 5, 7.* **S** 370/*16;* 379/P: *4;* 382/P: *2, 6.*	**11–1** 399–400/*1, 3, 5, 12, 16, 22, 24.* **S** 354/*18;* 376/*17.*
123	**9–9** 356/*1, 3.* **9–10** 357–8/*1, 3.* **S** 350/*19.*	**S** 372/*6;* 376/*8;* 377/P: *6;* 380/*2, 6;* 384–5/P: *2, 9.*	**11–2** 403/*4, 7, 21, 28.* **S** 381/*20;* 400/*19.*
124	**S** 342/*10;* 346/*10;* 352/*8;* 354/*9;* 356/*4.*	**10–9** 385/P: *1, 3, 5.* **S** 378/*4, 14;* 379/P: *6;* 382/P: *4.*	**11–3** 405–6/*4, 7, 11, 15, 19, 23.* **S** 400/*28;* 403/*8, 17, 22.*
125	**S** 345–6/*6, 15;* 352/*14;* 354/*6;* 358/*2.* *Announce chapter test.*	**S** 374/P: *8;* 379/P: *10;* 384/P: *4;* 385–6/P: *2, 7.*	**11–4** 410/*4, 11, 13, 16, 23.* **S** 403/*18;* 405/*12;* 406/*16, 24.*
126	*Administer chapter test.*	**10–10** 387/P: *1, 3, 5.* **S** 376/*14;* 380/*11.*	**11–5** 413–4/*3, 9,* P: *3, 5, 7.* **S** 395/*7;* 411/*25.*
127	*Discuss test.* **10–1** **R 10–1.** 370/*1, 3, 5.*	**S** 370/*8;* 377/P: *8;* 383/P: *12;* 385/P: *6;* 387/P: 2. *Announce chapter test.*	**R** *Inserts* M–R *following p.* 420. **S** 410/*7, 10, 22;* 411/P: *1, 3, 5, 8.*
128	**10–2** 372/*1, 3, 7, 11.* **S** 340/*15;* 354/*17;* 370/*2, 6, 17.*	*Administer chapter test.*	**11–6** 415–6/*3, 11, 15, 17, 19, 21.* **S** 406/*20, 26;* 410/*3, 5.*
129	**10–3** 374/P: *1, 3, 5.* **S** 342/*6;* 352/*4;* 356/*2;* 358/*3;* 372/*5, 9, 13, 15.*	*Discuss test.* **11–1** 399–400/*1–20 odd.*	**11–7** 418–9/*3, 9, 11, 19.* **S** 406/*21;* 411/P: *7;* 416/*22, 26.*
130	**S** 345/*10;* 370/*13;* 372/*2, 19, 21;* 374/P: *2, 7, 9.*	**11–2** 403/*1–20 odd.* **S** 374/P: *10;* 380/*12;* 385/P: *4;* 400/*10, 16, 21.*	**S** 411/*31;* 414/P: *8;* 416/*43;* 418–9/*1, 4, 6, 10, 13, 29.*

Lesson	Minimum Course	Average Course	Maximum Course
131	**10–4** 376/*1–12 odd.* **S** 348/*3;* 354/*13;* 372/*4, 6.*	**11–3** 405–6/*1–20 odd.* **S** 376/*8;* 377/P: *10;* 383/P: *8;* 403/*2, 6, 14.*	**11–8** 420/*1, 5, 7, 17.* **S** 377/*20;* 416/*27;* 419/*20, 25.*
132	**S** 374/P: *4, 10;* 376/*2, 4, 13;* 377/P: *1, 3, 5.*	**11–4** 401/*1, 5, 9, 13, 17, 19.* **S** 384/P: *8;* 387/P: *4;* 399/*2, 6;* 405–6/*2, 8, 14.*	**11–9** 422/*15, 17, 27, 29.* **S** 414/*9;* 416/*44;* 419/*27;* 420/*2, 6, 8, 14, 19, 21.*
133	**10–5** 378/*1–12 odd.* **S** 357/*5;* 370/*14;* 377/P: *2, 4, 6.*	**11–5** 413/*1, 3, 5, 9,* P: *1, 3.* **S** 378/*13;* 403/*4, 16, 21;* 410/*3, 15, 21;* 411/P: *2, 4.*	**S** 419/*22, 26;* 420/*20;* 422/*18, 28, 30, 33, 39;* 423/P: *5, 9.*
134	**S** 353/*15, 19;* 372/*8, 10;* 376/*14;* 377/P: *10;* 379/P: *1, 3, 5.*	**S** 379/P: *9;* 386/P: *8;* 406/*16;* 411/P: *1, 3;* 413/2, P: 2.	**S** 400/*27;* 406/*22;* 411/*26;* 414/P: *10;* 419/*28;* 420/*24;* 422/*31, 34, 40;* 423/P: *10.* *Announce chapter test.*
135	**10–7** 382/P: *1, 3, 5.* **S** 374/P: *6;* 378/2, *4, 14;* 380/*5.*	**11–6** 415–6/*1–25 even.* **S** 380/*13;* 400/*22, 27;* 410/*2, 12.*	*Administer chapter test.*
136	**S** 354/*10;* 378/*6, 10, 15;* 383/P: *7.*	**S** 383/P: *10;* 387/P: *6;* 403/*8, 22;* 413/*4;* 415–6/*1–25 odd;* 416/P: *1, 3, 5*	*Discuss test and assignment.* **S** 430–1/*15–34 odd.* **R 12–1.**
137	**10–8** 384/P: *1, 3, 5.* **S** 370/*7, 10;* 376/*10, 15.*	**11–7** 418–9/*1–16 odd.* **S** 384/P: *6;* 405/*6;* 416/P: *2.*	**12–1** 437–8/*2, 9, 11, 16, 19.* **S** 430/*8, 10, 14, 22.*
138	**S** 372/*12, 14;* 377/P: *7;* 382/P: *2, 4;* 384/P: *2, 4.*	**S** 406/*20, 24;* 410/*20;* 416/*26, 33;* 419/*5–22 even.*	**12–2** 441–2/*1, 6, 16, 24, 25.* **S** 430/*24, 28;* 438/*22;* 437–8/*4, 10, 17.*
139	**10–9** 385/P: *1, 3, 5.* **S** 374/P: *8;* 378/*8, 12;* 384/P: *6.*	**11–8** 420/*1–12 odd, 17, 19.* **S** 386/P: *9;* 399–400/*8, 14;* 414/P: *4.*	**12–3** 445–6/*3, 5, 14, 16, 20.* **S** 420/*23;* 441–2/*5, 23.*
140	**S** 382/P: *6, 8;* 384/P: *8;* 385/P: *2, 4.*	**S** 403/*10, 18;* 416/P: *4;* 418–9/*2, 4, 27;* 420/*2, 8, 14, 18, 20.*	**S** 413/P: *12;* 437/*8;* 446–7/P: *1, 3, 5, 9, 13.*

Lesson	Minimum Course	Average Course	Maximum Course
141	**10–10** 387/P: *1, 3, 5.* S 376/*8;* 377/P: *7;* 383/P: *9.*	**11–9** 422/*1–10 even;* 422–3/P: *2, 4.* S 387/P: *7;* 406/*18;* 416/*27, 30, 41, 49.*	**12–4** 450–1/*1, 3, 5, 7, 10.* S 445/*6, 19;* 447/P: *10, 12.*
142	S 370/*4;* 372/*16;* 385/P: *7;* 387/P: *2, 7.* *Announce chapter test.*	S 410/*22;* 411/P: *7;* 414/P: *5;* 420/*10.* 422/*1–20 odd,* P: *1, 3, 7.*	S 416/*29, 51;* 447/P: *14;* 451/P: *2, 4, 8, 10.*
143	*Administer chapter test.*	S 403/*29;* 419/*17, 19;* 422/*15–25 even;* 423/P: *5, 10.* *Announce chapter test.*	**12–5** 454/*2, 5, 7;* 455/P: *3, 5.* S 447/P: *11, 16;* 450–1/*2, 9, 14.* 451–2/P: *3, 11, 13, 17.*
144	*Discuss test.* **11–1 R 11–1.** 399–400/*1–20 odd.*	*Administer chapter test.*	S 445/*4;* 446/P: *4;* 450/*4;* 447/P: *15;* 451–2/P: *5, 12, 14;* 455/P: *2, 4, 8.* *Announce chapter test.*
145	**11–2** 403/*1–20 odd.* S 383/P: *10;* 386/P: *8;* 399–400/*2, 6, 10, 16, 21.*	*Discuss test.* **12–1 R 12–1.** 437/*1–12 odd.*	*Administer chapter test.* S 264/*2, 6, 19;* 266/*7.*
146	**11–3** 405–6/*1–18 odd.* S 377/P: *8;* 403/*4, 21.*	**12–2** 441/*1–12 odd.* S 405/*4;* 437/*4.*	*Discuss test and assignment.* S 264/*11, 20;* 266/*14, 16, 39.* **R 13–1.** 466–7/*5, 15, 30.*
147	**11–4** 410/*1, 5, 9, 13, 17.* S 385/P: *9;* 387/P: *6;* 399–400/*4, 12;* 405/*2, 8.*	**12–3** 445/*1–14 odd.* S 410/*23;* 419/*28;* 422/*23;* 441/*2.*	**13–1** 467/*10, 22, 23, 25, 37.* S 430/*16;* 455/P: *6.*
148	**11–5** 413/*1, 3, 5,* P: *1, 3.* S 378/*17;* 403/*2, 16;* 410/*2, 12, 19;* 411/P: *2.*	S 413/P: *2;* 437/*8;* 445/*2, 8;* 446–7/P: *1–10 odd.*	**13–2** 468–9/*1, 3, 5, 7, 14, 15, 19.* S 430/*18;* 467/*21, 36, 39.*
149	S 385/P: *6;* 406/*14, 19;* 410/*3, 11, 20;* 411/P: *1, 3;* 413/*2, 4,* P: *2.*	S 420/*6, 13, 21;* 441/*13;* 445/*4, 10, 16;* 446–7/P: *2, 10.*	S 466–7/*4, 7, 28, 38;* 468–9/*6, 10, 12, 17, 21.*
150	**11–6** 415–6/*1–20 odd.* S 383/P: *11;* 400/*18;* 410/*7;* 411/P: *5.*	**12–4** 450–1/*1–8 odd,* P: *1, 3, 5.* S 415–6/*29, 51;* 446/*20,* P: *6.*	**13–3** 471–2/*1, 5, 8, 11.* S 468–9/*8, 11, 16, 22.*

Lesson	Minimum Course	Average Course	Maximum Course
151	**S** 387/P: *4;* 403/*29;* 413/*7;* 415–6/*5–25 even.*	**S** 422/*12;* 437/*12;* 450–1/*2, 8;* 451–2/P: *7, 9,* *11, 13.*	**S** 456/P: *10;* 466/*8;* 468/*9;* 471–2/*3,* *15, 22,* P: *1, 3,* *5, 7.*
152	**11–7** 418–9/*1–16 odd.* **S** 385/P: *10;* 405–6/*4, 20.*	**12–5** 454–5/*1, 5, 7, 9;* 455/P: *1, 3, 5.* **S** 419/*21;* 441/*6.*	**13–4** 475/*2, 5, 9.* **S** 431/*34;* 472/*6, 23;* 472–3/P: *6, 8.*
153	**S** 410/*4, 21;* 411/P: *6;* 416/*26, 33,* P: *1;* 419/*6, 17, 27.*	**S** 445/*6, 12;* 450–1/*4, 9;* 454–5/*2, 4,* P: *2, 4.* *Announce chapter test.*	**S** 451/P: *9;* 467/*41;* 472/P: *2;* 475/*7, 17,* P: *2.*
154	**S** 386/P: *7;* 399–400/*8, 20;* 413/*8;* 414/P: *4;* 416/P: *3, 5;* 418–9/*2, 8.*	*Administer chapter test.* **S** 264/*6, 14;* 266/*26, 28.*	**13–5** 478/*2, 8;* 479/P: *9.* **S** 475/*3, 6, 10;* 476/P: *4, 6, 9.*
155	**11–8** 420/*1–12 odd, 17.* **S** 403/*8, 18;* 419/*12, 28.*	*Discuss test.* **13–1 R 13–1.** 466–7/*1–12 odd.*	**S** 354/*16;* 380/*5;* 468/*4;* 478/*1, 10,* *11, 15, 18;* 479/P: *2.*
156	**S** 387/P: *8;* 405/*6;* 415–6/*4, 21;* 420/*2, 19.*	**13–2** 468/*1–9 even.* **S** 422/*21;* 441/*10, 15;* 450–1/*6,* P: *6;* 466–7/*1–10 even.*	**13–6** 482/*2, 13, 19.* **S** 469/*20;* 475/*8, 16, 19;* 476/P: *7, 10;* 478/*4.*
157	**11–9** 422/*1–10 odd,* P: *1.* **S** 410/*8;* 411/P: *4;* 420/*4, 20.*	**S** 445/*14;* 446–7/P: *4, 13;* 468–9/*1–14 odd.*	**13–7** 484/*1, 9, 13.* **S** 438/*18;* 446/P: *2;* 479/P: *3, 5, 8;* 482/*9, 20.*
158	**S** 413/*6, 9;* 418–9/*4, 10;* 420/*12, 15, 18;* 422/*2, 12, 16, 18.*	**13–3** 471–2/*1–8 even.* **S** 454/*3;* 455/P: *7;* 467/*11–20 even.*	**S** 475/*4;* 478/*9;* 479/P: *7;* 482/*5, 10, 16, 21;* 484/*3, 17.*
159	**S** 400/*23;* 403/*6;* 406/*18;* 410/*6;* 416/*43.* *Announce chapter test.*	**S** 469/*10, 14;* 471–2/*1–12 odd;* 472/P: *2, 4.*	**S** 455/*9;* 472/*13;* 473/P: *9;* 475/P: *1;* 476/P: *8;* 479/P: *10;* 482/*14.* *Announce chapter test.*
160	*Administer chapter test.*	**S** 437/*2;* 450–1/*4,* P: *4;* 472/*9–18 even,* P: *1, 3, 5.*	*Administer chapter test.*
161	*Discuss test.* **12–1 R 12–1.** 437/*1–12 odd.*	**13–4** 475/*1–14 odd,* P: *1.* **S** 441/*4;* 467/*15, 27, 35;* 472/*13, 21.*	**14–1 R** 493–494 **14–2** 496–497/*1, 3, 5,* *7, 9, 17.* **S** 484/*2.*

Lesson	Minimum Course	Average Course	Maximum Course
162	**12–2** 441/*1–10 odd.* **S** 410/*14;* 419/*14;* 437/*12.*	**S** 454–5/*8*, P: *8;* 469/*19;* 475–6/*1–10 even*, P: *1–6 even.*	**14–3** 498–499/*1–9*, *10, 13, 15.* **S** 431/*30;* 472/*17;* 497/*11, 15, 19.*
163	**12–3** 445/*1–10 odd.* **S** 413/*10;* 422/*6;* 441/*2, 4, 11.*	**13–5** 478/*1, 3, 5.* **S** 441/*16;* 445/*15, 18.*	**14–4** 500/*1–11 odd.* **14–5** **R** 500–503. **S** 469/*18;* 498/*14.*
164	**S** 437/*4;* 445/*2, 4,* *11, 13, 16.*	**S** 472/*20, 26;* 475/*12, 15;* 478/*7, 9;* 479/P: *1, 3, 5.*	**14–6** 506/*1–11 odd.* **S** 500/*2–8 even.*
165	**S** 415–6/*2, 23, 49;* 420/*6;* 441/*6, 13,* *15.*	**13–6** 482/*1, 3, 5.* **S** 451/*12*, P: *10;* 467/*13, 25, 37.*	**14–7** 510–511/*1–7* *odd.* 507–8/P: *1, 6, 7.* **14–8** 512/*1–13 odd.* **S** 506/*2–12 even.*
166	**12–4** 450–1/*1–8 odd*, P: *1, 3, 5.* **S** 445/*6, 8*, 446/P: *1, 3, 6.*	**S** 469/*12, 15, 20;* 478/*4, 11;* 482/*7, 9, 11.*	**14–9** 515/P: *1–5 odd.* 516/P: *1, 3;* **S** 510–511/*9, 11,* *21;* 516/P: *1, 3.*
167	**S** 419/*18, 21;* 422/*4, 15, 21;* 437/*2, 6;* 450/*2,* *6;* 451/P: *2.*	**13–7** 484/*2, 4, 6.* **S** 454–5/*6*, P: *6;* 475/*14, 17, 19.*	**14–10** 518–519/*1–9* *odd.* 520/*1, 3.* **S** 515/P: *2, 4, 6.*
168	**12–5** 454/*1, 5, 7;* 455/P: *1, 3, 5.* **S** 441/*10, 12, 16;* 446/P: *2.*	**S** 472/*15, 19;* 482/*2, 8;* 484/*1, 7, 11.*	**14–11** 524/*1, 3, 9, 11*, 524–525/P: *1, 3.* **S** 517/P: *2, 4, 6;* 519/*8, 10.*
169	**S** 445/*10, 17;* 450/*4;* 451/P: *6;* 454/*2;* 455/P: *2.* *Announce chapter test.*	**S** 472/*17;* 478/*2, 14;* 479/P: *6;* 484/*3, 8.* *Announce chapter test.*	**14–12** 526/*1–7 odd.* **S** 430/*26;* 447/P: *8;* 479/P: *11;* 516/P: *2, 6;* 520/P: *2, 4.* *Announce chapter test.*
170	*Administer chapter test.*	*Administer chapter test.*	*Administer chapter test.*

This colophon represents a secondary mathematics program through the calculus.

The following pages are an annotated copy of the student textbook; annotations and answers are printed in blue. References to the Teacher's Manual are indicated by T.M.

HOUGHTON MIFFLIN · MODERN MATHEMATICS SERIES ·

MODERN ALGEBRA

Structure and Method: Book 1

Have all students read the "cover" and "title page" copy below.

COVER

These thousands of tiny, doughnut-shaped, ferromagnetic rings, threaded on wires, are memory units in a computer. They are a symbol of modern man's dependence on mathematics. An electrical current passing along the wire sets up a magnetic field which magnetizes the rings. Since current in the opposite direction reverses the magnetic field, the direction of the magnetic field may represent a 0 or 1, a + or −, a *yes* or *no* condition. This is the electronic mechanism for storing information in the binary number system used in most modern computers.

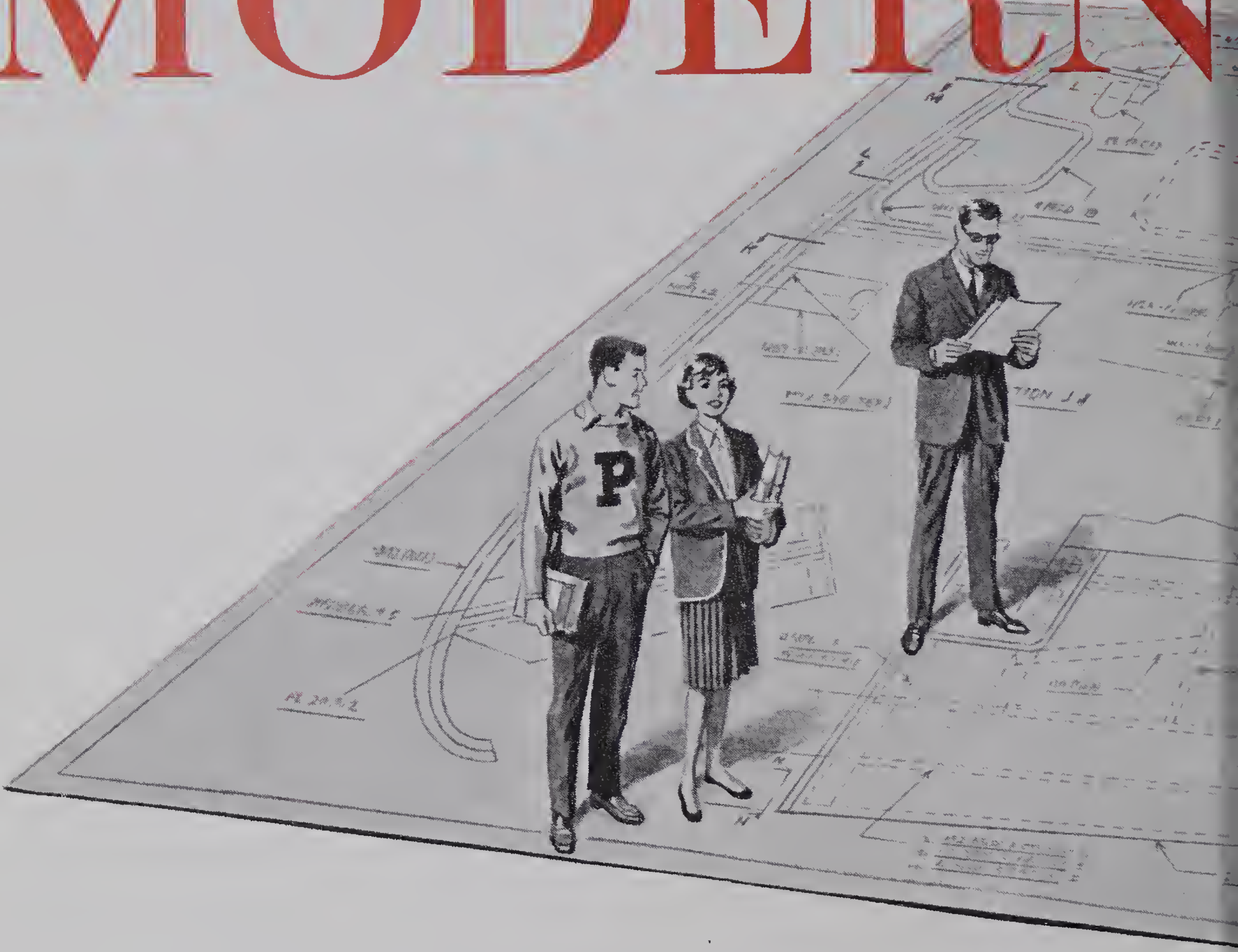
Knowledge of algebra has made possible these computers which initially were built to handle scientific and engineering problems. Today, electronic data processing systems are invaluable in business, industry, and research, where they help to untangle and to simplify, in a matter of seconds, calculations and paper work that formerly took days to do.

Emphasize this point.

TITLE PAGE

The illustration on the title pages indicates that future vocational plans are dependent upon your high school preparation. Are you interested in business, chemistry, medicine? Do you look forward to being an engineer, an architect, a home economist, a machinist, a housewife, or a psychologist? Regardless of your vocational plans, algebra is essential to the modern educated person. More important is the fact that algebra is essential to many future vocational opportunities that do not even exist today. Algebra is equipment you will need if you are to take your place as an educated person in the modern world of today and tomorrow.

MODERN



MARY P. DOLCIANI

SIMON L. BERMAN

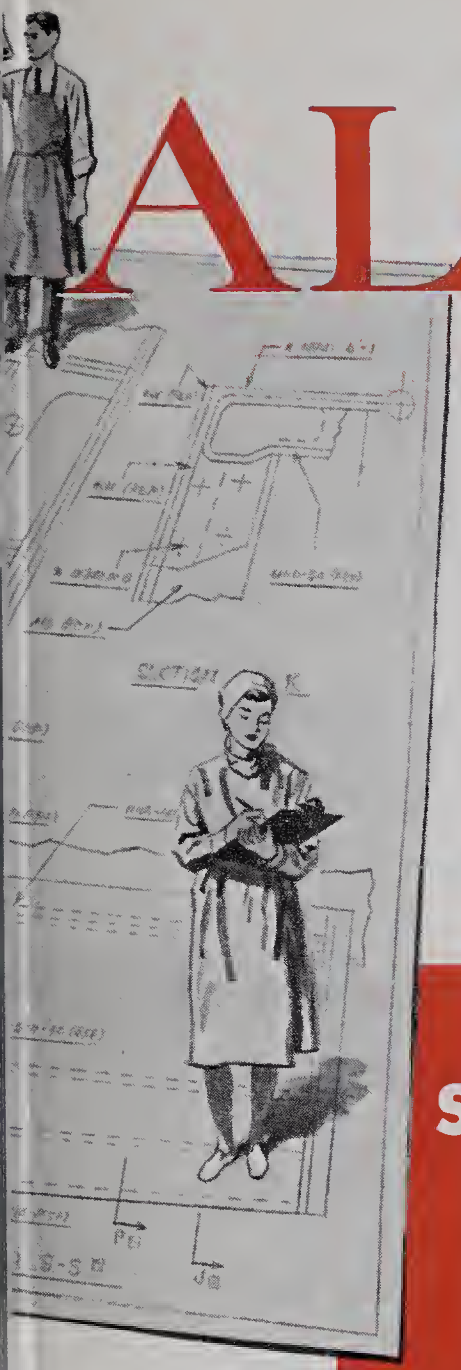
JULIUS FREILICH

●

EDITORIAL ADVISER

ALBERT E. MEDER, JR.

This set of plans suggests that all future vocational plans, ie. business, science research, medicine, etc., require a background in algebra.



ALGEBRA

STRUCTURE AND METHOD

·BOOK ONE·

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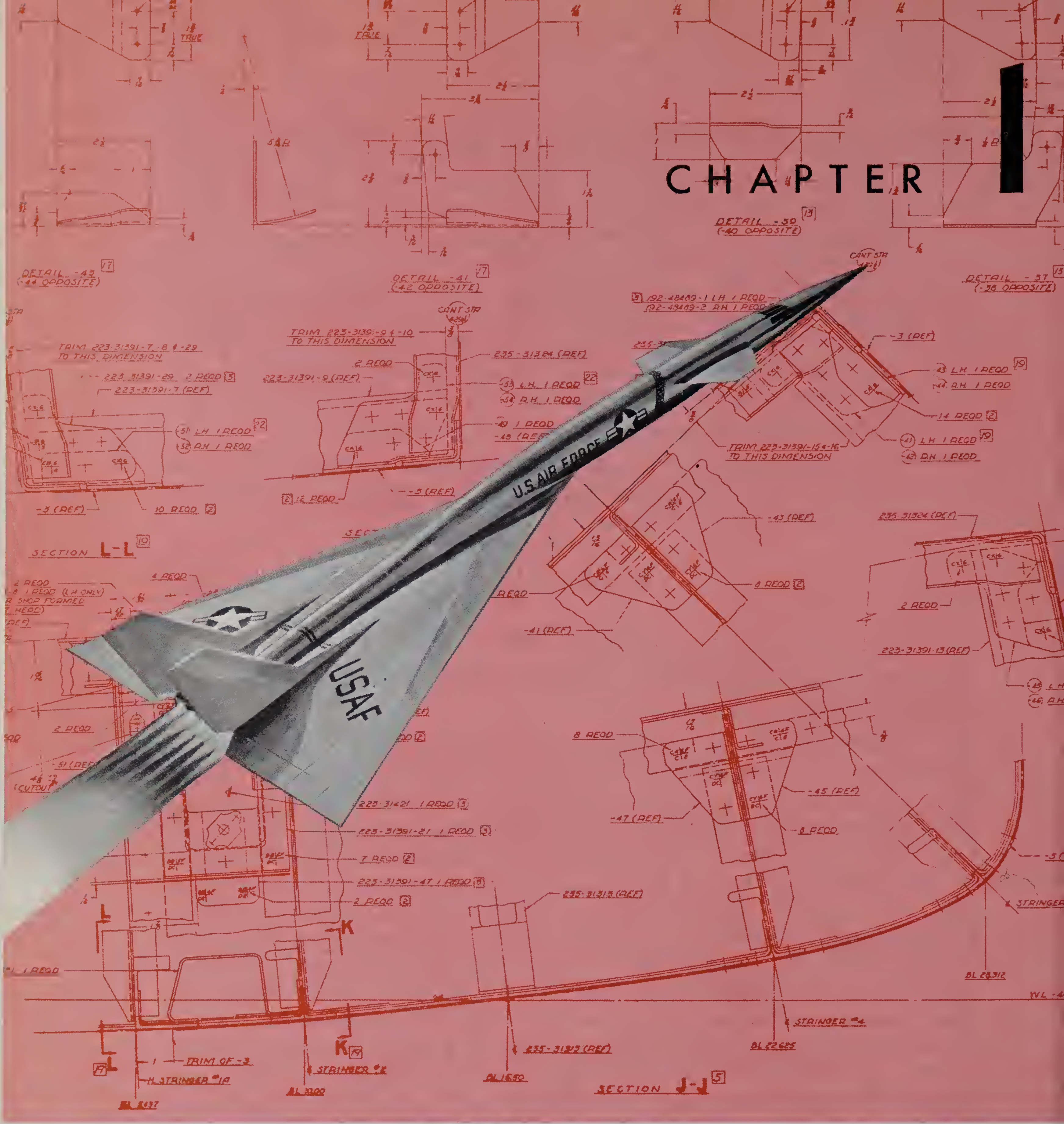
CHAPTER 1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Each chapter is introduced by a picture and appropriate copy which stimulate the student's interest in the mathematical ideas being presented.

Symbols and Sets

Sets are introduced in the very beginning and are used throughout the text to make ideas clear.

Where is science taking us? In the direction of our dreams. Mathematics, the language of science, is the language of dreamers who plan to achieve their dreams.

The man at the drawing board is taking one of the first steps toward making a dream come true. He is translating an idea into a set of drawings (as at the left). To do this he is using symbols which are understood universally by men who turn dreams into realities. Scientists of all languages exchange ideas by using the symbols of mathematics.

You need symbols not only to help you organize your own ideas, but also to explain your ideas to others. You already have learned something about communicating with mathematical symbols. Now you will add to that knowledge and learn how these symbols help you clarify your thinking.

These red centered headings divide each chapter into large blocks of connected material.

NUMBERS AND THEIR RELATIONSHIPS

These numbered side headings not only divide each large block into sections, but are a keying device for chapter tests and reviews (see pg. 27).

1–1 Representing Numbers on a Line: Order Relations

In arithmetic, you learned a good deal about how to use numbers. In algebra, your initial aim will be to discover some of the properties of the numbers of arithmetic. By the word "property" is meant a distinguishing trait or an essential quality. You use this word with meaning when you say, "Sweetness is a property of sugar; hardness is a property of diamonds."

As you will see, one of the properties of numbers — and you will learn many more — is illustrated in Figure 1–1 by the *number line*, sometimes also called the *number scale*. (This number scale has been drawn with an arrowhead to indicate the direction in which the line can be extended as far as you wish.)

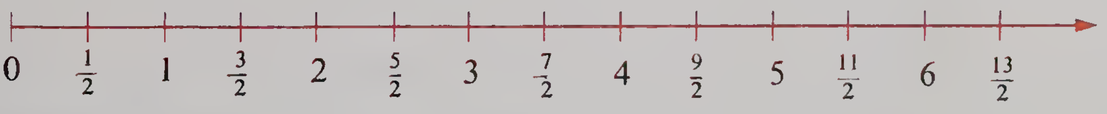
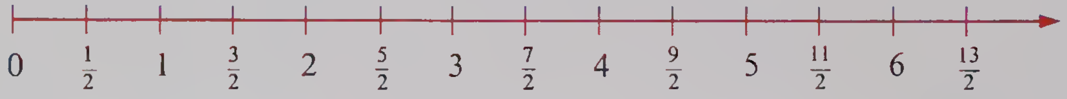


Figure 1–1

The number line is a basic tool in our study of the structure of the system of real numbers.

You will notice that some of the points on the line have been labeled. The starting point of the scale is labeled 0 (zero). The label for any point is a *numeral* or *name for a number* that is a measure of the distance of the point from zero. So, to label any point in Figure 1–1, all you have to know is its distance from 0. You take for granted that there is always a point which is the partner of an arithmetic (ar′ith-**met**-ic)* number. This means that every arithmetic number has exactly one corresponding point on the line. In Figure 1–1, the labeled points are paired with whole numbers or common fractions. Later you will find that many points are partners of numbers which are neither whole numbers nor common fractions.

Although you can freely choose the unit in terms of which you scale (mark) a line, it is important that the scale be *uniform*. Thus, the distance between the points labeled 0 and $\frac{1}{2}$ must be the same as the distance between the points labeled $\frac{3}{2}$ and 2. Uniformity of the scale is essential for illustrating addition on the number line and in picturing graphs.



Stress the connection between order of points on the line and order of numbers.

The number scale shows which of two numbers is the larger. For example, the number 1 is smaller than the number 5; the point labeled 1 lies to the left of the point labeled 5. Any number less than 5 corresponds to a point lying to the left of 5; whereas, any number greater than 5 corresponds to a point lying to the right of 5.

A new term is printed in red in the sentence giving its mathematical meaning.

You have just discovered that the first property of arithmetic numbers is that these numbers can be "lined up" or "ordered." That is, they can be arranged in *order of magnitude* (size) and associated with the points of a uniformly scaled line. The number that is paired with a point on the number line is called the **coordinate** (ko-**or**-din-it) of that point on the line. The point paired with a number is called the **graph** of that number.

What do you mean when you say 3 is *between* 1 and 5? You mean that 3 is greater than 1, but less than 5; or the graph of 3 is to the right of the graph of 1, but to the left of the graph of 5.

When you use a phrase like "the numbers between 1 and 5," you intend to include neither 1 nor 5. The phrase "the numbers between 1 and 5, inclusive," says that you want to include both 1 and 5. The phrase "the numbers between 1 and 5, including 5," says you want to include 5 but not 1. Be sure that students learn this use of "between" and "inclusive."

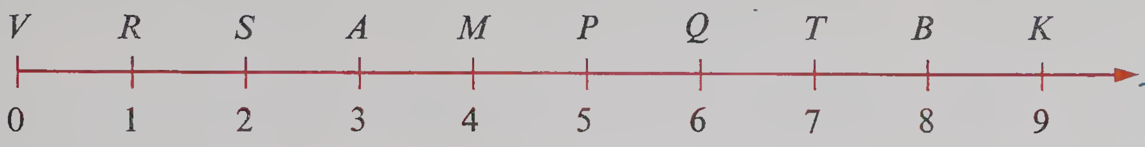
*Primary accent is shown by dark type (**met**), and secondary, by an accent mark (ar′).

Attention to phonetics aids in correct pronunciation.

In the Oral Exercises space limitations do not always permit giving as complete an answer as is expected from the student.

ORAL EXERCISES

Locate the points that are the graphs of the given numbers or state the coordinates of the given points. Refer to the accompanying number line.



The samples set the stage for student response

SAMPLE 1. B *What you say:* The coordinate is 8.

SAMPLE 2. $1\frac{1}{4}$ *What you say:* The point that is one-fourth of the way from R to S.

12. $\frac{1}{10}$ of way from R to S
15. $\frac{2}{3}$ of way from A to M

1. 5 P
2. P 5
3. 3 A
4. 0 V
5. $2\frac{1}{2}$ halfway betw. S + A
6. V 0
7. K 9
8. 9 K
9. $\frac{3}{4}$ $\frac{3}{4}$ of way from V to R
10. $\frac{6}{4}$ halfway betw. R and S
11. 3.25 $\frac{1}{4}$ of way from A to M
12. 1.1
13. 8.5 halfway betw. B and K
14. 2.8 $\frac{8}{10}$ of way from S to A
15. $\frac{11}{3}$
16. $\frac{9}{7}$ $\frac{2}{7}$ of way from R to S
17. $\frac{72}{10}$ $\frac{1}{5}$ of way from T to B
18. $8\frac{9}{11}$ $\frac{9}{10}$ of way from B to K
19. $\frac{54}{9}$ Q
20. 4.75 $\frac{3}{4}$ of way from M to P

21. The point halfway between V and S 1

22. The point halfway between S and B 5

23. The point halfway between V and K $4\frac{1}{2}$

24. The whole numbers less than 6 V, R, S, A, M, P

25. The whole numbers greater than $\frac{3}{4}$ and less than 5 R, S, A, M

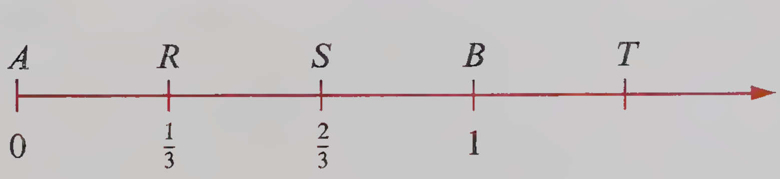
26. The whole numbers between 5 and 9, inclusive P, Q, T, B, K

A complete solution key provides the step-by-step solution for each of the Written Exercises, Problems, Extra for Experts, and Just for Fun questions.

WRITTEN EXERCISES

Give the coordinate for each R, S, and T on the following number lines.

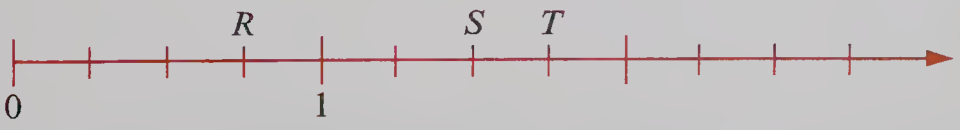
SAMPLE.



Solution: R: $\frac{1}{3}$ S: $\frac{2}{3}$ T: $1\frac{1}{3}$, **Answer.**

A

1.



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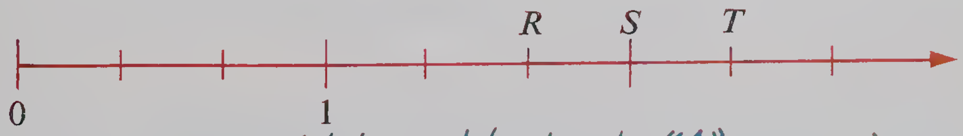
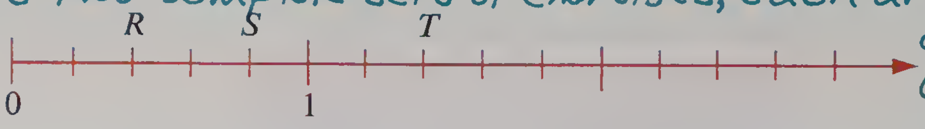


All students should be able to do "A" exercises.

In every set of Exercises and Problems, the even-numbered exercises are paralleled by an equivalent group of odd-numbered exercises. In these parallel "ramps" are provided two complete sets of exercises, each arranged in order of increasing difficulty.

3.



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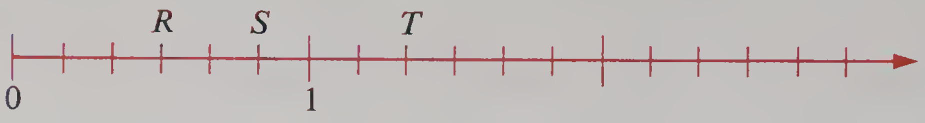

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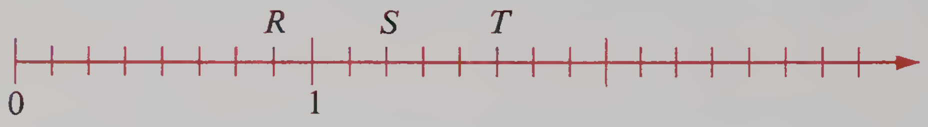

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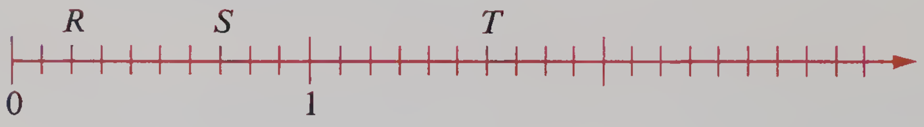
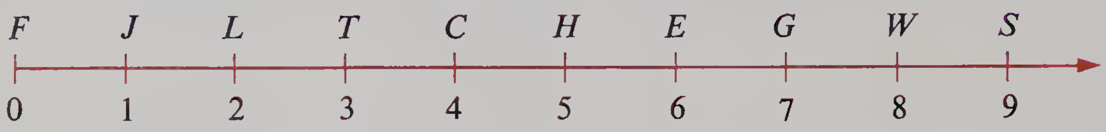


Name the coordinate of the point given. Refer to the number line.



SAMPLE. The point that is two-thirds of the distance from H to G

Solution: The length of the line from H to G is 2 units. The point is $\frac{2}{3} \times 2$ or $\frac{4}{3}$ units from H. Its coordinate is $5 + \frac{4}{3}$ or $\frac{19}{3}$.

Then $\frac{19}{3}$ or $6\frac{1}{3}$, **Answer.**

the "A-B-C" grouping of exercises provides you with a flexible program for individual differences.

7. The point H

8. The point F

9. The point halfway from J to T

10. The point halfway from L to W

11. The point one-fourth of the distance from T to G

12. The point halfway between T and C

13. The point halfway between L and H

14. The point one-third of the distance from J to L

15. The point 2 units to the right of E

16. The point 3 units to the left of T

17. The point 1.5 units to the right of J

18. The point $4\frac{1}{3}$ units to the left of G

Ⓑ **19.** The point one-sixth of the distance from E to H

20. The point one-fifth of the distance from T to L

Students with greater ability and interest should do these exercises.

21. The point two-thirds of the distance from J to G

22. The point three-quarters of the distance from W to F

23. The whole numbers greater than 2 and less than $4\frac{1}{4}$

24. The whole numbers between 7 and 8, inclusive

25. The whole numbers between 2 and 8

26. The whole numbers between 0 and 1, including 0

The "C" group is aimed to challenge the very best students.

C **27.** The whole numbers between $\frac{72}{12}$ and $\frac{6}{2}$, inclusive, that are exactly divisible by 3

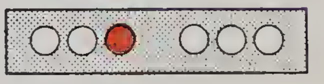
28. The whole numbers between F and H that are exactly divisible by 5

29. The point between T and H that is three times as far from H as it is from T

30. The point between F and L that is twice as far from F as it is from L

1–2 Comparing Numbers: The Sign of Equality

You already know at least two ways of writing any whole number. For example, you can represent eight by the Roman numeral VIII or by the Arabic numeral 8. In the Braille system for the blind, eight is shown by raised points arranged like this ⠓. The flashing red light of the binary computer in the margin indicates eight. In Morse code, – – – . . represents eight. Spanish boys write *ocho* as the name for eight. Each of these representations is a symbol for the number eight. Names or symbols for numbers are called **numerical expressions** or **numerals**.



You use symbols every day of your life. Can you explain the meaning of these familiar symbols?

eight $\quad \$ \quad 5+3 \quad 1+7 \quad = \quad \times \quad \div \quad + \quad -$

Of course, you easily recognize the meanings of these symbols. Notice particularly that the specific signs $+$, $-$, $\times$, $\div$ tell you what to do with numbers in arithmetic. Using these instruction symbols of arithmetic, you can express the number eight in many other ways. As a matter of fact, $\dfrac{18+22}{5}$ and $\dfrac{53-5}{2\times 3}$ each designates eight. Other numerical expressions for eight are $1+7$, $7+1$, $0+8$, $8+0$, 2×4, 4×2, $4+0+4$, X − II, $5+3$. But, the important symbol $=$ which stands for the word "equals" or for the words "is equal to" allows you to say that $1+7$ and $5+3$ designate the same number: $1+7=5+3$; in words, "one plus seven equals five plus three." On the other hand, if you wrote $1 + 7 = 4 + 5$, the statement would be false because $1 + 7$ and $4 + 5$ are not expressions for the same number. Stress that $a = b$ means that a and b name the same number.

These and later exercises review arithmetic processes and suggest number properties.

ORAL EXERCISES

Tell whether or not each statement is true. Give a reason for your answer.

Be sure that students understand the difference between *talking about* a statement and *using* it. See 1-2.T.M. pg. 7.

SAMPLE 1. $2 + 4 = 3 + 3$

What you say: True, because $2 + 4$ and $3 + 3$ each designates the number 6.

SAMPLE 2. $3 \times 9 = 9 \div 3$

What you say: False, because 3×9 designates 27, but $9 \div 3$ designates 3.

1. $2 \times 4 = 2 \times 2 \times 2$ T; 2×4 and $2 \times 2 \times 2$ each designates 8
2. $7 \times 5 = 5 \times 7$ T
3. $2 + 3 = 7 - 2$ T
4. $6 + 0 = 5 + 1$ T
5. $6 \times 0 = 5 + 1$ F
6. $5 - 4 = 1 + 0$ T
7. $8 \div 4 = 4 \div 8$ F
8. $2 + 10 = 5 + 7$ T
9. $3 \times 4 = 9 + 0 + \frac{9}{3}$ T
10. $9 \times 0 = 1 \times 0$ T
11. $49 - 9 = 4 \times 5 \times 2$ T
12. $2 + 2 + 2 = 3 \times 2$ T; $2+2+2$ and 3×2 each designates 6
13. $.001 + 8 = .008 + 1$ F; $.001+8$ designates 8.001, but $.008+1$ designates 1.008
14. $3 \times 1 = \frac{1}{8} \times 24$ T
15. $19 - 3 - 2 = 19 - 5$ T
16. $8 \times 1 = 2 \times 4 \times 1$ T
17. $6 \div 1 = \frac{6}{1}$ T
18. $10 \times \frac{1}{5} = 10 \times .2$ T
19. $1 + 5 + 4 = 10 - 1$ F
20. $12 \div 3 = 2 \times 5$ F
21. $19 - 2 - 2 - 2 = 19 - 6$ T
22. $19 - 6 = 19 - 2 - 2 - 2$ T
23. $\frac{8}{2} = 2 \times 1 \times 2$ T
24. $1 \div 5 = 5 \div 1$ F; $1 \div 5$ designates $\frac{1}{5}$, but $5 \div 1$ designates 5

WRITTEN EXERCISES

Tell whether or not each statement is true.

A

1. $8 \times 2.5 = 6 \times 4.5$
2. $.5 + .4 + .3 = .3 \times .4$
3. $\frac{1}{3} + \frac{1}{2} = \frac{3}{6} + \frac{2}{6}$
4. $\dfrac{1}{0.1} = \dfrac{2}{0.2}$
5. $\frac{5}{8} \div 15 = 24$
6. $6 \div \frac{2}{3} = 4$
7. $17 \times 1 \times 5 = 100 \div 1\frac{3}{17}$
8. $.1 \times 0 \times 8 = 0 \div \frac{4}{5}$
9. $\dfrac{42 - 15}{14 - 5} = \dfrac{5 \times 3}{5 \div 3}$
10. $\dfrac{51 + 14}{17 - 7} = 1\frac{2}{3} + 8\frac{1}{3} - 3\frac{1}{2}$

Exercise numbers are bold-faced to distinguish them from the content of the exercises.

In each case, find a numeral to replace the question mark and make the resulting statement true.

B **11.** $8 + 9 + ? = 31 - 12$
12. $32 + 19 + ? = 86 - 35$
13. $3\frac{2}{3} - ? = \frac{3}{4} + \frac{1}{3}$
14. $6\frac{1}{2} - ? = 2\frac{1}{3} + 1\frac{5}{6}$
15. $.7 - .3 = 2 \times ?$
16. $.7 + .3 = .7 \div ?$
17. $.24 \div .6 = 1 - ?$
18. $3.6 \div .9 = 1 \div ?$

1–3 Comparing Numbers: The Signs of Inequality

Note the openness of page and the use of special type faces to emphasize important material.

To change the false statement

$$\mathbf{7 = 6 - 3}$$

into a true one, you may use the symbol ≠, translated "is not equal to" or "does not equal." Thus, a true statement is

$$\mathbf{7 \neq 6 - 3.}$$

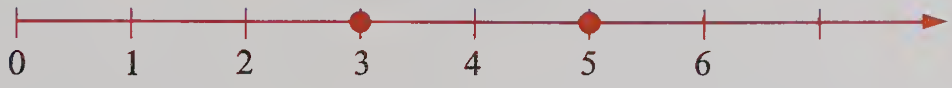
Can you tell why the following statement is also true?

$$\mathbf{\frac{12 + 3}{3} \neq 12 + 1}$$

Another inequality symbol you will use is $>$, which is read "is greater than." Thus, $5 > 3$ means "five is greater than three."

The symbol $<$ stands for the words "is less than." When you write $3 < 5$ you say, "three is less than five."

The statements $5 > 3$ and $3 < 5$ both give the same information: 5 is a larger number than 3, and the graph of 3 is to the left of the graph of 5, or the graph of 5 is to the right of the graph of 3. Emphasize that these statements compare numbers or the positions of points on a line. They do not compare numerals. See 1-3 T.M. pg. 7.



To avoid confusing the symbols $<$ and $>$, think of them as arrowheads always pointing to the numeral for the smaller number. For example,

$$\mathbf{39 - 33 < 14} \quad \text{and} \quad \mathbf{1000 > 66 \times 11}$$

are true statements,

The symbols ≮ (is not less than) and ≯ (is not greater than) may be introduced here at your discretion. Later in the text, the equivalent symbols ≥ (is greater than or equal to) and ≤ (is less than or equal to) are introduced. The two latter symbols are the less troublesome for students.

whereas

$$1.732 > 17 \quad \text{and} \quad 1 < 0.9999$$

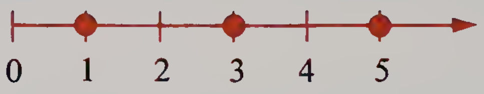
are false statements.

You remember that the statement "3 is between 1 and 5" means that 3 is greater than 1 and also that 3 is less than 5. In symbols, you would write

$$3 > 1 \quad \text{and} \quad 3 < 5.$$

It is customary to put this pair of statements together and write

$1 < 3 < 5$ or graph it.



This new expression is sometimes translated, "1 is less than 3, and 3 less than 5" or "3 is between 1 and 5." This expression illustrates the neatness of mathematical symbolism and its economy of space.

Graphs are used throughout the book as a visual aid to strengthen understanding.

ORAL EXERCISES

Tell whether or not the statement in each of the following exercises is true.

SAMPLE 1. $7 + 2 = 3 \times 3$

What you say: True, because $7 + 2$ and 3×3 each designates the number 9.

SAMPLE 2. $\frac{10 + 2}{2} \neq 10 + 1$

What you say: True, because $\frac{10 + 2}{2}$ designates 6, but $10 + 1$ designates 11.

1. $74 + 28 = 28 + 74$ T; 74+28 and 28+74 each designates 102
2. $48 \div 4 \neq 6 \times 2$ F; 48÷4 and 6x2 each designates 12
3. $3 \times 3 = 3 + 3 + 3$ T; 3x3 and 3+3+3 each designates 9
4. $5 + 3 = 5 - 3$ F; 5+3 designates 8, but 5-3 designates 2
5. $53 + 47 = 57 + 43$ T; 53+47 and 57+43 each designates 100
6. $4 + 4 = 2 \times 4$ T; 4+4 and 2x4 each designates 8
7. $1 \times 110 \neq 110$ F; 1x110 and 110 each designates 110
8. $2 \times 7 \neq 7 \times 7$ T; 2x7 designates 14, and 7x7 designates 49

9. $18 + 0 = 3 \times 6$ T

10. $7 + 5 < 4 \times 5$ T

11. $\frac{5 + 1}{2} = 4 - 1$ T

12. $\frac{12 + 8}{5} > 2 \times 2$ F

13. $\frac{30 - 18}{6} = 5 - 3$ T

14. $\frac{48 - 12}{12} < 48 - 1$ T

15. $\frac{15 + 18}{3} \neq 5 + 6$ F

16. $\frac{12 + 5}{2} > 10$ F

17. $50 =$ the number of states in the U.S.A. T

18. $47 \times 32 \neq 37 \times 42$ T

19. $8 \times 6 > 40 + 8$ F

20. $\frac{1}{2} + \frac{3}{6} > 1$ F

21. $12 + 0 \neq 15 + 0$ T

22. $\frac{23}{1} < 23$ F

23. $15 \times 0 = 15$ F

24. $15 \times 0 \neq 15 \times 0$ F

25. $\frac{3}{5} + \frac{2}{5} < \frac{2}{5} + \frac{3}{5}$ F

WRITTEN EXERCISES

Make a true statement by replacing each question mark with the sign $=$, $<$, or $>$.

SAMPLE 1. $5 \; ? \; 2$ *Solution:* $5 > 2$

A

1. $4 + 5 \; ? \; 10 - 1$

2. $13 \times 0 \; ? \; 16 + 0$

3. $7 \; ? \; 7$

4. $5 \; ? \; 6$

5. $0 \; ? \; 0$

6. $\frac{15}{3} \; ? \; 2 + 3$

7. $\frac{8 - 3}{5 - 2} \; ? \; 1 + .6$

8. $6 \times 0 \; ? \; 0$

9. $5 \times 1 \; ? \; 5 \times 0$

10. $2\frac{1}{8} \; ? \; 1 + 1\frac{2}{16}$

11. $\frac{1}{3} \; ? \; \frac{1}{6}$

12. $\frac{1}{5} + \frac{3}{4} \; ? \; \frac{6}{8} + .2$

13. $65 \times 1 \; ? \; 66$

14. $\frac{1}{3} + \frac{4}{6} \; ? \; \frac{5}{9}$

15. $3 \; ? \; 3 \times 0$

16. $\frac{1}{2} \; ? \; \frac{1}{4} + \frac{0}{4}$

17. $2 \times 10 \; ? \; 17 - 7$

18. $.75 + .25 \; ? \; \frac{3}{4} - \frac{1}{3}$

19. $\frac{6}{7} - \frac{2}{3} \; ? \; \frac{9}{10} + \frac{2}{3}$

20. $4 \div 2 \; ? \; 8 \div 4$

21. $\frac{1}{3} \div 4 \; ? \; \frac{4}{3}$

22. $\frac{1}{3} \times 4 \; ? \; \frac{3}{4}$

23. $\frac{21}{9} - \frac{4}{3} \; ? \; \frac{3}{3} \times 3$

24. $.6 - .2 \; ? \; .2 \times 2$

Copy each of the following statements. Replace each question mark by any numeral that makes the resulting statement true.

SAMPLE 2. $10 \times 2 \neq 20 + ?$

Solution: $10 \times 2 \neq 20 + \frac{1}{4}$ (or any numeral other than 0)

25. $4 + ? = 11 - 6$
26. $? + 1 \neq 8$
27. $5 + 3 = 3 + ?$
28. $8 \times ? = 0$
29. $? \div 10 = 0$
30. $5 + ? = 7$
31. $14 = 7 + ?$
32. $? + 7 = 7$

33. $2 \times ? < 17 - 7$
34. $? \times 0 = 0$
35. $3 \times ? = 9 + 6$
36. $12 \times ? > 4 \times 12$
37. $132 \neq 11 \times ?$
38. $? - 8 < 8$
39. $? \times 15 > 0$
40. $? \div 8 = 1$

Ⓑ **41.** $\dfrac{15 + ?}{8} = 2$
42. $\dfrac{24 + ?}{4} = 6 + 3$
43. $? \div ? = 1$
44. $? \times 10 = 1$
45. $1 < ? < 3$

46. $\frac{1}{3} < ? < \frac{2}{3}$
47. $0 < ? < .1$
48. $1.7 - 1.3 < ? < 1.8 - 1.3$
49. $\dfrac{3.2 + 3}{7.6} < ? < \dfrac{3.2 + 5}{7.6}$
50. $\dfrac{2.5 - 0.5}{1.09} < ? < \dfrac{2.5 - 0.5}{1.07}$

Ⓒ **51.** $1.54327 > ? > 1.53327$
52. $\dfrac{1.4867}{2.3864} < ? < \dfrac{1.4857}{2.3764}$

53. $\sqrt{7} > ? > \sqrt{6}$
54. $? \times 0 = 0$

The second basic tool in studying the structure of the real number system is the set concept. **GROUPING NUMBERS IN SETS AND SUBSETS**

1–4 Meaning of Membership in a Set

You are accustomed to talking about all sorts of collections or sets of objects. A set of dishes, a chemistry set, the high school crowd, a pair of shoes, all the whole numbers, are familiar examples of collections of objects. The mathematician calls any such collection of objects a *set*. Each object in a set is called a **member** or **element** of the set. For example, all the teachers in your school form a set, and your algebra teacher is a member or element belonging to that set. However, objects such as the letter r, the school custodian, and the number 9, are not elements in the set of all your teachers. Thus, a set is any collection of objects so well described that you can always tell whether or not an object belongs to the set.

Suppose a set is formed of any five whole numbers. Use a capital letter, say R, to name or refer to the set. You have no way of telling whether the number 3 is or is not an element of R until the five whole numbers are specified. If you specify the set by listing the objects forming the set within braces { }, then you may have

$$R = \{0, 3, 7, 8, 14\}$$

This says, "R is the set of numbers 0, 3, 7, 8, 14." You can easily see that the number 3 is a member of this set and that the number 4 is not. We use a special symbol, $\in$, to mean "is an element of," and $\notin$ to mean "is not an element of." Thus, $3 \in R$ and $4 \notin R$.

Specifying a set by listing its elements in braces gives you a **roster** or **list** of the set. The objects named in the listing, {our moon, the Constitution, the Alamo, California, Albert Einstein}, form a set. Note that the elements of a set need have no relation with one another other than being listed together. Furthermore, the order of listing the elements is unimportant. What is important is that each element be named in the listing.

Often a roster is an inconvenient way of specifying a set. For example, a roster of the set of states in the U.S.A. requires the listing of all 50 of the elements within braces. This inconvenience is overcome by writing within braces a **rule** which describes the elements of a set. Thus,

{the states of the U.S.A.}

says, "The set of the states of the U.S.A."

See 1-4 T.M. pg. 7.

EXAMPLE. **Specify the set of numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 by (a) roster, (b) rule.**

Solution: **(a) {1, 2, 3, 4, 5, 6, 7, 8, 9}**

(b) {the whole numbers between 0 and 10} or
{one-digit numbers except 0} or
{whole numbers from 1 to 9, inclusive}

Use illustrative examples to strengthen learning.

ORAL EXERCISES

Specify each of the following sets by a roster.

SAMPLE 1. {the letters in the word Mississippi}

What you say: {i, m, p, s}

1. {the letters in the word *freshman*} {a, e, f, h, m, n, r, s}
2. {the letters in your given name} Answers will differ
3. {the numerals on the face of a clock} {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}
4. {the whole numbers less than 20} {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19}
5. {students in your row in the algebra class} Answers will differ
6. {states of the U.S.A. on the Gulf of Mexico} {Tex., La., Miss., Ala., Fla.}

Specify each of the following sets by a rule.

20. {every fraction with numerator 1 and denominator an even no. < 11}

SAMPLE 2. $\{\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}\}$

25. {whole nos. from 1 to 13 inclusive obtained by starting with 1 and adding 3 successively}

What you say: {every fraction whose numerator is 1 and whose denominator is an odd number less than 8}

7. {Alaska, Hawaii} {the 49th and 50th states to enter the Union}
8. {California, Washington, Oregon} {states of the U.S.A. bordering the Pacific Ocean}
9. {2, 3, 4} {whole nos. betw. 1 and 5}
10. {2, 4, 6} {even nos. between 1 and 7}
11. {Eisenhower, Truman} {presidents of the U.S.A in office betw. 1944 and 1961}
12. {Dolciani, Freilich, Berman} {authors of your algebra book}
13. {1, 3, 5, 7} {odd nos. from 1 to 7 inclusive}
14. $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ {every fract. with numerator 1 and denom. a whole no. betw. 1 and 6}
15. {Illinois, Indiana, Idaho, Iowa} {states of the U.S.A. that begin with I}
16. {Los Angeles, San Francisco} {the two cities of Calif. with the largest population}
17. {a, e, i, o, u} {vowels}
18. {Saturday, Sunday} {days of the week that begin with S}
19. {20, 10, 5, 15} {multiples of 5 betw. 4 and 21}
20. $\{\frac{1}{2}, \frac{1}{6}, \frac{1}{8}, \frac{1}{4}, \frac{1}{10}\}$
21. {Washington (D.C.)} {the capita of the U.S.A}
22. {London, Paris} {the capitals of England and France}
23. {x, w, z, y} {the last 4 letters of the alphabet}
24. {Jefferson Davis} {the president of the Confederate States of America}
25. {1, 7, 4, 13, 10}
26. {16, 1, 11, 6, 21} {whole nos. fro 1 to 21 inclusive obtained by starting with 1 and adding 5 successively

Tell whether or not each statement is true. Give a reason for each answer.

27. $5 \in$ {whole numbers less than 5} F; 5 is not a whole no. < 5
28. $5 \notin \{15, 20, 25\}$ T; 5 is not an element of given set
29. $3 \notin$ {whole numbers less than 5} F; 3 is a whole no. < 5
30. $8 \in \{0, 8, 9\}$ T; 8 is an element of the given set
31. $\frac{1}{2} \in$ {multiples of $\frac{1}{4}$} T; 1/2 is a multiple of 1/4 namely 2×1/4
32. $12 \notin$ {even numbers} F; 12 is an even no.
33. $\frac{1}{2} \in \{.25, .5, .75\}$ T; $\frac{1}{2}$ is an element of the set, namely the no. designated by .
34. $7 \notin \{1, 9, 12, 7, 21, 15\}$

34. F; 7 is an element of the given set

1–5 Kinds of Sets

In counting the number of eggs in the basket in Figure 1–2 — one, two, three, four — you really pair each egg with a number as shown, and conclude that there are as many eggs as there are numbers in {1, 2, 3, 4}. This pairing of eggs with numbers is a *one-to-one correspondence.* Two sets are in **one-to-one correspondence** when each member of one set has one partner in the other set, and no element in either set is without a partner. The pairing of point and number on a number line is another example of one-to-one correspondence.

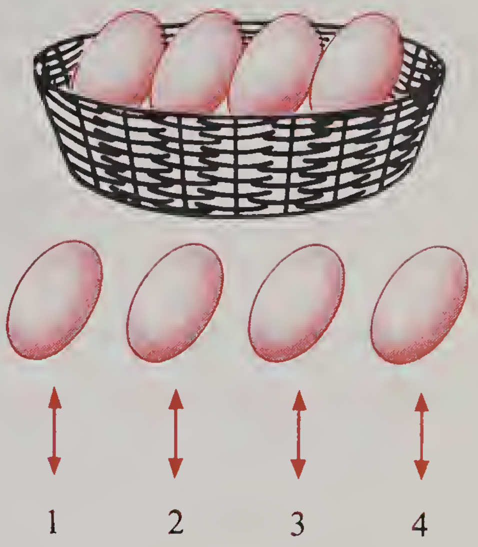


Figure 1–2

Each element of either set has exactly one partner in the other set.

Can you list *all* the members of the set of whole numbers? If you start to write

{0, 1, 2, 3, 4, 5, 6, . . .}

you will never come to the end of the list. The three dots after the 6 are the mathematician's way of indicating that the roster continues without end. A set which has so many elements that the process of counting them would never come to an end is called an **infinite set**. For example, you cannot list the members of

{all the fractions between 0 and 1}

although the rule enables you to identify them. Another infinite set is the set of points on a line. A set containing a large number of elements is not always an infinite set. Thus, {the grains of sand on the beach at Waikiki} is not an infinite set, even though it has many members.

A set is **finite**, or has a finite number of elements, if the process of counting the elements comes to an end. Such a set is

{two-digit numbers} = {10, 11, 12, . . . , 99}.

In this example, the three dots mean *and so on through.*

Can a set have no elements? Consider the set of whole numbers between 8 and 9. This set contains no elements and is called the **empty set** or **null set**. Notice that this symbol {0} does not designate the empty set. It contains the number 0. Empty braces { } might be used, but a special symbol ∅, written without braces, usually is used to designate the null set.

The symbol ϕ is an adaptation of a letter of the Danish alphabet. Be sure that students distinguish between ϕ and {ϕ}. See 1-5 T.M. pg. 8.

The notion of the empty set may seem strange at first. Still, how often have you reached into your pocket or purse to find it empty of coins? The set of coins in your pocket or purse was the empty set. By agreement there is only one null set. Thus, the set of whole numbers between 8 and 9 and the set of coins contained in your empty purse or pocket are one and the same set.

ORAL EXERCISES

Use a roster to specify each of the following sets.

SAMPLE. {Persons now 2000 years of age}

What you say: The empty set, $\emptyset$.

1. {living dogs with wings} $\emptyset$

2. {even numbers} {2, 4, 6, 8, ...}

3. {multiples of 7} {0, 7, 14, 21, 28, ...}

4. {letters of the alphabet} {a, b, c, d, e, f, g, ..., z}

5. {unit fractions} $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$

6. {three-digit numbers} {100, 101, 102, ..., 999}

7. {multiples of 12 less than 200} {0, 12, 24, 36, 48, ..., 192}

8. {leap years from 1900 to 2000} {1904, 1908, 1912, 1916, 1920, ..., 1996}

Tell whether the members of the given sets may be paired so that the sets will be in one-to-one correspondence.

9. $\{a, b, c\}$ and $\{c, b, a\}$ yes

10. {0, 1, 3, 5} and {1, 3, 5} no

11. {vowels} and {a, e, u, o, i} yes

12. $\{\Delta, \pi\}$ and $\{\pi, \Delta\}$ yes

13. $\emptyset$ and {0} no

14. $\{\frac{1}{4}, \frac{1}{2}, \frac{1}{3}\}$ and {.5, .3, .25} yes

15. {2, 4, 6, 8} and $\{\frac{2}{1}, \frac{12}{2}, \frac{32}{4}, 4\}$ yes

16. $\{C, A, T\}$ and $\{K, A, T\}$ yes

WRITTEN EXERCISES

Give a roster for each set, and state whether it is finite.

SAMPLE. {multiples of 4 between 0 and 17}

Solution: {4, 8, 12, 16}, finite

(A multiple of a whole number is any product of that whole number by a whole number.)

A

1. {the whole numbers between 7 and 10}

2. {the whole numbers between 0 and 5, inclusive}

3. {the vowels in the word *field*}

4. {all five-headed people}

5. {U.S. cities each with populations greater than 20 million}

6. {the odd numbers between 0 and 10}
7. {the even numbers less than 20 and greater than 9}
8. {the months of the year which have fewer than 30 days}
9. {the whole numbers greater than 0 but less than 1776}
10. {the even numbers between 200 and 1000}

B 11. {the multiples of 3 between 3 and 15, inclusive}
12. {the multiples of 25 greater than 0}
13. {the leap years after 1960}
14. {the multiples of 20 between 31 and 53}
15. {every fraction whose denominator is an even number chosen from {1, 2, 3} and whose numerator is 1}
16. {all the odd numbers}
17. {all the multiples of 2 greater than 0}
18. {the United States Presidents who have served more than 4 terms}

C 19. {all odd whole numbers whose squares are less than 40}
20. {whole numbers less than 19 which are squares}
21. {all whole numbers between 7 and 8}
22. {all numbers between 3 and 8 that divide 13 exactly}

In Column II find a designation for each set in Column I.

I

23. $\{x, y, z\}$
24. {0, 1}
25. {1, 9, 25, . . .}
26. {0, 4, 16, 36, . . .}
27. $\emptyset$
28. {0}
29. {1, 3, 5, 7}
30. {2, 4, 6, 8}
31. {0, 7, 14, 21, . . .}
32. {45}
33. {36}
34. {the digits in the numeral for the product of 9 and 3}

II

a. {whole numbers between 1 and 2}
b. {odd numbers between 1 and 7, inclusive}
c. {all multiples of 7}
d. {numbers each of which equals its square}
e. {the sum of 24 and 12}
f. $\{z, y, x\}$
g. {zero}
h. {squares of odd numbers}
i. {even numbers between 2 and 8, inclusive}
j. {squares of even numbers}
k. {the product of 15 and 3}
l. {the digits in the numeral for the sum of 68 and 4}

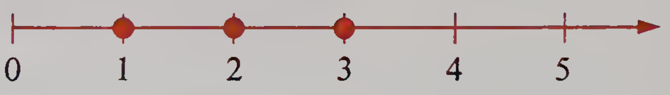
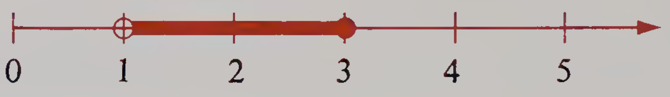
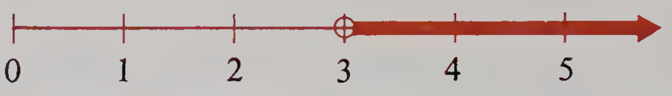
the matching exercises above are an example of the variety of types of exercises in the text.

1–6 The Graph of a Set

Here the basic concepts of number line and set are tied together.

Another way of specifying a set of numbers is by showing the numbers as points on the number line. The set of points corresponding to a set of numbers is called the **graph** of the set.

EXAMPLES.

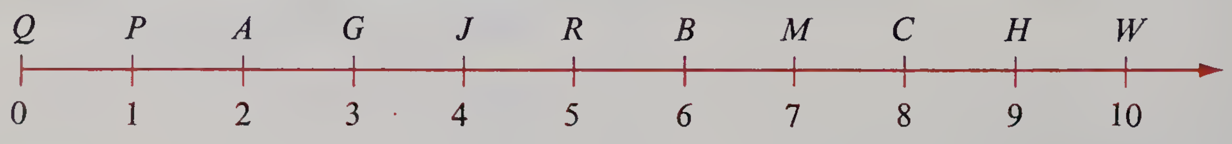
Set	Graph
{1, 2, 3}	
{the numbers between 1 and 3, including 3}	
{the numbers greater than 3}	

Note: A darkened circle ● represents a point corresponding to a number in a set. A darkened line ▬ is used to show that all points on it belong to the graph. Points not belonging to the graph are indicated by open circles ○ or appear on undarkened lines. A darkened arrow indicates that the graph continues indefinitely.

Specifying a set: identifying its elements by
1. the roster method — listing the elements, or
2. the rule method — describing the elements, or
3. the graphic method — locating the elements on the number line.

ORAL EXERCISES

Specify the graph of each set by referring to the number line below.



SAMPLE {the even whole numbers between 3 and 9, inclusive}

What you say: *J*, *B*, *C*.

1. {8, 2, 5} C, A, R
2. {1, 8, 10} P, C, W
3. {7, 3, 4} M, G, J
4. ∅ no point
5. {0} Q
6. {4, 2, 7} J, A, M
7. {even whole numbers between 3 and 7} J, B

Tint blocks highlight the methods which students should understand.

8. {odd whole numbers between 3 and 7} R

9. {even whole numbers between 5 and 6} no point

10. {whole numbers greater than 2 but less than 3} no point

11. {whole numbers less than or equal to 2} Q, P, A

12. {whole numbers between 1 and 6, inclusive} P, A, G, J, R, B

13. {whole numbers between 1 and 5} A, G, J

14. {multiples of 2 between 0 and 1} no point

15. {multiples of 5 between 4 and 6} R

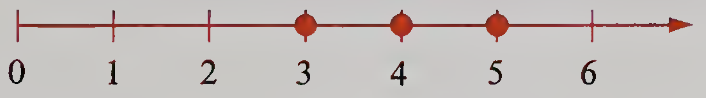
16. {whole numbers between 1 and 9} A, G, J, R, B, M, C

WRITTEN EXERCISES

Draw the graph of each given set.

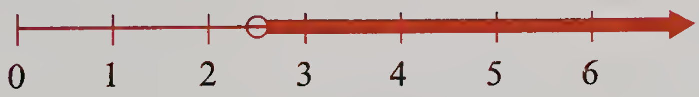
SAMPLE 1. {the whole numbers between 3 and 5, inclusive}

Solution:



SAMPLE 2. {the numbers greater than $2\frac{1}{2}$}

Solution:



A **1.** {the whole numbers between 2 and 8, inclusive}

2. {the whole numbers less than 5}

3. {the whole numbers less than or equal to 5}

4. {the numbers greater than 5}

5. {the numbers less than 5}

6. {the numbers greater than $7\frac{1}{2}$}

7. {the numbers greater than $3\frac{1}{3}$ but less than 10}

8. {the numbers between 3 and 6, including 6}

9. {the numbers between $4\frac{1}{2}$ and 5, including $4\frac{1}{2}$}

10. {the numbers less than 6 and greater than or equal to 2}

11. {the whole numbers less than 8 and greater than 3}

12. {the whole numbers between 3 and 4}

13. {3, 9, 2}

14. $\{\frac{1}{2}, \frac{3}{2}\}$

15. {0}

16. $\{\frac{0}{4}\}$

17. {2, 4, 6}

18. {1, 2, 3, 4, 5, . . . , 9}

1–7 How Subsets Relate to Sets

Suppose you form another set by removing one of the elements of $R = \{0, 1, 2\}$. For example, remove the element 1, and form a new set,

$$M = \{\mathbf{0}, \mathbf{2}\}.$$

Notice that every element in set M is also an element in set R. We say that M is a subset of R. Whenever a set, such as M, contains only elements which are also elements of another set, such as R, the set M is said to be a **subset** of set R. A subset, such as M, which does not contain all the elements of the given set is called a **proper subset** of the set. Thus, {junior high school teachers} is a proper subset of {all teachers}.

See T.M. pg. 8 for an alternative definition of subset and a suggestion for using diagrams.

See how many subsets of R you can find. By removing either one or two elements from R you can form six proper subsets of R:

$$\{\mathbf{0}, \mathbf{2}\} \qquad \{\mathbf{0}, \mathbf{1}\} \qquad \{\mathbf{1}, \mathbf{2}\} \qquad \{\mathbf{0}\} \qquad \{\mathbf{1}\} \qquad \{\mathbf{2}\}.$$

Notice again that every element in each subset is also an element of the set R, and that none of these subsets contains all the elements of R.

When you remove all the elements of R, you obtain the set with no elements, the empty set, $\emptyset$. You see that $\emptyset$ is a proper subset of every set, except of itself.

Another subset of R is formed when you remove no elements. Thus, the full set

$$\{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$$

is also a subset of R, but it is called an *improper subset.* You can see that every set is a subset of itself.

The notion of set appears everywhere in human thought; in mathematics the notion is consciously developed as a basic, unifying idea that you will meet over and over again as you increase your mathematical knowledge.

ORAL EXERCISES

Tell which statements are true and which are false. Justify your answers.

1. {1, 3, 6} is a subset of {7, 6, 5, 4, 3, 2, 1} *T*
2. {0, 2} is a subset of {1, 3, 2, 4} *F; 0 is not an element of the second set*
3. {the New England states} is a subset of {the states of the U.S.A.} *T*

5. F; since not all high school students study algebra.

4. {red-haired people} is not a subset of {women} T; not every element of the first set is an element of the second set, e.g., a red-haired boy.
5. {high school students} is a subset of {people studying algebra}
6. {people studying algebra} is a subset of {people studying mathematics} T; every student of algebra is a student of mathematics
7. {0, 1, 2, 3, . . . , 9} is not a subset of {all the digits} F; every element of the first set is an element of the second set.
8. {1, 3, 5, 7, 9, . . .} is a subset of {1, 2, 3, 4, 5, . . .} T
9. {5} is a subset of {2, 3} F
10. {0} is not a subset of {10, 6, 18} T; 0 is not an element of the given set.
11. {0} is a subset of {0} T
12. ∅ is not a subset of {0} F; ∅ is a subset of every set.

For each set, list the largest subset of (a) odd numbers, (b) even numbers.

13. {1, 2, 3, 4, 5} (a) {1, 3, 5} (b) {2, 4}
14. {18, 19, 20, 21, 22} (a) {19, 21} (b) {18, 20, 22}
15. {8, 16, 32, 64} (a) ∅ (b) {8, 16, 32, 64}
16. {5, 25, 125, 625} (a) {5, 25, 125, 625} (b) ∅
17. {5, 6, 7, 8, . . .} (a) {5, 7, 9, ...} (b) {6, 8, 10, ...}
18. {1, 4, 9, 16, . . .} (a) {1, 9, 25, 49, ...} (b) {4, 16, 36, ...}
19. {1, 9, 25, . . .} (a) {1, 9, 25, ...} (b) ∅
20. {4, 16, 36, 64, . . .} (a) ∅ (b) {4, 16, 36, 64, ...}

WRITTEN EXERCISES

Let $U = \{3, 15, 10\}$. List all of the subsets of U that:

Ⓐ

1. have exactly one element
2. have exactly two elements
3. have three elements
4. have no elements
5. have multiples of 5 for elements
6. consist of elements each less than 3
7. have even numbers for elements
8. have odd numbers for elements

Specify, by rule or roster, a set of which the given set is a proper subset.

9. {the numbers greater than 5}
10. {the whole numbers}
11. {the fractions between 0 and 1}
12. {Babe Ruth}
13. {Juneau, Nome, Anchorage}
14. {Hawaii, Oahu, Molokai, Maui}

USING NUMBERS IN ONE OR MORE OPERATIONS

1–8 Punctuation Marks in Algebra

Some students were asked to rewrite the following set of words in the same order, but to insert punctuation marks and capital letters to produce a grammatically correct and meaningful sentence:

paul said the teacher is very intelligent

Mary wrote:

"Paul," said the teacher, "is very intelligent."

Henry wrote:

Paul said, "The teacher is very intelligent."

Both Mary and Henry had produced correct and meaningful sentences. The differences in punctuation, however, had produced a world of difference in meaning. Without punctuation marks the original statement could be interpreted in more than one way.

In mathematics you eliminate statements which could be interpreted in more than one way by using mathematical punctuation marks, much as you use punctuation marks in English composition. For example, what number is represented by the set of symbols

$$3 \times 2 + 4?$$

Is it 10 or is it 18? An expression such as $3 \times 2 + 4$ could be called *ambiguous* (am-**big**-u-us) because of the different interpretations.

In mathematics, one way you avoid ambiguous statements is by using parentheses. When you punctuate $3 \times 2 + 4$ as follows:

$$(3 \times 2) + 4,$$

you mean 10. When you write

$$3 \times (2 + 4),$$

you mean 18.

Stress the fact that symbols of inclusion are used to aid in the interpretation, without ambiguity, of numerical expressions. See 1-8 T.M. pg. 8.

A pair of parentheses is called a **symbol of inclusion** because it is used to enclose, or include, an expression for a particular number. The parentheses in the numerical expression $3 \times (2 + 4)$ serve to group the numerals 2 and 4 together with the symbol + and to indicate that the *sum* of 2 and 4 is to be multiplied by 3.

In writing $3 \times (2 + 4)$, it is customary to omit the symbol × and to write simply

$$3(2 + 4).$$

Similarly, the product 3×6 may be expressed in any of the forms

$$3(6), \qquad (3)6, \qquad \text{or} \qquad (3)(6).$$

Brackets, braces, and a bar are used for the same purpose:

Parentheses	*Brackets*	*Braces*	*Bar*
$3(2 + 4)$	$3[2 + 4]$	$3\{2 + 4\}$	$3\,\overline{2 + 4}$

In working with fractions you have seen that the bar acts as a division sign, as well as a symbol of inclusion. For example, in the expression below, the bar groups the 16 and 4; it also groups the 4 and 1. The bar tells you that the number (16 − 4) is to be divided by the number (4 − 1).

$$\frac{16-4}{4-1}=\frac{12}{3}=4$$

Each part of this statement designates the same number as the one before it, but as you carry out the operations in the indicated order the numeral becomes simpler. In simplifying an expression, you use the signs of grouping to determine the order of operation.

When you see a grouping inside of another grouping, such as $5[3+(7\times 2)]$, you always simplify the numeral in the innermost symbol of inclusion and proceed to work toward the outermost grouping until all symbols of inclusion are removed, thus,

$$\begin{aligned}5[3+(7\times 2)] &= 5[3+(14)]\\ &= 5[17]\\ &= 85.\end{aligned}$$

ORAL EXERCISES

Simplify each of the following expressions.

SAMPLE. $(8\div 2)+7$ *What you say:* $(8\div 2)+7=4+7=11$

1. $5+(3\times 2)$ 11
2. $(8+3)+5$ 16
3. $(5+3)\times 2$ 16
4. $8+(3+5)$ 16
5. $4(3+7)$ 40
6. $\dfrac{15+4}{6+4}$ $\frac{19}{10}$
7. $\dfrac{10-1}{1+2}$ 3
8. $([3\times 2]\div 4)$ $\frac{3}{2}$
9. $\dfrac{(\frac{20}{2})}{5}$ 2
10. $\dfrac{64}{[\frac{8}{2}]}$ 16
11. $\dfrac{7}{(\frac{21}{3})}$ 1
12. $\dfrac{17+1}{8-2}$ 3
13. $\overline{5+3}+\frac{8}{2}$ 12
14. $(4\times 3)+(4\times 7)$ 40
15. $2\times\{6+5+2\}$ 26
16. $5[2+1-2]+6$ 11
17. $\overline{5-2}\times\overline{5+2}$ 21
18. $\dfrac{6\times 2}{12-4}$ $\frac{3}{2}$
19. $\dfrac{13-10}{5\times 2}$ $\frac{3}{10}$
20. $\dfrac{15-5}{2\times 10}$ $\frac{1}{2}$

WRITTEN EXERCISES

Simplify each of the following expressions.

SAMPLE. $[(14 \times 2) + 5] \div 11$

Solution: $$\begin{aligned}[(14 \times 2) + 5] \div 11 &= [28 + 5] \div 11 \\ &= [33] \div 11 \\ &= 3\end{aligned}$$

A

1. $0 \times [5 + (2 \times 3)]$

2. $5 \times [5 - (0 \times 3)]$

3. $17 \times \overline{15 \times 2}$

4. $\overline{17 \times 15} \times 2$

5. $\dfrac{49 - 25}{7 + 5}$

6. $\dfrac{25 - 9}{5 - 3}$

7. $\dfrac{100 - 64}{10 - 8}$

8. $\dfrac{100 - 64}{10 + 8}$

9. $(17 + 3)18$

10. $(17 \times 18) + (3 \times 18)$

11. $(16 + 4) \div ([3 \times 2] \div 4)$

12. $[30 \div (5 \times 2)] \div 3$

B

13. $(27 \div 9) + [(27 + 9) \div 3]$

14. $[(16 + 4) \div (3 \times 2)] \div 13$

15. $\left[\dfrac{100 - 64}{10 + 8}\right] \div 12$

16. $\left[\left(\dfrac{50 + 25}{15}\right) \div 5\right] + 1$

17. $[\overline{5 \times 20} + \overline{6 \div 3}] - 51$

18. $\left[\dfrac{12 + 8}{9 \times 6} - \dfrac{1}{54}\right] - \dfrac{1}{54}$

19. $(3 \times 9) + [\overline{9 \times 3} + 2]$

20. $\{4 \times 3\} - \{4 \div 3\} \div 8$

C

21. $\left[\left\{\left(\dfrac{100 - 36}{1 + 7} - 8\right) \times 3\right\} + 5\right] \times 200$

22. $\{(\overline{1776 - 324} \times 5) - 5\} \times 100$

23. $[(\overline{4 \times 2} + \overline{6 \times 10} - \overline{3 \times 20}) + 56] \div 13$

24. $\left[\left\{\left(\dfrac{579 + 682}{39}\right) \times 27\right\} + \overline{9 \times 3}\right] \div 27$

1–9 Order of Operations

We use this convention to simplify expressions and because students will encounter it elsewhere. See T.M. pg. 8.

Parentheses and the other symbols of inclusion are the customary means used to make clear the meaning of a numerical expression. However, mathematicians have agreed on a rule to fall back on if someone omits punctuation marks. This rule gives the order to be followed in performing the operations indicated in the expression. The agreement implies, for example, that

Color is used for teaching emphasis.

$5 + 3 \times 2$ means $5 + (3 \times 2) = 5 + 6 = 11$

$7 \times 4 + 3$ means $(7 \times 4) + 3 = 28 + 3 = 31$

$36 \div 4 - 1$ means $(36 \div 4) - 1 = 9 - 1 = 8$

$6 \times 8 - 7 \times 2$ means $(6 \times 8) - (7 \times 2) = 48 - 14 = 34$

$30 \div 10 \times 3$ means $(30 \div 10) \times 3 = 3 \times 3 = 9$

$5 \times 3 \times 4$ means $(5 \times 3) \times 4 = 15 \times 4 = 60$

$4 + 3 + 2$ means $(4 + 3) + 2 = 7 + 2 = 9$

$7 - 3 - 2$ means $(7 - 3) - 2 = 4 - 2 = 2$

$5 + 2(4 - 3)$ means $5 + 2(1) = 5 + 2 = 7$

> In a numerical expression containing a series of numerals connected by symbols of operation, you agree to follow this order:
>
> 1. simplify the expression within each symbol of inclusion;
> 2. perform the multiplications and divisions in order from left to right;
> 3. finally, do the additions and subtractions in order from left to right.

ORAL EXERCISES

Simplify each of the following expressions.

1. $5 - 3 - 2$ 0
2. $12 - 0 + 1$ 13
3. $3(4) + 7$ 19
4. $8 + 6 \times \frac{1}{3}$ 10
5. $20 \div 4 \div 2$ $\frac{5}{2}$
6. $10 \div 5 \times 4$ 8
7. $10 + 9 + 3$ 22
8. $49 + 5 \times 0$ 49
9. $14 \div 7 \times 1$ 2
10. $14 \div 1 \times 7$ 98
11. $80 \div 8 - 80 \div 10$ 2
12. $12(4) - 16 \div 2$ 40
13. $17 \times 25 \times 4$ 1700
14. $\frac{1}{5} \times 12\frac{1}{2} \times 2$ 5
15. $6 \times 2 - 2 \div 2$ 11
16. $12 \div 12 + 6 \times 2$ 13
17. $18 - 6(3) + 5$ 5
18. $5 - 2 + 4 \times 3$ 15
19. $12 \times 6 + 12 \times 4$ 120
20. $39 \times 9 + 39 \times 1$ 390

WRITTEN EXERCISES

Simplify each of the following expressions.

SAMPLE. $11 + 2(6 + 4) - 3(1 + 3)$

Solution: $11 + 2(6 + 4) - 3(1 + 3)$

Step 1: $11 + 2(10) - 3(4)$
Step 2: $11 + 20 - 12$
Step 3: $31 - 12$
19, **Answer.**

A

1. $(7 + 3 + 2) \div 3 + 1$

2. $(7 + 3 + 2) \div (3 + 1)$

3. $5(7 + 9) \div 4 + 3$

4. $21 + 5(7 + 3) - 20$

5. $7 + 3(5 - 1) \div 6$

6. $7 + (15 - 3) \div 6$

7. $8 \div 2 + 6 \div 3$

8. $30 - 3(7 - 2)$

9. $\dfrac{32 \div 8 + 3}{7}$

10. $\dfrac{5(3) + 20}{5 + 2}$

11. $\dfrac{2 \times 3 + 21}{2 + 1}$

12. $\dfrac{13 \times 5 - 3 \times 5}{8 + 2}$

13. $27 + 10 - (11 + 3)$

14. $27 + 10 - 11 - 3$

15. $5 + 4 - (3 - 1)$

16. $5 + 4 - 3 + 1$

B

17. $64 \div 8 \div 4 \div 2$

18. $12 \times 6 \div 3 \times 2 \div 48$

19. $6(7 + 2) - 15 \div 5$

20. $5(7 - 4) \div 3 + 2$

21. $\dfrac{9 - 5(3 - 2)}{3 + 5}$

22. $\dfrac{4(3 + 1) - 1}{4 + 1}$

23. $\dfrac{15 \div 3 + 2 \times 3}{2(5 + 6)}$

24. $6(6 + 8) + 8 - 32 \div 16$

25. $(12 - 2)(18 - 3)$

26. $(2 \times 3 - 12 \div 3) \div 2$

27. $(3 + 4.3)(4 + 5)(5 + 4)$

28. $3.3 + 3(6)(5 - 1) - 2$

29. $(15 - 3 + 8 \div 2) \div 8 \times 5$

C

30. $3 + 48 - 16 - 35 \div 7$

31. $\frac{1}{2}(8 \div 2 \times 4) \div (8 \times 4 \div 2)$

32. $\dfrac{\overline{4 \times 5} \times 20 - 6 \div 2}{13 + 3 \times 1 \times 5}$

33. $\dfrac{8 \div 4 \times 3 - 2 + 16}{(1 \times 2) \times 5 \times 2 \div 5}$

34. $\dfrac{1 + 44 \div 4 + 12 \times 44}{3 \times 3 - 3 \div 3 + 2}$

These interesting historical vignettes should lead students to further reading about the history of math. Note the use of = in the page design.

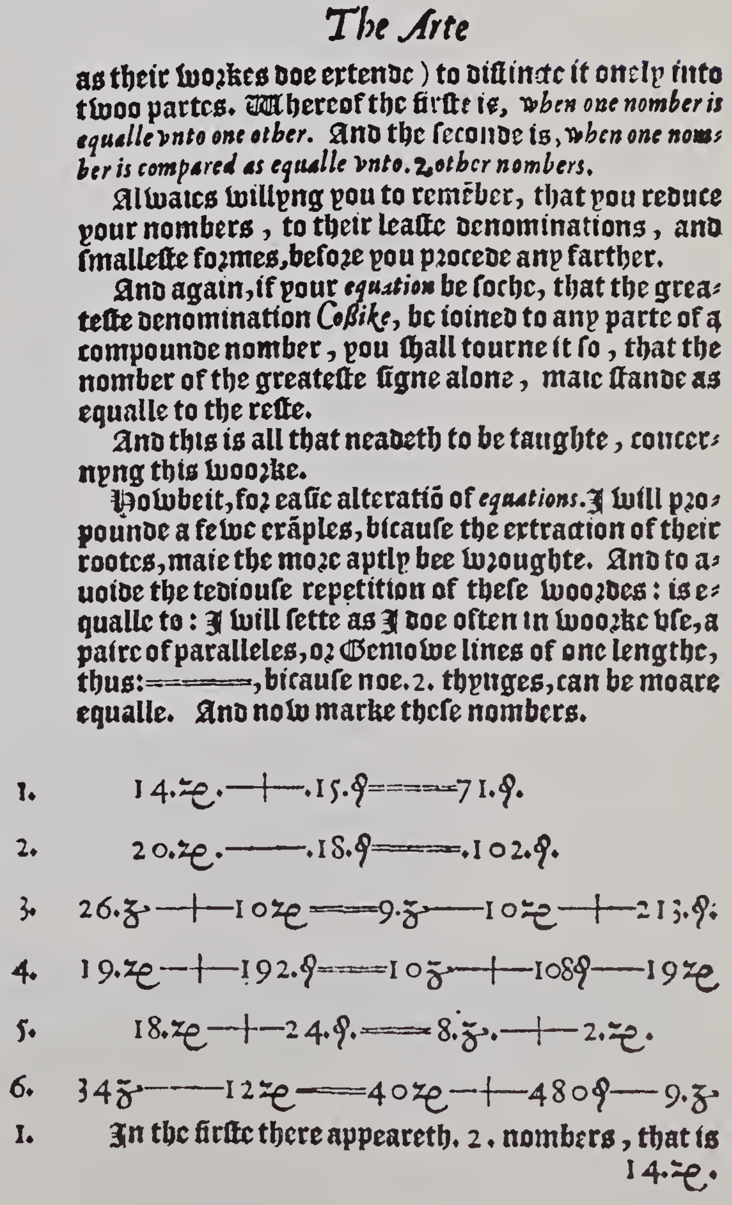
THE HUMAN
EQUATION

A Reserved Table

A casual visitor to the Tower of London in the year 1606 might have witnessed an unexpected sight. In the midst of this infamous prison, at a table reserved for their use, a group of men, all friends and guests of one of the prison's inmates, would congregate to discuss mathematics. The host of this unusual party was no lesser personage than the Earl of Northumberland. The leading figure in the discussions was an accomplished astronomer and mathematician, Thomas Harriot.

Harriot had come to his place at the Earl's table in the Tower by way of an eventful life. Born in 1560, he was caught up in the spirit of vigor and creativity which pervaded England during the reign of Elizabeth I. His career began with studies at Oxford, and soon after, he served as Sir Walter Raleigh's tutor in mathematics. It was Raleigh who appointed Harriot to the office of surveyor with the second expedition to Virginia. After returning to England and his mathematical studies, Harriot was awarded a life pension by the Earl of Northumberland, himself an amateur mathematician. So it was that in 1606, when the Earl came into disfavor with the Crown and was imprisoned in the Tower, Harriot was among the honored guests at his table.

Although Harriot's last years were beset by cancer, he continued to demonstrate remarkable mathematical talents. The use of the sign = for equality, though introduced by another mathematician, Recorde, is partly due to Harriot, who helped persuade other mathematicians of the day to adopt this notation. To Harriot alone, moreover, we owe two of the most useful mathematical notations, the symbols > and <.



The Arte

as their woꝛkes doe extende) to diſtincte it onely into twoo partes. Whereof the firſte is, *when one nomber is equalle vnto one other.* And the ſeconde is, *when one nomber is compared as equalle vnto. 2. other nombers.*

Alwaies willyng you to remēber, that you reduce your nombers , to their leaſte denominations , and ſmalleſte foꝛmes, befoꝛe you pꝛocede any farther.

And again, if your *equation* be ſoche, that the greateſte denomination *Coßike*, be ioined to any parte of a compounde nomber , you ſhall tourne it ſo , that the nomber of the greateſte ſigne alone , maie ſtande as equalle to the reſte.

And this is all that neadeth to be taughte , concernyng this woozke.

Howbeit, foꝛ eaſie alteratiō of *equations*. I will pꝛopounde a fewe exāples, bicauſe the extraction of their rootes, maie the moꝛe aptly bee wꝛoughte. And to auoide the tediouſe repetition of theſe woozdes : is equalle to : I will ſette as I doe often in woozke vſe, a paire of paralleles, oꝛ Gemowe lines of one lengthe, thus: ═══════, bicauſe noe. 2. thynges, can be moare equalle. And now marke theſe nombers.

1. 14.ʓ.—+—.15.ϙ═══71.ϙ.
2. 20.ʓ.———.18.ϙ═══.102.ϙ.
3. 26.ʒ—+—10ʓ═══9.ʒ———10ʓ—+—213.ϙ.
4. 19.ʓ—+—192.ϙ═══10ʒ—+—108ϙ———19ʓ
5. 18.ʓ—+—24.ϙ.═══8.ʒ.—+—2.ʓ.
6. 34ʒ———12ʓ═══40ʓ—+—480ϙ———9.ʒ

1. In the firſte there appeareth. 2. nombers, that is 14.ʓ.

A page from Robert Recorde's Whetstone of Witte. *This was the earliest algebra written in English and contains the first use of the = sign, which Thomas Harriot later helped popularize.*

This section is important for review; key points are emphasized by bold-faced type.

Chapter Summary

Inventory of Structure and Method

1. **Arithmetic numbers** can be arranged in *order* of size. Each number can be paired with a point on the **number line.** A pairing, like that of point and number, is called a **1-to-1 correspondence.**

2. In **comparing numbers,** symbols of equality (=) and inequality ($\neq$, $>$, $<$) are used. Numbers may be compared in terms of their location on the number line; larger numbers are to the right. To **specify a set,** identify its elements by (a) roster (list the elements) or (b) rule (describe the elements) or (c) graph (locate the elements on the number line).

3. An expression within a **symbol of inclusion** (grouping), such as $(9 - 5)$ in $3(9 - 5)$, is to be treated as one quantity or numeral. The *order of operations* is as follows:

   a. Simplify within each symbol of inclusion.

   b. Perform multiplications and divisions in order from left to right.

   c. Do additions and subtractions in order from left to right.

Vocabulary and Spelling

Pronounce, spell, and give the mathematical meaning or symbol of the words and expressions. The number refers to the page on which each word is first introduced.

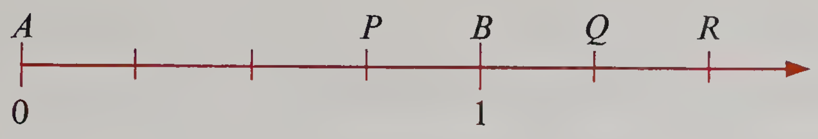
property (*p. 1*)
number line (*p. 1*)
numeral (*p. 2*)
arithmetic number (*p. 2*)
uniform scale (*p. 2*)
order of magnitude (*p. 2*)
coordinate of a point (*p. 2*)
graph of a number (*p. 2*)
between (*p. 2*)
numerical expression (*p. 5*)
sign of equality (*p. 5*)
sign of inequality (*p. 7*)
set (*p. 10*)
member of set (*p. 10*)
element of set (*p. 10*)
braces (*p. 11*)
($\in$), ($\notin$) (*p. 11*)
specifying a set (*p. 11*)
roster of a set (*p. 11*)
rule for a set (*p. 11*)
one-to-one correspondence (*p. 13*)
infinite set (*p. 13*)
finite set (*p. 13*)
empty or null set ($\emptyset$) (*p. 13*)
graph of a set (*p. 16*)
subset (*p. 18*)
proper subset (*p. 18*)
improper subset (*p. 18*)
symbol of inclusion (*p. 20*)
parentheses (*p. 20*)
brackets (*p. 20*)
bar (*p. 20*)
simplify (*p. 21*)
order of operations (*p. 23*)

This cross reference to pages provides the student with a convenient check on his interpretations.

This is a diagnostic test with keyed section numbers. If a student misses any items on the test, he knows at once which section to restudy. After restudy, he answers the questions in the appropriate section of the Chapter Review.

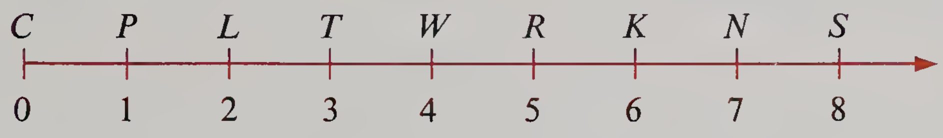
Chapter Test

1–1 Find the numeral corresponding to each of the following division points on the number line below:



1. P **2.** Q **3.** R

Use the number line below to find the coordinate of each of the following points:



4. The point halfway between P and N

5. The point one-fourth of the distance from T to R

6. The point $3\frac{1}{3}$ units to the left of K

1–2 For each statement, tell whether or not it is true, and give a reason for your answer.

7. $4 \times 5 = 8 + 12$

8. $16 - 7 = 3 \times 4$

9. $3 + 5 + 9 = 8 + 9$

10. $12\frac{3}{5} = 11\frac{13}{5}$

1–3 For each statement, find a numeral that may replace the question mark and make the resulting statement true.

11. $8 \times 4 \neq 30 + ?$

12. $? - 10 > 10$

13. $2 < ? < 4$

14. $\frac{27}{2.35} > ? > \frac{2.7}{.245}$

1–4 Make a roster for each of the following sets:

15. {the multiples of 6 between 5 and 26}

16. {the three-digit numerals}

1–5 Identify the following sets as being finite, infinite, or the empty set:

17. $\{0, 1, 2, \ldots\}$

18. $\{4, 6, 8\}$

19. {whole numbers between 0 and 1}

This is an important part of the extensive built-in testing program in this text.

1–6 Draw the graph of each of the following sets:

20. {the whole numbers between 1 and 4, inclusive}

21. {the numbers between 1 and 4}

22. {the numbers greater than $1\frac{1}{2}$}

1–7 **23.** List that subset of A, where $A = \{2, 4, 6, 8\}$, that consists of all the elements of A that are multiples of 4.

1–8 **24.** Simplify: $[60 \div (3 \times 2)] \div 10$

1–9 **25.** Simplify: $10 + 15 \div 5$ **26.** Simplify: $25 \times 2 - 7 \times 7$

Before You Go on to Chapter 2

Did you miss any of the test items? If so, note the section number that corresponds to each item you missed. Restudy that section in the chapter. Then find the section number in the Chapter Review, and do the exercises under it.

Did you get all the items correct? If so, you may turn to page 30 and enjoy the Extra for Experts.

This section consists of reteaching and additional drill. The student will use the sections of the review which correspond to the test items which he failed.

Chapter Review

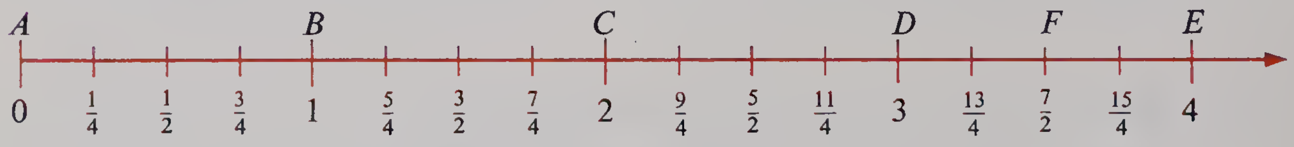
1–1 Representing Numbers on a Line: Order Relations *Pages 1–5*

1. A numeral is a name for a ___?___.

2. The starting point of the number line is labeled ___?___.

3. On a number line, arithmetic numbers appear in order of ___?___.

Exercises 4–7 refer to the following number line:



4. To label any point, you must know its ___?___ from zero.

5. The distance between the points labeled $\frac{3}{2}$ and $\frac{9}{4}$ is the same as the distance between the points labeled 0 and ___?___.

6. The coordinate of the point halfway between B and F is ___?___.

7. The points that are the graphs of the whole numbers between 1 and 4, including 4, are C, D, and ___?___.

The pages refer to topic areas in the text.

1-2 Comparing Numbers: The Sign of Equality *Pages 5–7*

8. Any name or symbol for a number is called a __?__ expression or __?__.

9. When two expressions represent the same number, they are said to be __?__.

10. The symbol for *equals* or *is equal to* is __?__.

1-3 Comparing Numbers: The Signs of Inequality *Pages 7–10*

11. The symbol for *does not equal* or *is not equal to* is __?__.

12. $7 > 5$ is read "7 is __?__ than 5."

13. "Five is less than seven" may be written in symbols as __?__.

14. $1 < 8 < 9$ reads "8 is greater than __?__ and less than __?__."

15. Replace the question mark by a whole number to make the resulting statement true: $4 <$ __?__ < 6.

1-4 Meaning of Membership in a Set *Pages 10–12*

16. Any collection of objects is called a __?__.

17. Each object in a set is called a(n) __?__ of the set.

18. The fact that 3 belongs to the set of numbers 1, 2, 3, 4, can be written 3 __?__ {1, 2, 3, 4}.

19. The true statement $3 \notin$ {all the even numbers} means that 3 is __?__ an even number.

20. When you identify a set by listing its elements within braces, you are making a __?__.

21. When you specify a set by describing the elements within braces, you are giving a __?__.

22. {the multiples of 3 between 10 and 20} = {__?__, __?__, __?__}.

1-5 Kinds of Sets *Pages 13–15*

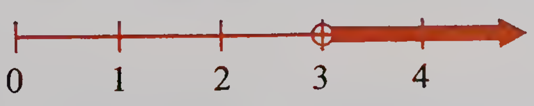
23. A nonending set of numbers is said to be __?__.

24. A set containing no elements is the __?__ set.

25. A set with a specific number of elements is a __?__ set.

26. $\emptyset$ represents the __?__ set.

1-6 The Graph of a Set *Pages 16–17*

27. 

represents the graph of {all numbers greater than __?__}.

28. On a number line, show the graph of {all the numbers between 0 and 2, including 2}.

1-7 How Subsets Relate to Sets *Pages 18–19*

29. If every element of set A is also an element of set B, then set A is a ___?___ of set B.

30. The subset of {4, 10, 20} which consists of elements that are multiples of both 4 and 5 is {___?___}.

1-8 Punctuation Marks in Algebra *Pages 19–22*

31. Parentheses, ___?___, ___?___, and the ___?___ are called symbols of inclusion.

32. When an expression is enclosed in parentheses, it is to be treated as ___?___ quantity or numeral.

33. Simplify: $2 + \overline{3 \times 4}$

34. Simplify: $\dfrac{(8 \div 4) \div 2}{8 \div (4 \div 2)}$

1-9 Order of Operations *Pages 23–24*

35. In performing a series of operations, first simplify within each symbol of ___?___, then perform the indicated ___?___ and ___?___; finally do the ___?___ and ___?___.

36. $72 \div 6 - 2 \times 3 =$ ___?___

37. $3 + 6 \times 4 =$ ___?___

38. $\frac{24}{4} - 2 + 3 \times 5 =$ ___?___

39. $45 - 5.5 \div 10 =$ ___?___

40. $\dfrac{18 \div 9 + 3 \times 4}{2(3 + 4)} =$ ___?___

The "Extras" are designed to challenge the rapid learner. They provide enrichment in the form of additional concepts or advanced techniques, usually extending ideas presented in the chapter.

Extra for Experts

The Arithmetic of Sets: Intersection

The *intersection* of two sets consists of the elements they have in common. For example, if $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$, the intersection of these sets would be {3, 4, 5}, which could be designated set C. The symbol for intersection is ∩ (read "cap").

In words: The intersection of set $\boldsymbol{A}$ and set $\boldsymbol{B}$ is set $\boldsymbol{C}$.

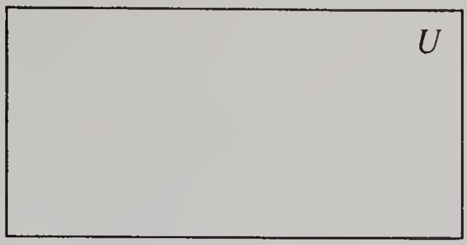
In symbols: $\boldsymbol{A \cap B = C}$

or: $\{\mathbf{1, 2, 3, 4, 5}\} \cap \{\mathbf{3, 4, 5, 6, 7}\} = \{\mathbf{3, 4, 5}\}$.

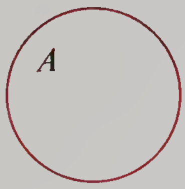
It should be noted that the intersection of two sets is a subset of each set.

Intersection may also be represented pictorially by closed figures called Venn diagrams. The region within a Venn diagram is assumed to represent the set being illustrated. For example,

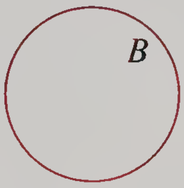
Let $U = \{\text{whole numbers}\}$:



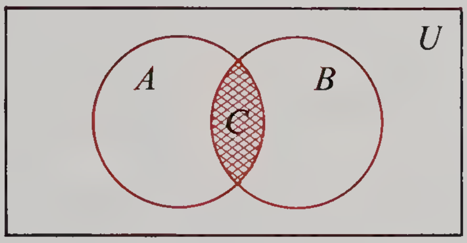
Let $A = \{1, 2, 3, 4, 5\}$:



Let $B = \{3, 4, 5, 6, 7\}$:



Then $C = A \cap B$:
(the cross-hatched region,
common to the circles)



Because each of the sets used in the problem is a subset of U, U is called the universe or universal set. In the above, {people over 21 years of age} would not be a suitable universe since A and B would not be subsets of U. By changing the universal set, you change the nature of the problem.

Questions

1. If $A = \{1, 2, 3\}$ and $B = \{0, 2, 3, 4\}$, give the roster of $A \cap B$ and use Venn diagrams to picture it.

- **a.** What is the relationship between $A \cap B$ and A? and B? Why?
- **b.** Give a rule explaining when $A \cap B = A$ will hold.

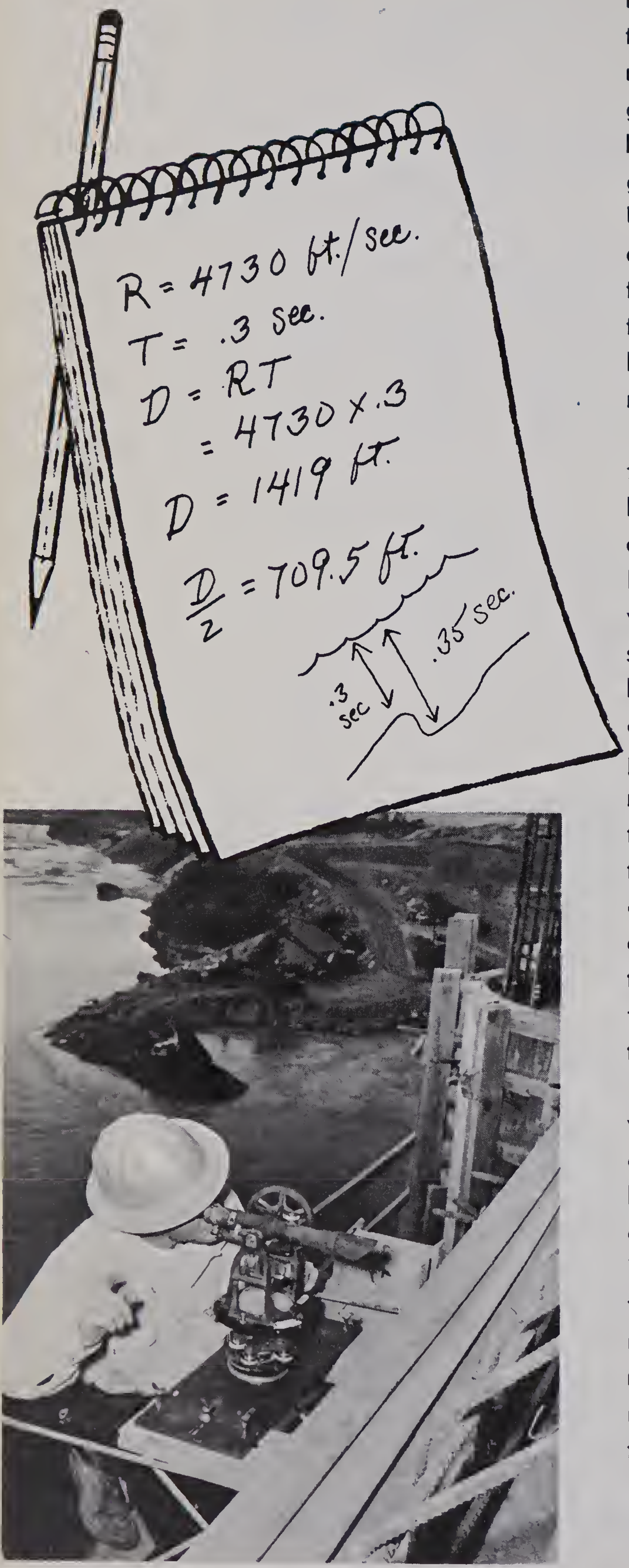
2. If $A = \{q, r, s, y\}$ and $B = \{t, u, v, w\}$, specify $A \cap B$. Use Venn diagrams to picture A and B.

- **a.** Explain the nature of $A \cap B$ when A and B have no common elements.
- **b.** What is the relationship between $A \cap B$ and A? and B? Why?
- **c.** Explain why the names *disjoint sets* or *mutually exclusive sets* are appropriate for sets of this nature.

Solutions to these questions will be found in the Solution Key.

Surveyors and Mathematics

Vocations in which mathematics have a part are emphasized by these featured layouts.


R = 4730 ft./sec.
T = .3 sec.
D = RT
= 4730 × .3
D = 1419 ft.
D/2 = 709.5 ft.
.3 sec
.35 sec.

The earliest known surveys probably were made to determine property boundaries. Although these simple land surveys are still the most familiar type, surveying is now used in a great variety of situations. Whether building highways, dams, tunnels, or skyscrapers, engineers work hand in hand with surveyors. Underground surveys, for example, guide the engineer in selecting the best position for a tunnel. On the basis of these surveys, in fact, teams excavating the tunnel from opposite banks of a river meet midway under the river, within inches of each other.

Hydrographic surveys, which provide information about the form of lake and river beds as well as the deepest parts of the ocean floor, involve certain special techniques. Instead of sighting through a telescopic device, the surveyor may use sonar to send out sound waves and receive the echo which bounces off the ocean floor. He then makes a computation like that illustrated on the work pad. Since sound waves travel at a constant rate (*R*) through water, the total distance (*D*) they travel is equal to the rate (*R*) times the time (*T*) elapsed between sending and receiving the sound. Dividing the "round-trip" distance by two gives the depth to the ocean floor. A series of such soundings and computations reveals depth variations and thus reflects the configuration of the ocean floor.

Not all surveying is large scale. The surveyor in the photograph is using a transit to determine the position of the girders of a bridge. Since steel contracts when cold and expands when hot, expansion joints, that adjust to temperature changes, are often used to join steel girders. The engineers try to merge the girders within an accuracy of a millimeter and depend on surveyors' measurements during the construction of the bridge to be certain that the girders are aligned.

Just for Fun

Cryptography

A modern use of letter symbols for numbers has become established in cryptology, the science of constructing and deciphering coded messages. Though the intelligence service uses numbers for letters, some private businesses use letters for numbers on price tags to code the initial cost of goods.

Suppose a store used the letters of the word *davenports* for the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, respectively. An *X* indicates that the digit preceding it should be repeated. Find the initial cost of the following articles: Coat: \$89.95 (NTNS); Hat: \$12.95 (RSX); Refrigerator \$625 (VANXX).

Examine the code table and the decoded messages at the right.

0	1	2	3	4	5	6	7	8	9
a	b	c	d	e	f	g	h	i	j
k	l	m	n	o	p	q	r	s	t
u	v	w	x	y	z				

857836 = 7474
SPRING IS HERE

1829474 = 4078
VICTORY IS OURS

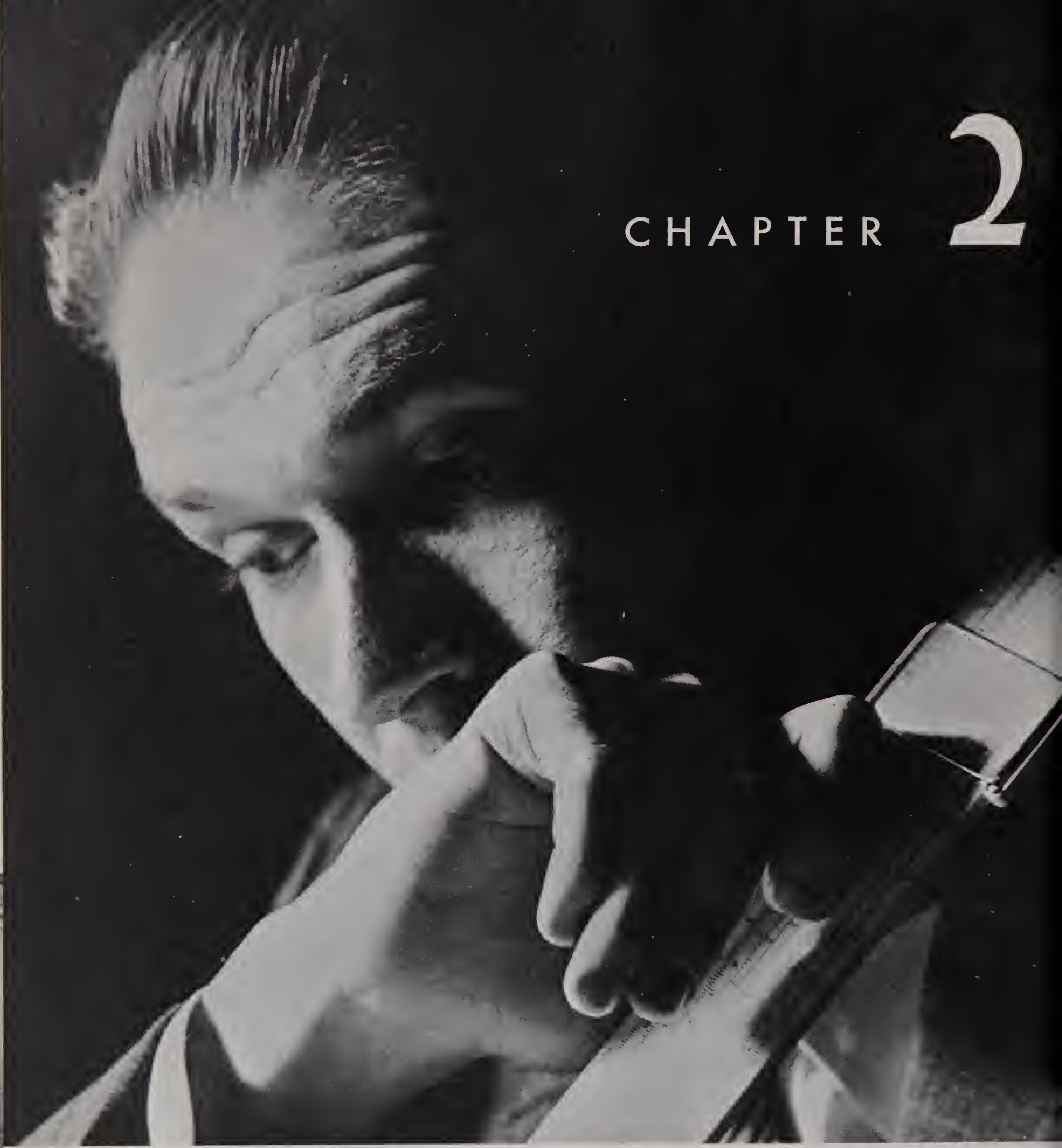
Any letter or digit in one of the columns may stand for any other letter or digit in the same column.

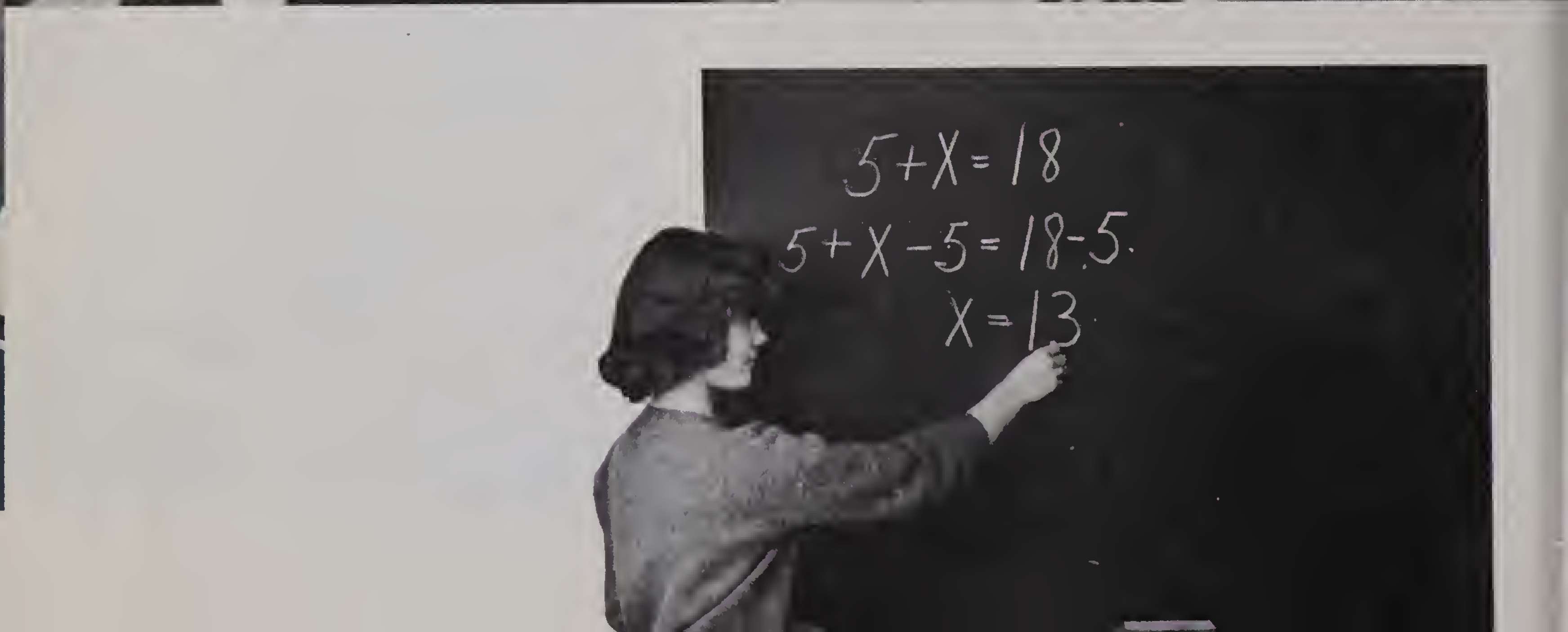
To decode a message, list all possible replacements under each numeral, then select those letters that form words. Can you decipher these messages?

1. 7083 = 249 **2.** 54024 = 507024039 **3.** WYNOI = PAD

The Just for Fun sections are recreational and cultural in nature and are intended to give all students an opportunity to widen their familiarity with mathematical ideas.

CHAPTER 2




5+X=18
5+X−5=18−5
X=13

Variables and Open Sentences

Investigate before you invest. Most people have to make decisions frequently. You usually base an important conclusion upon the investigation and consideration of several related conditions. In mathematics you are faced with decisions concerning numbers which must meet specified conditions, as the pictures indicate. In this course, you will learn a systematic procedure for using such conditions in algebraic form to find the desired numbers.

Although organized decision making is taught as part of your mathematical education, its methods are appropriate for many purposes. Analyzing conditions, evaluating them, and selecting a suitable course of action are skills which you will find useful whether your decisions are concerned with buying clothes, choosing a college, or investing a million dollars. Thus, mathematics offers opportunities for investigation and for investment in your future.

ANALYZING ALGEBRAIC STATEMENTS

2–1 Evaluating Algebraic Expressions Containing Variables

See 2-1 T.M. pg. 9 for different uses of the term "variable."

A sentence you probably have heard is, "Vegetables are good for you." The word "vegetables" is used as a general noun which refers to elements of a set of edible plants. Replace the general noun by the name of an element of the set, and you get a specific instance of the general statement. Sentences such as

"String beans are good for you."
"Carrots are good for you."

are combined in the single statement, "Vegetables are good for you."

Now, instead of vegetables, suppose you wanted to represent a multiple of three. Consider the set of multiples of three:

$$A = \{0, 3, 6, 9, \ldots\}$$

You may wish to ask better students to write expressions representing elements of such sets as: $\{0, 4, 8, 12, \ldots\}$ ($4n$) and $\{0, 1/3, 2/3, 1, \ldots\}$ ($x/3$), each replacement set being {whole nos.}.

You can express the elements of set A as "three times zero," "three times one," "three times two," "three times three," and so on.

$$\{3 \times 0, 3 \times 1, 3 \times 2, 3 \times 3, \ldots\}$$

Can you write one general expression that will include all these individual elements as special cases? To do this, you need a symbol which will assume the role that was played by the word "vegetables."

For this purpose, a letter is generally used. Thus, an element in set A can be written as $3 \times n$ where n stands for a member of $\{0, 1, 2, 3, \ldots\}$. It is very important that you notice that the letter n is not another name for *one* specific number, but rather that it represents any number which is an element of $\{0, 1, 2, 3, \ldots\}$. When used in this way, n is called a *variable*. A **variable** is a symbol which may represent any of the elements of a specified set. A set whose elements serve as replacements for a variable is called the **replacement set** for the variable. This set is also called the **domain** of the variable. The individual members of the replacement set, such as 0, 1, 2, 3, . . . , are called the **values** of the variable. A variable with just one value is called a **constant**.

Be sure that students see the need for specifying the domain of the variable.

When a product involves a variable, it is customary to omit the symbol × of multiplication. Thus, $3 \times n$ is written $3n$ and means three times n, and $a \times b$ is written ab and means a times b. Observe that in a product like 8×7 or $8(7)$ you do not omit the symbol of multiplication, for 87 means eighty-seven not 8 times 7. A raised dot also is used to show multiplication. Thus, $2 \cdot n$ means $2 \times n$ and $8 \cdot 7$ means 8×7.

When you replace the variable in the expression $3n$ by each element of its replacement set in turn, the expression represents different numbers. If you replace n by 0, you get an expression for 0. If you replace n by 1, you get an expression for 3. If you replace n by 2, you get an expression for 6. An expression which contains a variable is called a **variable expression** or an **open expression** because you have left open the decision of what number to specify for the variable. In other words, you do not know what number $3n$ names until you specify the value of n. Any variable expression or numerical expression is known as an **algebraic expression**.

If you let r represent, in turn, each of the elements of the replacement set $\{2, 3, 5, 8\}$, you *evaluate* the expressions $\frac{1}{2}r$ and $5r$ as follows:

$$\frac{1}{2}r = \frac{1}{2}(2) = 1 \qquad 5r = 5(2) = 10$$
$$\frac{1}{2}r = \frac{1}{2}(3) = \frac{3}{2} \qquad 5r = 5(3) = 15$$
$$\frac{1}{2}r = \frac{1}{2}(5) = \frac{5}{2} \qquad 5r = 5(5) = 25$$
$$\frac{1}{2}r = \frac{1}{2}(8) = 4 \qquad 5r = 5(8) = 40$$

This process of determining the number an algebraic expression represents is called **evaluating the expression** or **finding its value**. An algebraic expression written as a product or quotient of numerals or variables or both is called a **term**. The expression $3n$ is a term, and so are 7, $2d$, $3xy$, and $\frac{m}{n-1}$. In $7xy + 3(x - y) + \frac{y-x}{5}$ there are three terms, $7xy$, $3(x - y)$, and $\frac{y-x}{5}$.

Pupils should see that $3(x-y)$ is one term, while $3x-3y$ consists of two terms. Also, $\frac{y-x}{5}$ is a single term even though the expression in the numerator contains two terms.

ORAL EXERCISES

If a, b, and x have the values 15, 3, and 2, respectively, tell the value of each of the following; also, tell how many terms each contains.

1. $3a$ 45; 1
2. $7x$ 14; 1
3. $a + x$ 17; 2
4. $a - b$ 12; 2
5. $\frac{1}{3}a$ 5; 1
6. $30 \div x$ 15; 1
7. ab 45; 1
8. xx 4; 1
9. $2x - b$ 1; 2
10. $\frac{1}{6}(a + b)$ 3; 1
11. $42 \div (a - 1) + x$ 5; 2
12. bbb 27; 1
13. $b + b + b$ 9; 3
14. $3b$ 9; 1
15. $a \div b + x$ 7; 2
16. $a \div (b + x)$ 3; 1
17. $ab - x$ 43; 2
18. $ax - b$ 27; 2
19. $ax + bx$ 36; 2
20. $ab - bx$ 39; 2
21. $\frac{bx}{a}$ $\frac{2}{5}$; 1
22. $\frac{ax}{b}$ 10; 1
23. $\frac{a+b}{x}$ 9; 1
24. $\frac{a-b}{x}$ 6; 1

WRITTEN EXERCISES

In each algebraic expression, let $r = 1$, $s = 3$, $t = 12$, $u = 0$, $v = 5$, and $w = \frac{1}{2}$. Tell how many terms each expression contains; then evaluate it.

SAMPLE. $\frac{5s + 3r}{v + 2w}$ *Solution:* $\frac{5 \cdot 3 + 3 \cdot 1}{5 + 2 \cdot \frac{1}{2}} = \frac{15 + 3}{5 + 1} = \frac{18}{6} = 3$

Number of terms, 1; value, 3, **Answer.**

A

1. $6r + 2s$
2. $3t - 5v$
3. $2(3r + s)$
4. $(st)(st)$
5. $(ss)(tt)$
6. $2u(t - 2r)$
7. $2st - 4sr$
8. $r + r + u + u + u$
9. $5s - wt$

These exercises review arithmetic operations and give experience in evaluating expressions.

Ⓑ **10.** $(vv - v) - v$

11. $(2w - r)(2w + r)$

12. $(3r + t)(3r + t)$

13. $\dfrac{4w + 3r}{7}$

14. $\dfrac{6w + 5}{7v}$

15. $\dfrac{3v + t}{5s - t}$

16. $\dfrac{16sw}{(4s)(2r)}$

17. $\dfrac{t(2s + v)}{t(2s - v)}$

18. $\dfrac{2w(s + r)}{2w(s - r)}$

19. $\dfrac{5t - u}{5t + u}$

By assigning several of these problems you help students see simple, practical applications of algebraic expressions.

PROBLEMS

Evaluate the following algebraic expressions found in geometry, statistics, physics, and machine shop work by replacing the variables as indicated.

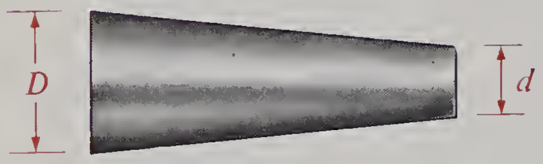
Ⓐ **1.** Tailstock offset for cutting a full-length taper: $\dfrac{D - d}{2}$; let $D = 4.375''$ and $d = 3.625''$.

2. Specific gravity of an object which floats in water: $\dfrac{U}{W - V}$; let $U = 5$ lb., $W = 12$ lb., $V = 6$ lb.

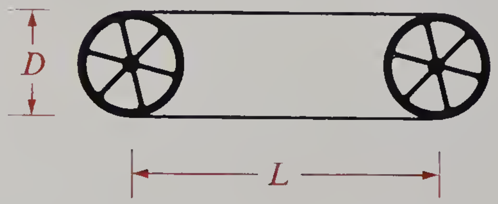
3. Degrees centigrade to degrees Fahrenheit: $\frac{9}{5}C + 32$; let $C = 20°$.

4. Length of an open belt over pulleys of equal diameters: $2L + \pi D$; let $L = 12.250'$, $D = 3.000'$, $\pi = 3.142$.

Ex. 1

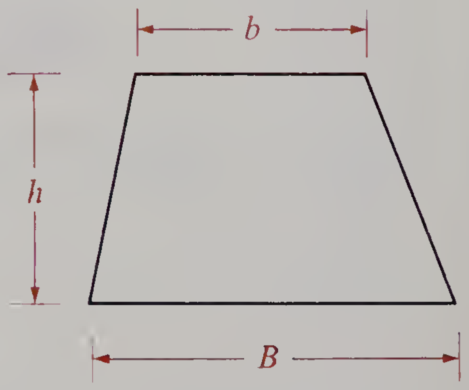


Ex. 4



Ⓑ **5.** Area of a trapezoid: $\dfrac{h(b + B)}{2}$; let $h = 4''$, $b = 6''$, $B = 7''$.

Ex. 5



6. Sum, $1 + 2 + 3 + \cdots + n$: $\dfrac{n(n + 1)}{2}$; let $n = 7$.

7. Degrees Fahrenheit to degrees centigrade: $\frac{5}{9}(F - 32)$; let $F = 77°$.

8. Distance traveled in a given second under acceleration: $\dfrac{a}{2}(2t - 1)$; let $a = 20$ ft./sec./sec., $t = 6$ sec.

9. Mechanical advantage of a differential pulley: $\dfrac{2C}{C - c}$; let $C = 15''$, $c = 12''$.

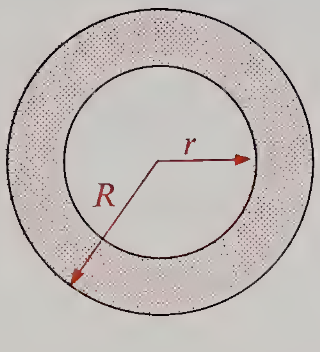
10. Slope of a line: $\dfrac{Y - y}{X - x}$; let $Y = 7$, $y = 3$, $X = 6$, $x = 2$.

11. Interior angle of a regular polygon: $180°\left(\dfrac{n - 2}{n}\right)$; let $n = 12$.

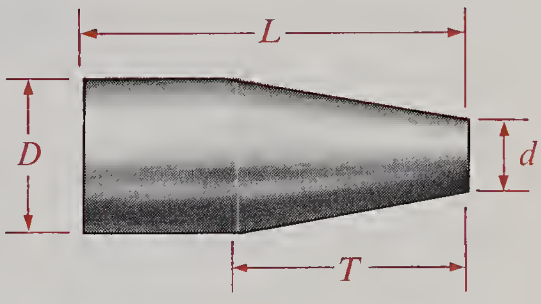
12. Linear expansion of a heated rod: $al(T - t)$; let $a = .000023$, $l = 10'$, $T = 80°$, $t = 20°$.

C **13.** Area of a circular cross section: $\pi(R - r)(R + r)$; let $R = 24''$, $r = 20''$, $\pi = \frac{22}{7}$.

Ex. 13



Ex. 17



14. Square of the area of a triangle: $s(s - a)(s - b)(s - c)$; let $a = 12''$, $b = 10''$, $c = 8''$, $s = 15''$.

15. Surface area of a box: $2(lw + wh + lh)$; let $l = 7'$, $w = 6'$, $h = 5'$.

16. Electric current through three resistances in parallel: $E\left(\dfrac{1}{A} + \dfrac{1}{B} + \dfrac{1}{C}\right)$; let $E = 110$ volts, $A = .10$ ohm, $B = .25$ ohm, $C = .20$ ohm.

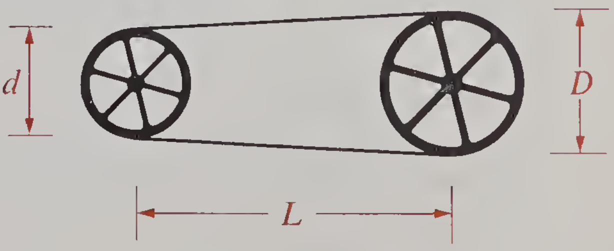
17. Tailstock offset for cutting a partial length taper: $\dfrac{L}{T}\left(\dfrac{D - d}{2}\right)$; let $L = 12.250''$, $T = 6.125''$, $D = 1.875''$, $d = 1.125''$.

18. Electric current delivered by battery cells in series: $\dfrac{nE}{R + nr}$; let $n = 3$, $E = 1.5$ volt, $R = 12$ ohms, $r = 0.1$ ohm.

19. Sum, $1 + 4 + 9 + \cdots + n^2$: $\dfrac{n(n + 1)(2n + 1)}{6}$; let $n = 7$.

20. Approximate length of an open belt over pulleys of unequal diameters: $2L + 3.25\left(\dfrac{D + d}{2}\right)$; let $L = 14'$, $D = 2.5'$, $d = 1.2'$.

Ex. 20



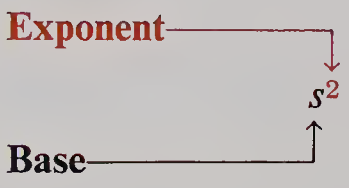
See T.M. pg. 9 (1).

2–2 Identifying Factors, Coefficients, and Exponents

When two or more numbers are multiplied, each of the numbers is called a **factor** of the product. Thus, 3 and 7 are factors of 21; two other factors are 1 and 21. Note that in factoring whole numbers you usually consider only whole number factors. Thus, the product $6x$ has 1, 2, 3, 6, x, $2x$, $3x$, and, of course, $6x$ itself as factors.

Each factor of a product is the **coefficient** (ko′-e-**fish**-ent) of the product of the other factors. In the product $\frac{1}{2}xy$, $\frac{1}{2}$ is the coefficient of xy, $\frac{1}{2}x$ is the coefficient of y, and $\frac{1}{2}y$ is the coefficient of x. Frequently, the numerical part of a term is called the coefficient of the term. For example, the coefficient of $343x^2$ is 343. Also, the coefficient of a is 1, since $a = 1a$. Have students differentiate between *term* and *factor*.

Sometimes a number appears more than once as a factor in a product. The product $s \cdot s$ is commonly written s^2. The term s^2 may be read: s squared or s-square. The small raised number is an **exponent** (ek-**spo**-nent). It shows that s, which is called the *base*, is to be used twice as a factor. The **base** is the expression used as a factor one or more times (as indicated by the exponent).



To compare an exponent with a coefficient, compare s^2 and $2s$ when you replace s by 15.

$$\begin{aligned} s^2 &= s \cdot s \\ s^2 &= 15 \cdot 15 \\ s^2 &= 225 \end{aligned}$$

$$\begin{aligned} 2s &= 2 \cdot s \\ 2s &= 2 \cdot 15 \\ 2s &= 30 \end{aligned}$$

An exponent tells how many times another number, called the base, is to be used as a factor.

A coefficient is a factor.

Do not permit students to say that the exponent shows the number of times a base is multiplied by itself. See T.M. pg. 9 (2).

A number which can be expressed by means of a base and exponent is called a **power**. The exponent 1, which is seldom written, means that the base is used only once; therefore, x^1 — the first power of x — is the same as x. Here are some other powers of x:

Third power: $x^3 = x \cdot x \cdot x$ (read x cubed or x-cube)

Fourth power: $x^4 = x \cdot x \cdot x \cdot x$ (read x fourth or x exponent 4)

Fifth power: $x^5 = x \cdot x \cdot x \cdot x \cdot x$ (read x fifth or x exponent 5)

In an expression such as $3a^2$, the 2 is the exponent of the base a. In an expression such as $(3a)^2$, the 2 is the exponent of the base $3a$, because you enclosed the expression in a symbol of inclusion. Compare the examples that follow:

$rs^3 = r \cdot s \cdot s \cdot s$

$(rs)^3 = rs \cdot rs \cdot rs$

$4 \cdot 5^3 = 4 \cdot 5 \cdot 5 \cdot 5 = 500$

$(4 \cdot 5)^3 = 20 \cdot 20 \cdot 20 = 8000$

$5 - n^2 = 5 - (n \cdot n)$

$(5 - n)^2 = (5 - n)(5 - n)$

$15 - 3^2 = 15 - 9 = 6$

$(15 - 3)^2 = (12)^2 = 144$

ORAL EXERCISES

Read each of the following expressions as a product.

SAMPLE 1. $7(y + 3)$ *What you say:* 7 times the sum y plus 3.

SAMPLE 2. $\frac{a}{4}$ *What you say:* One-fourth times a.

1. $2y$ $2 \times y$

2. $3x$ $3 \times x$

3. $11n$ $11 \times n$

4. $\frac{1}{2}z$ $\frac{1}{2} \times z$

5. $\frac{1}{3}r$ $\frac{1}{3} \times r$

6. $.3b$ $.3 \times b$

7. cd $c \times d$

8. $2(x + 5)$ 2 × the sum $x + 5$

9. $3(m - 6)$ 3 × the difference $m - 6$

10. $\frac{1}{2}y$ $\frac{1}{2} \times y$

11. $.7ab$ $.7 \times a \times b$

12. $1.5qr$ $1.5 \times q \times r$

13. $\frac{3y}{5}$ (two ways) $\frac{3}{5} \times y$; 3 × the quotient $y \div 5$

14. $\frac{2f}{17}$ (two ways) $\frac{2}{17} \times f$; 2 × the quotient $f \div 17$

In Exercises 15–22, name at least two factors of the given terms.

15. 14 1, 2, 7, 14

16. 15 1, 3, 5, 15

17. 12 1, 2, 3, 4, 6, 12

18. 7 1, 7

19. ab^2 a, b, ab, b^2, ab^2

20. $2c^2d^2$ $1, 2, c, d, c^2, d^2, 2c, 2d, 2c^2, 2d^2, cd, c^2d, cd^2, c^2d^2, 2cd, 2c^2d, 2cd^2, 2c^2d^2$

21. $\frac{ab^2c^3}{2}$ some factors: $a, b, c, ab, ac, bc, b^2, c^2, c^3, ab^2c^3, \frac{1}{2}, abc$

22. $\frac{rs^4}{3}$ some factors: $\frac{1}{3}, r, s, rs, s^2, s^3, s^4, rs^2, rs^3, rs^4$

In Exercises 23–34, name the coefficient of z.

23. $4z$ 4

24. $19z$ 19

25. $\frac{1}{8}z$ $\frac{1}{8}$

26. $\frac{1}{2}z$ $\frac{1}{2}$

27. z 1

28. xyz xy

29. $z(3 + 1)$ 4

30. $(a + b)z$ $a+b$

31. $.7z$ .7

32. $1.4z$ 1.4

33. $2z + 5$ 2

34. $a + bz$ b

In Exercises 35–44, name the numerical coefficient, the base, and the exponent.
In ex. 35-44 answers will be listed in the order: numerical coefficient, base, exponent

SAMPLE. $9(u+6)^2$ *What you say:* The coefficient is 9; the base is $(u+6)$; the exponent is 2.

35. $2z^2$ 2; z; 2
36. $4y^3$ 4; y; 3
37. x^7 1; x; 7
38. w^6 1; w; 6
39. $3(V+2)^3$ 3; (V+2); 3
40. $4(u-3)^2$ 4; (u−3); 2
41. $(a+b)^3$ 1; (a+b); 3
42. $(a-b)^4$ 1; (a−b); 4
43. $5t$ 5; t; 1
44. $17s$ 17; s; 1

Tell the meaning of each of the following terms; then give its value.
Answers to ex. 45-56 will be given in the order: meaning, value.

SAMPLE. $y^2;\ y=9.$
With these exercises you begin very simply to translate from algebra to English.

What you say: y^2 means y times y; when $y=9$, $y^2=9\cdot 9=81$.

45. $k^2;\ k=5$ K×K; 25
46. $n^3;\ n=10$ n×n×n; 1000
47. $a^1;\ a=176$ a; 176
48. $u^4;\ u=2$ u×u×u×u; 16
49. $(3x)^3;\ x=2$ 3x×3x×3x; 216
50. $(2y)^2;\ y=3$ 2y×2y; 36
51. $5h^2;\ h=10$ 5×h×h; 500
52. $2J^2;\ J=5$ 2×J×J; 50
53. $(a+2)^2;\ a=3$ (a+2)×(a+2); 25
54. $(b-7)^3;\ b=9$ (b−7)×(b- ×(b−7); 8
55. $(m-9)^2;\ m=13$ (m−9×(m- 16
56. $(5+Q)^2;\ Q=4$ (5+Q)×(5+Q); 81

WRITTEN EXERCISES

Rewrite each of the following expressions in a shorter form.

A

1. $b\cdot b$
2. $c\cdot c\cdot c$
3. a cubed
4. d squared
5. $7\cdot n\cdot n\cdot n$
6. $14\cdot m\cdot m\cdot m$
7. nnn
8. $8nnn$
9. E fourth
10. F cubed
11. R used as a factor 5 times
12. s used as a factor 6 times

Here you translate from English to algebra.

13. Five times the cube of y
14. Eight times the square of z
15. One-half the second power of g
16. One-fourth the fifth power of h
17. The square of $2P$
18. The square of $8i$
19. The cube of xy
20. The cube of ab
21. The cube of $(a-1)$
22. The cube of $(1-a)$
23. The cube of the sum r plus 2
24. The square of the sum t plus 7

Find the value of each of the following expressions.

B

25. $m^2;\ m=\frac{1}{2}$
26. $n^2;\ n=\frac{1}{4}$
27. $4p^2;\ p=3$
28. $8r^2;\ r=5$
29. $(9x)^2;\ x=\frac{1}{3}$
30. $(8y)^2;\ y=\frac{1}{4}$

31. $2x^2 + 4x + 5; x = 3$

32. $5y^2 - 3y + 4; y = 1$

33. $7z^3 + z^2 - z; z = 2$

34. $a^3 - 2a^2 + a + 4; a = 5$

35. $v^5 + 3v^4 - v^3 + v; v = 0$

36. $w^{10} + w^5 + w + 9; w = 0$

Let $x = 5$, $y = 2$, and $z = 3$, and evaluate the following expressions.

C **37.** $x^2 + y^2 + z^2$

38. $x - y^2 + z^2$

39. $x^2 + y + z^2$

40. $x - y + z^2$

41. $x^2 + y^2 - 2z^2$

42. $x^2 + y - z^2$

43. $x^2 - y^2 + z^2$

44. $\dfrac{2x^2 + xy}{20z}$

45. $\dfrac{z^3 - 27}{xy}$

46. $\dfrac{x^2 + z^2}{y^2}$

47. $\dfrac{7y^2z - 2x^2}{xy}$

48. $(xz)^3 + y^6$

49. $\left(\dfrac{6y}{z}\right)^3 + 3x^2$

50. $\dfrac{(x - z)^4 - y^4}{x - y^2}$

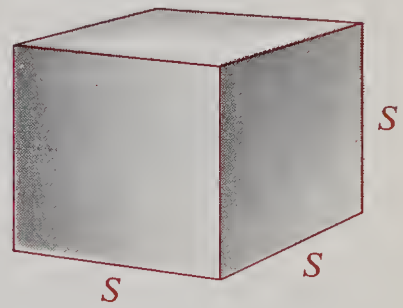
51. $\dfrac{(2y - z)^3 + y^3}{(2x - z + y)^3}$

These problems show more applications of algebra in science and industry.

PROBLEMS

Evaluate each expression.

Ex. 2



A **1.** Area of a square: s^2; let $s = 15$ cm.

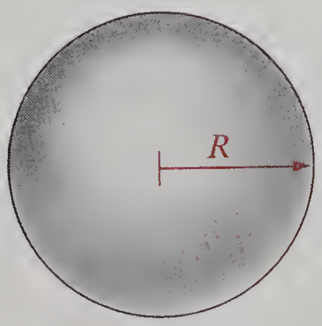
2. Volume of a cube: s^3; let $s = 15$ cm.

3. Conversion of mass to energy: mc^2; let $c = 300{,}000{,}000$ meters per second and $m = 254$ g.

4. Electrical power: I^2R; let $I = 15$ amp., $R = .01$ ohm.

5. Distance traveled during acceleration: $\dfrac{at^2}{2}$; let $a = 13.7$ ft./sec./sec., $t = 25$ sec.

Ex. 6



B **6.** Volume of a sphere: $\dfrac{4\pi R^3}{3}$; let $R = 24''$, $\pi = 3.14$.

7. Heat radiation: kT^4; let $k = .000000822$, $T = 6000°$K.

8. Kinetic energy: $\dfrac{mv^2}{2}$; let $m = 25$ g., $v = 100$ cm./sec.

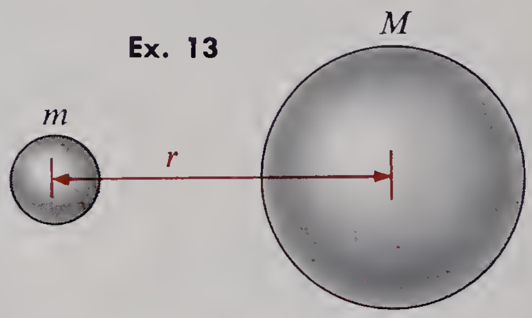
9. Volume of a circular cylinder: $\pi r^2 h$; let $\pi = 3.14$, $r = 1.25''$, $h = 12.0''$.

10. Illumination: $\dfrac{C}{D^2}$; let $C = 300$ candle power, $D = 500$ feet.

11. Centripetal force: $\dfrac{mv^2}{r}$; let $m = 15$ lb., $v = 20$ ft./sec., $r = 10$ ft.

Ⓒ **12.** Resistance of an electrical conductor: $\dfrac{kl}{d^2}$; let $k = 10.37$, $l = 200$ ft., $d = 25$ mil.

13. Law of gravitation: $\dfrac{GmM}{r^2}$; let
$G = .0000000667$, $m = 100{,}000$ g.,
$M = 900{,}000$ g., $r = 1000$ cm.


Ex. 13

14. Heat energy from electricity: $0.238I^2Rt$; let $I = 20$ amp., $R = 10$ ohms, $t = 300$ sec.

15. Length of a pendulum: $g\left(\dfrac{t}{2\pi}\right)^2$; let $g = 32.2$ ft./sec./sec., $t = .50$ sec.

2–3 Solving Open Sentences

See T.M. pg. 9 (3).

Consider this sentence:

w is a city in Texas.

This sentence, as written, is neither true nor false. Suppose the replacement set of *w* is the cities of the U.S.A. Replacing *w* by *Dallas* produces a true statement. Putting *New York* in place of *w* leads to a false statement. This sentence becomes true or false as the variable is replaced by one of the values from its replacement set. In general, a sentence containing a variable may be neither true nor false, as the value of the variable is left open. Consequently, a sentence containing a variable is called an **open sentence**. The open sentence serves as a pattern for the various sentences, some true, some false, which you obtain by substituting in it the different values of the variable.

An **algebraic sentence** is a statement composed of algebraic expressions related by one of the symbols $=$, $\neq$, $>$, $<$, $\leq$, or $\geq$. ($\leq$ is read is less than or equal to; $\geq$ is read is greater than or equal to.) These symbols of relationship are equivalent to the verbs in a sentence.

Any sentence using the symbol $=$ is called an **equation.** Consider the equation

$$\underset{\text{Left member}}{\mathbf{3x + 1}} \quad = \quad \underset{\text{Right member}}{\mathbf{16}}$$

The equation states that the left member (L.M.) expression, $3x + 1$, and the right member (R.M.) expression, 16, designate the same number. If $\{5, 6\}$ is the replacement set for x, you can determine the values of x which make the given equation a true statement, as follows:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
3x + 1 &=& 16 & 3x + 1 &=& 16 \\
3 \cdot 5 + 1 &=& 16 & 3 \cdot 6 + 1 &=& 16 \\
15 + 1 &=& 16 & 18 + 1 &=& 16 \\
16 &=& 16, \text{ True} & 19 &=& 16, \text{ False}
\end{array}$$

The subset of the domain of the variable consisting of the elements of the domain which make the open sentence true is called the **solution set** of the open sentence. The solution set of $3x + 1 = 16$ is $\{5\}$. The process of determining the solution set is called **solving** the open sentence. Each member of the solution set is called a **root** of the open sentence; thus 5 is a root of $3x + 1 = 16$. "Truth set" is another name for "solution set."

Consider the open sentence $3x + 1 > 16$, and let $x \in \{5, 6\}$. It is called an **inequality** because it uses one of the symbols $\neq$, $>$, $<$, $\geq$, $\leq$ to relate the left member expression and the right member expression.

To solve the inequality, you may proceed as follows: See T.M. pg. 9 (4) for *identity vs. conditional equation.*

$$\begin{array}{rcl@{\qquad\qquad}rcl}
3x + 1 &>& 16 & 3x + 1 &>& 16 \\
3 \cdot 5 + 1 &>& 16 & 3 \cdot 6 + 1 &>& 16 \\
15 + 1 &>& 16 & 18 + 1 &>& 16 \\
16 &>& 16, \text{ False} & 19 &>& 16, \text{ True}
\end{array}$$

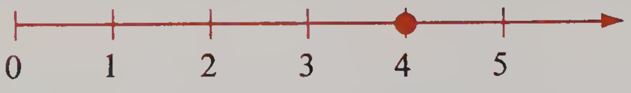
$\therefore$ (Read "therefore") The solution set is $\{6\}$.

The solution set of an equation or of an inequality may be shown in rule form, roster form, or by graph. The graph of the solution set of an open sentence is called the graph of the sentence.

EXAMPLE 1. $3t + 2 = 14;\ t \in \{3, 4\}$

Solution:

$$3t + 2 = 14 \qquad 3t + 2 = 14$$
$$3 \cdot 3 + 2 = 14 \qquad 3 \cdot 4 + 2 = 14$$
$$11 = 14, \text{ False} \qquad 14 = 14, \text{ True}$$



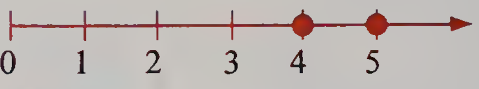
∴ The solution set is $\{4\}$, Answer.

EXAMPLE 2. $2 < x \le 5;\ x \in \{4, 5, 8\}$

Solution:

$$2 < x \le 5$$
$$2 < 4 \le 5, \text{ True}$$
$$2 < 5 \le 5, \text{ True}$$
$$2 < 8 \le 5, \text{ False}$$

∴ The solution set is $\{4, 5\}$, Answer.



Point out that in both cases, each w is replaced by the same numeral. In general, when any variable is replaced by a numeral, the same replacement is made wherever that variable occurs.

ORAL EXERCISES

In Exercises 1–16, replace the variable by names of elements of the given replacement set, and tell whether the resulting sentences are true.

SAMPLE 1. $2w - 3 < w + 1;\ w \in \{5, 4\}$

What you say: $2 \cdot 5 - 3 < 5 + 1$, False

$2 \cdot 4 - 3 < 4 + 1$, False

1. $t - 7 = 6;\ t \in \{9, 13\}$

$9 - 7 = 6$, F; $13 - 7 = 6$, T

2. $10 = n + 5;\ n \in \{5, 2\}$

$10 = 5 + 5$, T; $10 = 2 + 5$, F

3. $2y + 7 = 17;\ y \in \{0, 4\}$

$2 \cdot 0 + 7 = 17$, F; $2 \cdot 4 + 7 = 17$, F

4. $9 \ne 3q - 6;\ q \in \{5, 8\}$

4. $9 \ne 3 \cdot 5 - 6$, F; $9 \ne 3 \cdot 8 - 6$, T

5. $d + 3 > 5;\ d \in \{3, 4\}$

$3 + 3 > 5$, T; $4 + 3 > 5$, T

6. $6 < f - 1;\ f \in \{10, 9\}$

$6 < 10 - 1$, T; $6 < 9 - 1$, T

7. $2k = 3 + k;\ k \in \{2, 4\}$

$2 \cdot 2 = 3 + 2$, F; $2 \cdot 4 = 3 + 4$, F

8. $s - 2 = s \div 2;\ s \in \{8, 2\}$

$8 - 2 = 8 \div 2$, F; $2 - 2 = 2 \div 2$ F

9. $m \div 2 > m + 4;\ m \in \{2, 0\}$

$2 \div 2 > 2 + 4$, F; $0 \div 2 > 0 + 4$, F

10. $18 \div b < 18 + b;\ b \in \{9, \frac{1}{2}\}$

$18 \div 9 < 18 + 9$, T; $18 \div \frac{1}{2} < 18 + \frac{1}{2}$, F

11. $n + 5 < 100;\ n \in \{2, 4, 6, \ldots, 100\}$

$n + 5 < 100$, $n \in \{2, 4, 6 \ldots, 94\}$, T; $n + 5 < 100$, $n \in \{96, 98, 100\}$, F

12. $4m > 6;\ m \in \{1, 2, 3, 4, \ldots, 10\}$

$4m > 6$, $m = 1$, F; $4m > 6$, $m \in \{2, 3, 4, \ldots, 10\}$, T

13. $\frac{1}{2}u + 1 > \frac{1}{2}(u + 2);\ u \in \{2, 4, 6, 8\}$

$\frac{1}{2}u + 1 > \frac{1}{2}(u + 2)$; $u \in \{2, 4, 6, 8\}$, F

14. $\dfrac{s}{2} - 1 < 10;\ s \in \{2, 4, 6, \ldots, 20\}$

$\frac{s}{2} - 1 < 10$; $s \in \{2, 4, 6, \cdots, 20\}$, T

15. $2(a + 3) = 2 \times a + 3;\ a \in \{1, 5\}$

$2(1 + 3) = 2 \times 1 + 3$, F; $2(5 + 3) = 2 \times 5 + 3$, F

16. $2(b + 3) = 2 \times b + 2 \times 3;\ b \in \{1, 5\}$

$2(1 + 3) = 2 \times 1 + 2 \times 3$, T; $2(5 + 3) = 2 \times 5 + 2 \times 3$, T

In Exercises 17–30, determine the solution set from the replacement set of {the numbers of arithmetic}. See T.M. pg. 9 (5) for an example of "guessing" solution sets.

SAMPLE 2. $1 < x \le 2$ *What you say:* The solution set is {the numbers between 1 and 2, including 2}.

SAMPLE 3. $y + 3 = y + 2$ *What you say:* The solution set is the empty set.

17. $x + 4 = 9$ {5}
18. $z - 3 \ne 7$ {the nos. of arith. except 10}
19. $v + 5 \ne 11$ {the nos. of arith. except 6}
20. $r - 2 = 10$ {12}
21. $m > 6$ {the nos. of arith. > 6}
22. $0 < d < 4$ {the nos. of arith. betw. 0 and 4}
23. $v + 21 = 32$ {11}
24. $29 - p = 16$ {13}
25. $4x \ne 20$ {the nos. of arith. except 5}
26. $7y = 46$ $\{\frac{46}{7}\}$
27. $0 \le x \le 2$ {the nos. of arith. between 0 and 2 inclusive}
28. $3 \le r \le 5$ {the nos. of arith. betw. 3 and 5 inclusive}
29. $\frac{1}{2}r > 8$ {the nos. of arith. > 16}
30. $3t > 9$ {the nos. of arith. > 3}

WRITTEN EXERCISES

In Exercises 1–10, substitute members of the given replacement set in the open sentences, tell whether the resulting sentences are true, and give each solution set.

SAMPLE 1. $1 < x < 7;\ x \in \{0, 2, 4, 6, 8\}$

Solution:
$1 < 0 < 7$, False
$1 < 2 < 7$, True
$1 < 4 < 7$, True
$1 < 6 < 7$, True
$1 < 8 < 7$, False
$\therefore$ The solution set is $\{2, 4, 6\}$.

A

1. $x + 1 = 5;\ x \in \{1, 2, 3, 4, 5\}$
2. $x > 3;\ x \in \{0, 2, 4, 6, 8\}$
3. $y < 1;\ y \in \{0, 1, 2, 3, 4\}$
4. $2z \ne 8;\ z \in \{2, 4, 6, 8\}$
5. $2x \ne 0;\ x \in \{0, 2, 4, 6\}$
6. $2y + 1 = 5;\ y \in \{0, 2, 4, 6\}$
7. $2z < 20;\ z \in \{8, 9, 10, 11\}$
8. $3y > 25;\ y \in \{7, 8, 9, 10\}$
9. $x + (y + 2) = (x + y) + 2$
  a. $x \in \{10\},\ y \in \{7\}$
  b. $x \in \{2\},\ y \in \{0\}$
10. $x - (y + 1) > (x - y) + 1$
  a. $x \in \{10\},\ y \in \{6\}$
  b. $x \in \{5\},\ y \in \{3\}$

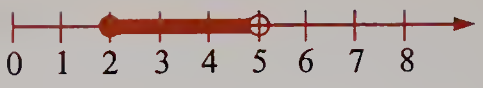
Determine the solution set from {numbers of arithmetic}. Give the solution set of each equation in roster form. Graph the solution set of each inequality.

SAMPLE 2. $3t + 1 = 10$

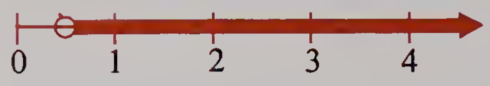
Solution: $3 \cdot 3 + 1 = 10$, $\therefore \{3\}$, **Answer.**

Be sure to assign at least one exercise in which the solution set is ϕ (Ex. 23, 24, 33) and at least one in which the solution set is {nos. of arithmetic} (Ex. 26, 27, 28).

SAMPLE 3. $2 \leq x < 5$ *Solution:*



SAMPLE 4. $2n > 1$ *Solution:*



11. $t + 2 = 5$

12. $5 - z = 4$

13. $m + 40 = 51$

14. $2 = 5 - t$

15. $x = 0$

16. $14 - s = 14$

17. $k \leq 5$

18. $x \leq 6$

19. $1 < x < 3$

20. $7 \leq y \leq 9$

21. $2 \leq r < 7$

22. $5 < n \leq 8$

23. $z + 1 = z + 4$

24. $y - 4 = y - 5$

25. $2 > m$

26. $x \geq 0$

27. $1 + x = x + 1$

28. $y + y = 2y$

29. $2t + 7 = 23$

30. $3k + 11 = 26$

31. $x + 1 > 5$

32. $y - 1 < 2$

33. $4 < a < 4$

34. $0 \leq k \leq 0$

35. $3w - 1 = 20$

36. $1 = 4s - 3$

37. $3s + 7 \neq 10$

38. $v \neq 3v$

39. $2m + 1 > 5$

40. $3k - 1 \leq 8$

In Exercises 41–56, substitute the members of the given replacement set in the open sentences, and tell whether the resulting sentences are true.

These exercises prepare students for the discovery of the properties of numbers.

SAMPLE 5. $5t + 9v = 14tv;\ t \in \{1, 2\},\ v \in \{4, 5\}$

Solution:

a. $5 \cdot 1 + 9 \cdot 4 = 14 \cdot 1 \cdot 4$, False

b. $5 \cdot 1 + 9 \cdot 5 = 14 \cdot 1 \cdot 5$, False

c. $5 \cdot 2 + 9 \cdot 4 = 14 \cdot 2 \cdot 4$, False

d. $5 \cdot 2 + 9 \cdot 5 = 14 \cdot 2 \cdot 5$, False

The idea of a counter-example to show an algebraic error is suggested in Ex. 46, 47, 49.

Ⓑ **41.** $9(m + t) = 9m + 9t;\ m \in \{3, 5\},\ t \in \{2, 4\}$

42. $12(cd) = (12c)d;\ c \in \{1, 3\},\ d \in \{9, 11\}$

43. $(r + 1)(k + 3) > 3rk;\ k \in \{10, 21\},\ r \in \{2, 4\}$

44. $12d + 8 < 20d + 8e;\ d \in \{4, 7, 0\},\ e \in \{0, 1\}$

45. $(t)(u + v) = tu + tv;\ t \in \{1, 2\},\ u \in \{1, 2\},\ v \in \{1, 2\}$

46. $x + yz = (x + y)z;\ x \in \{0, 2\},\ y \in \{3\},\ z \in \{5, 3\}$

47. $5t + 9v = 14tv;\ t \in \{2, 1\},\ v \in \{3, 1\}$

48. $12d + 8d < 20d;\ d \in \{4, 7\}$

49. $(9 \div m) \div n = 9 \div (m \div n);\ m \in \{3, 9\},\ n \in \{3, 1\}$

50. $10 + 8y - 2 < 2y + 12 + 4y;\ y \in \{2, 1\}$

Ⓒ **51.** $10 - t \div s + 2 > 8 \div t + 8(2 \div s)$; $t \in \{4, 8\}$, $s \in \{2, 4\}$

52. $\frac{dd - hh}{d + h} = d - h$; $h \in \{4, 8\}$, $d \in \{6, 8\}$

53. $7r + 5s + 2t > 12r + t + t$

a. 3 for r; 3 for s; 0 for t **b.** 4 for r; 1 for s; 2 for t

54. $(5 + x) + 7(2 + y) + (r + y) = 19 + x + 8y + r$

a. 2 for x; 3 for y; 5 for r **b.** 5 for x; 2 for y; 3 for r

55. $k(m - n) \neq km - n$

a. 5 for k; 12 for m; 4 for n **b.** 1 for k; 6 for m; 0 for n

56. $\frac{hk + m}{h} = k + m$

a. 15 for k; 5 for m; 4 for h **b.** 12 for k; 2 for m; 1 for h

The first step toward facility in solving word problems is taken by translating from algebra to English. See T.M. pg. 10 (6).

PROBLEMS SOLVED WITH VARIABLES

2–4 Thinking with Variables: From Symbols to Words

Can you find an English phrase that is represented by the algebraic expression

$$x + 1?$$

You may say: "Ron is a year older than his sister Sue. If x represents Sue's age in years, then $x + 1$ represents Ron's age."

Another interpretation: "Let x represent a number; then $x + 1$ represents that number increased by 1 or it represents the sum of x and 1 or it is the number one more than x."

Why may the variable x, in the second interpretation, represent any number in the set of numbers of arithmetic, while in the first case x may not be 0?

Be sure that pupils understand that the variable always stands for a number, as: x represents a *number* of years and w, a *number* of feet. Do not permit students to say "x is Sue" or "w is the width."

The expression

$$2w - 3,$$

that is, the difference between twice a number w and 3, could arise in the following way. The length of a room is 3 feet less than twice its width. Hence, if w represents the number of feet in the width of the room, $\mathbf{2w - 3}$ represents **the number of feet in its length**. Since $2w - 3$ must be greater than 0, $2w$ must be greater than 3. Thus, w must represent a number greater than $1\frac{1}{2}$. Why?

Encourage students to devise a wide variety of interpretations for each expression. Have different pupils tell the class their interpretations.

ORAL EXERCISES

Note: Not all the problems in the following exercise can be worked because of space limitations.

Find two interpretations for each of the following algebraic expressions.

1. $2x$

2. $10y$

3. $z + 3$

4. $n - 1$

5. $\frac{g}{3}$

6. $\frac{1}{2}h$

7. $d + 5$

8. $u - 1.7$

9. $2p + 1$

10. $2q - 1$

11. $\frac{1}{2}s + \frac{1}{2}$

12. $\frac{1}{3}r + \frac{1}{3}$

13. $3m - 2$

14. $5n + 6$

15. $\frac{u + 1}{3}$

16. $\frac{v - 1}{4}$

17. $2t + 7$

18. $5w - 3$

19. $9s + 42s$

20. $30r - 19r$

a. Ron is twice as old as his sister Sue. b. $2x$ represents a no. twice as large as a no. represented by x.

a. The length of a rectangle is 10 times its width. b. $10y$ represents a no. 10 times as large as another no. represented by y.

a. If g represents a no., then $\frac{g}{3}$ represents a no. $\frac{1}{3}$ as large. b. Paul caught $\frac{1}{3}$ as many fish as Jerry.

3. a. Jane is 3 yr older than Mary. b. The length of a room is 3 ft more than its width.

4. a. $n - 1$ represents 1 less than the no. represented by n. b. James can walk at a rate wich is 1 mph less than Bill's rate.

WRITTEN EXERCISES

Give two interpretations for each of the following algebraic expressions. In each case identify the replacement set of the variable.

SAMPLE. $\frac{v}{2} - 5$

Solution: 1. Half a given number v decreased by 5, with $v \geq 10$.

2. Mike and Bill belong to different boys' clubs. Bill's club has 5 fewer than half as many members as Mike's club has. Let v represent the number of boys in Mike's club. Then $\frac{v}{2} - 5$ represents the number in Bill's club. Here, the value of v can be any *even* whole number exceeding 10.

A

1. $7x$

2. $9y$

3. $x + 15$

4. $y - 10$

5. $2a + 3$

6. $3b - 7$

7. $4(x + 1)$

8. $5(y - 2)$

9. $\frac{x}{4} - 2$

10. $\frac{1}{3}n - 3$

11. $2(2v + 1)$

12. $u(3u - 1)$

13. $z + (2z + 5)$

14. $m + 2(m + 5)$

Ⓑ **15.** $\left(\frac{s}{5} - 3\right) + s$

16. $r - \left(\frac{r}{2} - 3\right)$

17. $x + (100 - x)$

18. $n + (n + 1)$

19. $s^2 - 4$

20. $4t^2 - 1$

21. $w - w^2$

22. $t^2 - t$

23. $8 - x^3$

24. $\frac{y^3}{8} - 1$

The experience of the preceding section should clarify the translation process. If a student translates incorrectly ask him or another to translate the proposed algebraic expression back into English.

2–5 Thinking with Variables: From Words to Symbols

In solving a problem your initial job will be to translate English phrases into the language of algebra. Consider the following examples:

EXAMPLE 1. **If whole milk costs 5 cents more per quart than skimmed milk, express the cost of a quart of whole milk in terms of the cost of a quart of skimmed milk.**

Solution: **Since the cost in cents of skimmed milk could be any number greater than 0 (the milkman is not giving it away), use a variable to represent the cost:**

let m = the cost in cents of 1 qt. of skimmed milk

$\therefore m + 5$ = the cost in cents of 1 qt. of whole milk

Here you are employing the symbol = in place of the word *represent* or *represents*.

Emphasize the need for stating exactly what number the variable represents. See 2-5 T.M. pg. 10.

EXAMPLE 2. **Ken has 4 fewer than one-third as many stamps as Len has. Write an algebraic expression for the number of Ken's stamps.**

Solution: **Let s = the number of Len's stamps.**

Then, $\frac{s}{3}$ = one-third the number of Len's stamps.

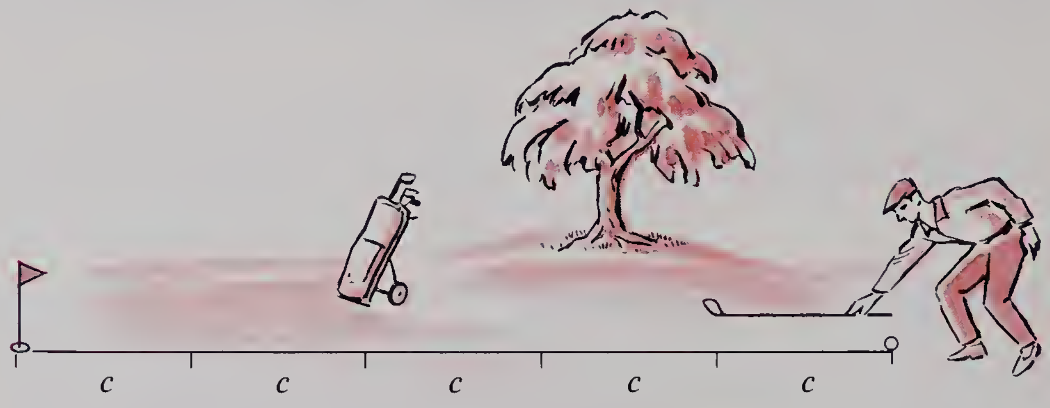
$\therefore \frac{s}{3} - 4$ = the number of Ken's stamps.

Here s is a variable whose value may be any of the multiples of 3 greater than or equal to 12. Why?

ORAL EXERCISES

Answer each question by giving an algebraic expression. In each case identify a suitable replacement set of the variable.

SAMPLE. After reaching the green, a golfer used his putter to measure the distance of his ball from the cup. He found the distance to be 5 club lengths. Using a variable, represent the number of feet from the ball to the cup.



What you say: Let c = the number of feet in 1 club length.

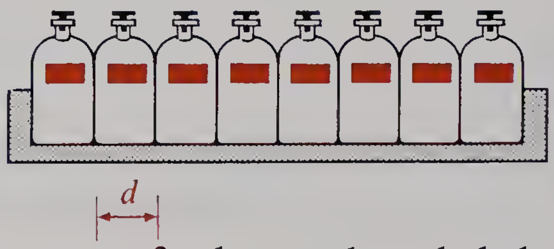
Then $5c$ = the number of feet in 5 club lengths.

$\therefore\ 5c$ = the number of feet from the ball to the cup.

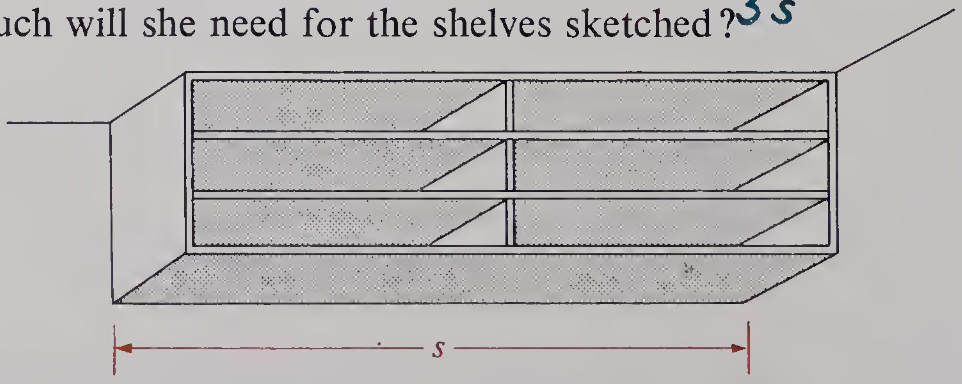
The replacement set of c is the set of the numbers of arithmetic appropriate for the length of a putter.

In ex. 1-12 the replacement set of the variable = {nos. of arith. appropriate for lengt of item being considered}.

1. A chemistry teacher wishes to have a shelf built which will hold exactly 8 reagent bottles. If each bottle is d inches in diameter, how long should the shelf be? 8 d

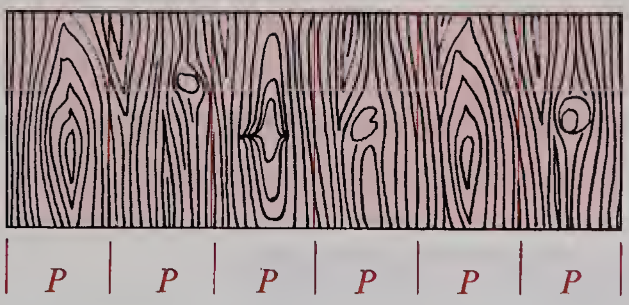


2. Mrs. Lund wishes to buy some fancy paper for her cupboard shelves. How much will she need for the shelves sketched? 3 s

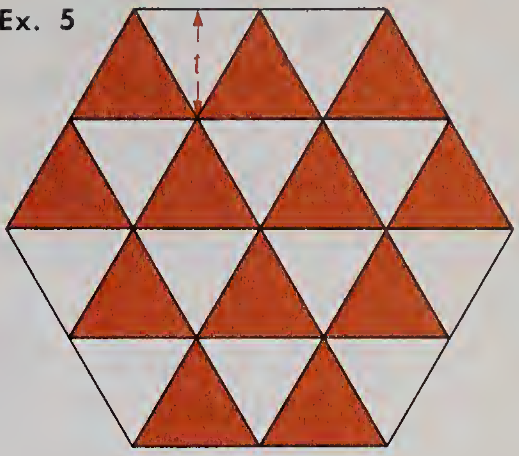


3. Each of the 13 stripes of a flag is w inches wide. What algebraic expression represents the hoist (width) of the entire flag? 13 w

Ex. 4



Ex. 5

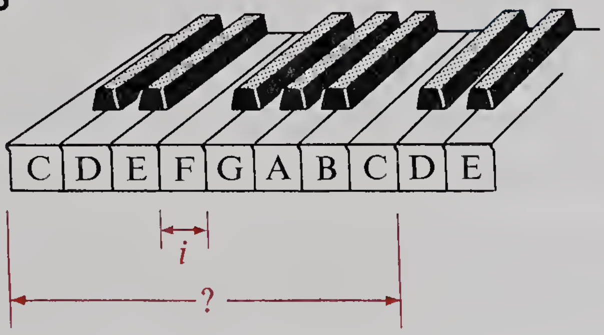


4. To panel one wall of a recreation room required 6 sections of paneling, each p inches wide. How long was the wall? $6p$

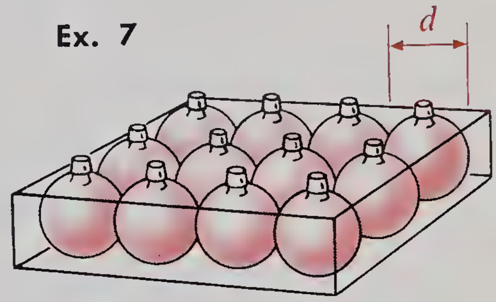
5. A hexagonal entry is floored with tile as shown. If each tile is t inches high, what is the width of the entry? $4t$

6. On a piano keyboard, a white key is i inches wide. How many inches are there in the width of an octave, such as the stretch of 8 white keys from C to C, inclusive? $8i$

Ex. 6



Ex. 7



7. A box holds 12 Christmas ornaments, each d inches in diameter. State the length, width, and height of the box. Length $= 4d$; width $= 3d$; height $= d$

8. One tree is half as tall as another. Let h be the height of the taller tree. Represent the height of the smaller tree. $\frac{1}{2}h$

9. The distance across a baseball diamond from first base to third base is 1.4 times the distance from home plate to third base, which is x feet. Represent the distance from first base to third base. $1.4x$

10. The linoleum floor in a kitchen is made up of squares each k inches on a side. If the floor has eight tiles on one side and twelve on the other, what are its dimensions? $8K \times 12K$

11. A farmer made his chicken lot one-fourth as wide as it was long. Let q be the number of feet in its length. Express the number of feet in its width in terms of q. $\frac{1}{4}q$

12. The height in inches of a certain triangle is 2 less than the number of inches in its base. Represent the number of inches in the base of this triangle; then, the number of inches in its height, using that variable. Let $b =$ no. of inches in base. Then $b-2 =$ no. of inches in height.

13. John owns k books, and Sue owns 1 more than twice as many. What algebraic expression stands for the number of Sue's books? $2K+1$; replacement set: $\{$whole nos. $\geq 0\}$

14. Joy is x years old. Tom is 1 year younger. What algebraic expression represents Tom's age? $x-1$; $\{$nos. of arith. $> 1\}$

$\frac{1}{10}d$; {nos. of arith., dependent on monthly income}

15. A man earns d dollars a month. He gives one-tenth of his earnings to charity. Represent the amount he gives per month.

16. In making pastry, the amount of fat used is one-third the amount of flour. Represent the amount of fat to be used with c cups of flour.

$\frac{1}{3}c$; {nos. of arith., dependent on pastry recipe}

WRITTEN EXERCISES

Each pupil should do 2 or 3 of the harder exercises, as well as several of the easier ones.

Translate into symbols.

A

1. 5 added to x
2. c added to d
3. $2x$ increased by 4
4. $5y$ increased by $2x$
5. 15 decreased by n
6. $3x$ decreased by 2
7. 1 less than z
8. d less than x
9. The difference between $2x$ and 5
10. The difference between $5k$ and 5
11. $3x$ times y
12. The product of 6 and a
13. The sum $r + 6$, divided by 3
14. One-third of the difference, $r - 6$
15. One-half of the sum, $x + y$
16. The difference $a - b$, divided by c
17. 5 times the sum of 2 and y
18. a times the difference between a and b
19. 5 more than the product of x and the sum $x + 2$
20. The difference between twice u and half of v

B

21. 5 divided by the sum of 4 times x and 3 times k
22. The sum of a and b, decreased by their product
23. 1 divided by twice the sum of a and b
24. The product of m and n divided by 3 times their difference
25. The quotient of three times the difference of r and s divided by twice their sum

No fewer than 8 or 10 of these problems should be assigned. The more of the that are handled successfully, the better. In at least a few cases, ask students for suitable domains for the variables.

PROBLEMS

Wherever a drawing will help you answer a question, make one.

A

1. A runway at a large airfield is twice as long as a runway at a smaller field. Draw lines to represent the two runways, and express the length of each runway in terms of the same variable.

2. Amy used some cotton material to make a blouse. She required a piece three times as long to make a dress. Draw lines to represent the two lengths of material. Label the lines in terms of the same variable.

3. The floor of a schoolroom is made of 90 boards, laid side by side. If w equals the number of inches in the width of one board, express the width of the room.

4. There are five steps at the entrance to the school building, each i inches in height. Use an algebraic expression to represent the height of the door sill above the street level.

5. Mr. Landon divided his rectangular piece of land into three lots, each w feet in width. Express the width of the land before it was divided.

6. Mr. Paxton bought a box of apples which contained three layers. Each layer contained n apples. How many apples did he buy?

7. In a ten-year period, the total number of miles of airmail routes in the United States was tripled. Represent the number of miles in the routes at the beginning and at the end of the period.

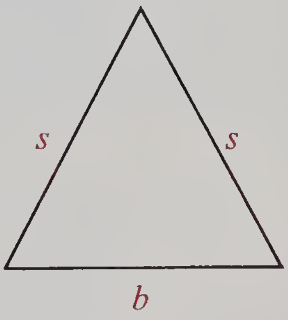
8. There are twelve rows of seats in a high-school stadium. Each row is higher than the row below it by the distance the bottom row is above the ground. If a first row seat is s inches above the ground, how far above the ground is a seat in the tenth row?

9. During its second year of operation a company produced 400 units of a certain commodity. If its production had increased by q units during the second year, express its production during the first year.

10. A firm produced h units of a certain commodity during its first year. Production increased by 5000 units during each succeeding year; express the number of units produced during the third business year.

11. A tennis court is 6 feet longer than twice its width. Represent the number of feet in (a) the width, (b) the length.

12. Express the number that is 17 less than 3 times a given number.



B **13.** The base of a triangle with two equal sides is 4 feet less than the sum of the two equal sides. Represent the number of inches in (a) each of the equal sides, (b) the base, (c) the perimeter.

14. The sum of two numbers is 57. Let w represent the smaller number. Using these facts, find an algebraic expression for the larger number.

15. One number is $\frac{4}{5}$ of another number. Using s to represent the larger number, write an equation stating that the sum of the two is 108.

16. Represent the number that is 5 more than the sum of 3 times a certain number and 7 times that number.

17. A notebook costs 28 cents more than a pencil. Represent the cost of 5 pencils and 2 notebooks.

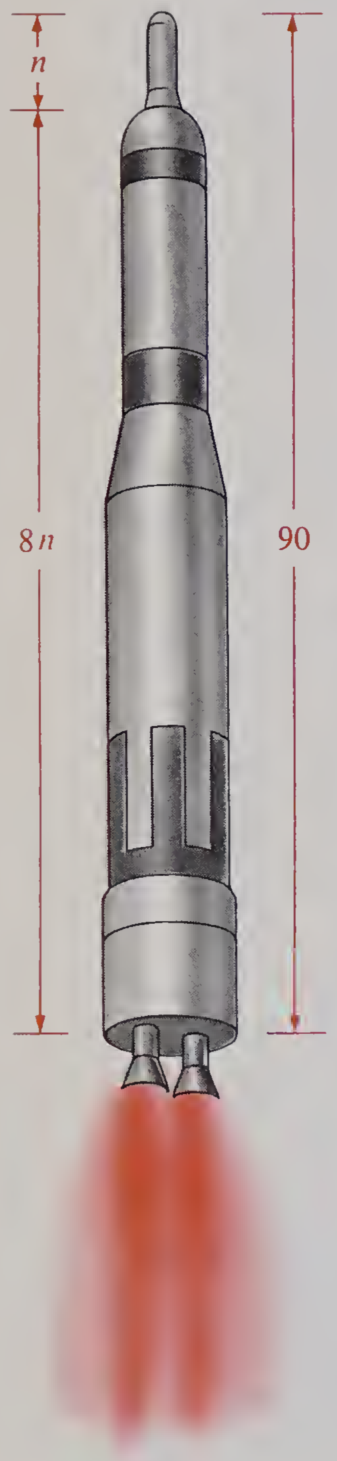
18. Bob is twice as old as Emma. Kent is 3 years older than Bob. If Emma is x years old, how old is (a) Bob? (b) Kent?

2–6 Solving Problems with Open Sentences

In a "word problem" you are told how numbers, some of which are described by word phrases, are related to one another. To solve the problem, you determine the described numbers so that the indicated relationships will be true. See T.M. pg. 10 (7) for a useful teaching device.

EXAMPLE 1. **The main body of the Air Force Titan missile in 1960 was eight times as long as the nose cone. The entire missile was 90 feet in length. How long was the nose cone?**

Solution: Used throughout the book, the four-step method of problem solving aims at an orderly but not mechanical solution.



1. **The first step in solving this problem is to choose a symbol to represent the number you want to find: the number of feet in the nose cone. Let *n* represent this number. You know that the main body of the missile is 8 times as long as the nose cone; so 8*n* represents the number of feet in the main body of the Titan. The only values admissible for *n* are the numbers of arithmetic, since *n* represents a measurement (number of feet).**

Here, you should emphasize Steps 1 and 2.

2. **The second step is to write an open sentence. You show the two expressions for the total length of the missile as:**

$$\underbrace{\text{length of cone}}_{n}\quad \underbrace{\text{added to}}_{+}\quad \underbrace{\text{length of body}}_{8n}\quad \underbrace{\text{equals}}_{=}\quad \underbrace{\text{entire length}}_{90}$$

or $$9n = 90$$

3. **The third step is to determine the solution set of the open sentence. The only value in the replacement set of *n* that produces a true statement for the sentence $9n = 90$ is 10. The possible length of the nose cone is, therefore, 10 feet.**

4. **The fourth step is to use the words of the problem to check your answer.**

Length of nose cone:	10 feet
Length of main body (8 times as much):	80 feet
Entire length:	90 feet ✓

∴ 10 feet, Answer.

EXAMPLE 2. **The number of boys on a special committee is supposed to be more than three times the number of girls. If 12 boys are put on the committee, show on the number line the possible values for the number of girls.**

Solution: Small, red arrows identify the four steps of problem solving.

1 Let x = the number of girls; $x \in$ {whole numbers}

$\therefore 3x$ = three times the number of girls

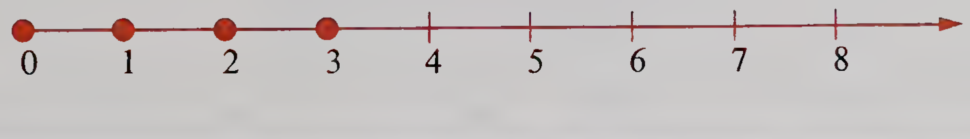
Have students distinguish between expressions like *greater than* which translates into + and *is greater than* which translates into >.

2 Form an open sentence.

$$\underbrace{\text{The number of boys}}_{12}\quad \underbrace{\text{is more than}}_{>}\quad \underbrace{\text{three times the number of girls}}_{3x}$$

3 Find the solution set. Since x must represent a whole number, the solution set is {0, 1, 2, 3}.

4 Since the largest value for the number of girls is 3, and since 12 is greater than 3 times 3, each element of the solution set does satisfy the requirements of the problem. Why is 4 not an element of the solution set? On the number line:



Answer.

Four possible steps in solving a problem:

1. Choose a variable with an appropriate replacement set, and use the variable in representing each described number.
2. Form an open sentence by using facts given in the problem.
3. Find the solution set of the open sentence.
4. Check your answer with the words of the problem.

Encourage students to follow these steps intelligently but not slavishly. Also encourage them to suggest alternative solutions, as no solution is the only solution.

ORAL EXERCISES

Give an open sentence to fit each of the following exercises.

SAMPLE.

$$\underbrace{\text{Multiply a number by 3,}}_{3n}\ \underbrace{\text{then add}}_{+}\ \underbrace{\text{8 to the product}}_{8}\ \underbrace{\text{and you get}}_{=}\ \underbrace{23.}_{23}$$

What you say: $3n + 8 = 23$

1. Multiply a number by 2, and you get 6. $2n = 6$

2. Multiply a number by 9, and you get 45. $9m = 45$

3. Double a number, and you get 52. $2x = 52$
4. Add 5 to a number, and you get 11. $y + 5 = 11$
5. Subtract 4 from a number, and you get 15. $z - 4 = 15$
6. Add 20 to a number, and you get 39. $t + 20 = 39$
7. Subtract 2 from a number, and you get 5. $p - 2 = 5$
8. Multiply a number by one, and you get 8. $r = 8$
9. Multiply a number by 3, then multiply this product by 2, and you get 30. $2(3s) =$ or $6s =$
10. Multiply a number by 5, then multiply the product by $\frac{1}{5}$, and you get 13. $\frac{1}{5}(5a) =$ or $a =$
11. Add 4 to a number, subtract 4 from the sum, and you get 39. $(K+4)-4=39$ or $K=3$
12. Add 7 to a number, then subtract 4 from the sum, and you get 5. $(u+7)-4=5$ or $u+3=5$
13. Multiply half a number by 2, and you get 9. $2(\frac{1}{2}v)=9$ or $v=9$
14. Double a number, add 2 to the product, and you get 15. $2w+2=15$

The emphasis now should be on writing the open sentence, rather than on finding the answer. You may wish to have students complete the solution in only a few problems.

PROBLEMS

Solve the following problems by the four-step method on page 57.

SAMPLE. Jim is 3 years more than twice as old as his sister June. If June is 6 years old, how old is Jim?

Solution:

1 Let x represent Jim's age; $x \in \{\text{whole numbers}\}$

2 $x = 2(6) + 3$

3 $x = 15$

4 15 equals 3 more than 2 times 6. ✓

$\therefore$ Jim's age is 15 years, **Answer.**

A

1. A baseball team won 3 times as many games as it lost. It won 84 games. How many games did it lose? (Let x represent the number lost.)
2. A class assessed each member 5 cents to buy flowers for an entertainment. The total was 170 cents. How many members were there?
3. Mr. Jonas got a roll of 50 pennies to use only for parking meters. If he used 5 pennies daily, how many days did the roll last?

4. The number of boys in a certain class is seven times the number of girls. The number of boys is 28. How many girls are in the class?

5. A house cost $18,200. It cost seven times as much as the lot on which it was built. What was the cost of the lot?

6. A man takes a position at a monthly salary. At the end of nine months he has earned $4050. What is his monthly salary?

7. Si's age is $\frac{1}{7}$ that of his aunt. If Si is 8 years old, how old is his aunt?

8. In a certain city $\frac{1}{3}$ of the girls are blonde. Find the number of girls in the city if 10,195 of them are blonde.

9. The perimeter of a square is 50 inches. How long is one side?

10. The area of a rectangle that is 4 feet wide is 68 square feet. Find the length of the rectangle.

11. After Jack deposited $55.25, his total bank balance was $1342.70. How much did Jack have in his account before that deposit?

12. After a $15.75 bank withdrawal Phil's balance was $672.39. How much did he have on deposit before making the withdrawal?

13. Jane's weight is 11 pounds more than normal for her height and age. If Jane weighs 109 pounds, what is the normal weight?

14. A thermometer reads 56 degrees. What is the temperature, if this reading is 3.7 degrees less than the true reading?

15. Fred earns $7.50 per week more than Bill. If Fred's weekly salary is $115, what does Bill earn per week?

16. Dave's golf score was 3 less than Mark's. If Dave scored 89, what was Mark's score?

B 17. A man traveled a certain number of miles by automobile, and then nine times as far by airplane. His total trip was 500 miles in length. How far did he travel by automobile?

18. The number of Central High School freshmen studying French is one-fourth the number studying Spanish. The total number of students enrolled in these languages is 150. How many freshmen elect Spanish?

19. A certain number was doubled. Then the product was multiplied by 3. If the result was 84, find the number.

20. After delivering his first dozen bottles of milk, a milkman had fewer than 75 bottles left. At most, how many bottles had he originally?

21. Sue owns 1 more than twice as many books as John. If Sue owns 59 books, how many books does John own?

22. The number of bolts produced daily by machine A is 600 less than four times the number produced by machine B. If machine A's output is 4800 bolts per day, what is B's daily output?

23. Mary's bowling score was 10 more than half Jay's score. If Mary bowled 100, find Jay's score.

24. If one-third of a certain number is diminished by 16, the result is 21. Find the number.

C

25. The length of a picture is 4 feet less than twice its width. To frame it, 76 feet of framing are needed. Find the dimensions of the picture.

26. Mr. Tripp completed a journey of 640 miles. The average speed of the jet plane taken by Mr. Tripp was 15 times that of the automobile he used to get to the airport. If he traveled an hour by auto and an hour by jet, how far did he travel by automobile?

27. Linda said: "I sold 3 more than twice the number of tickets Jo sold." Maria replied: "I sold 32 tickets, and that is more than you sold." What is the largest possible number of tickets Jo sold?

28. Helen weighs twice as much as her sister. If the sum of their weights is less than or equal to 165 pounds, what is the most that Helen may weigh?

29. There are three numbers such that the second is twice the first and the third is 1 less than three times the first. If the sum of the numbers is 35, find the largest number.

30. One side of a triangular lot is 13 feet less than 3 times the second. The third side is 18 feet more than the second. To fence the lot 130 feet of fencing are required. Find the length of each side of the lot.

Extra for Experts

The Arithmetic of Sets: Union

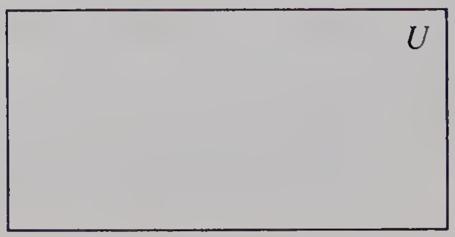
The **union** of two sets consists of all the elements of both sets, but no element is listed more than once. For example, if the union of the sets $A = \{2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$ is called set D, then:

$$\begin{array}{ccccc} A & \cup & B & = & D \\ \{2, 3, 4, 5\} & \cup & \{3, 4, 5, 6, 7\} & = & \{2, 3, 4, 5, 6, 7\} \end{array}$$

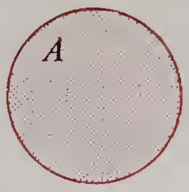
(∪ is read "cup.")

This example may be illustrated pictorially:

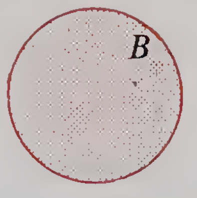
Universe: $U = \{\text{whole numbers}\}$:



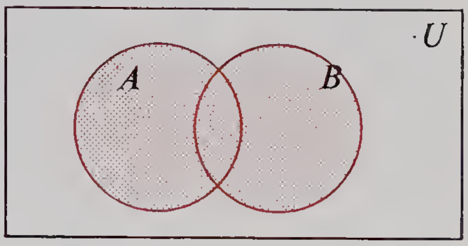
$A = \{2, 3, 4, 5\}$:



$B = \{3, 4, 5, 6, 7\}$:



$D = \quad A \cup B$:



Note that the union of two sets has each of the original sets as a subset; also, that the symbol for union is a stylized U.

Questions

1. If $A = \{1, 2, 3\}$ and $B = \{0, 2, 4, 3\}$, give the roster of $A \cup B$, and show the union pictorially.

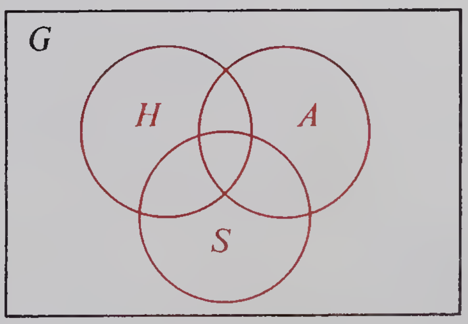
- **a.** What is the relationship between $A \cup B$ and A? and B? Why?
- **b.** Give a rule to determine when $A \cup B = B$ will hold. Illustrate.

2. Under what conditions would $A \cup B = A \cap B$?

- **a.** Define a set A and a set B which would satisfy this condition.
- **b.** For these sets, draw Venn diagrams of $A \cup B$ and $A \cap B$.

3. In the accompanying diagram, the universal set G = {all graduates of Newville High School}, H = {honor graduates}, A = {award winners}, and S = {scholarship recipients}. For each item, copy the diagram, and shade it to show the indicated subset.

- **a.** Honor graduates
- **b.** Award-winning honor graduates
- **c.** Award-winning honor graduates who received scholarships
- **d.** Award-winning graduates and honor graduates



- **e.** Graduates receiving no honors, awards, or scholarships

THE HUMAN EQUATION

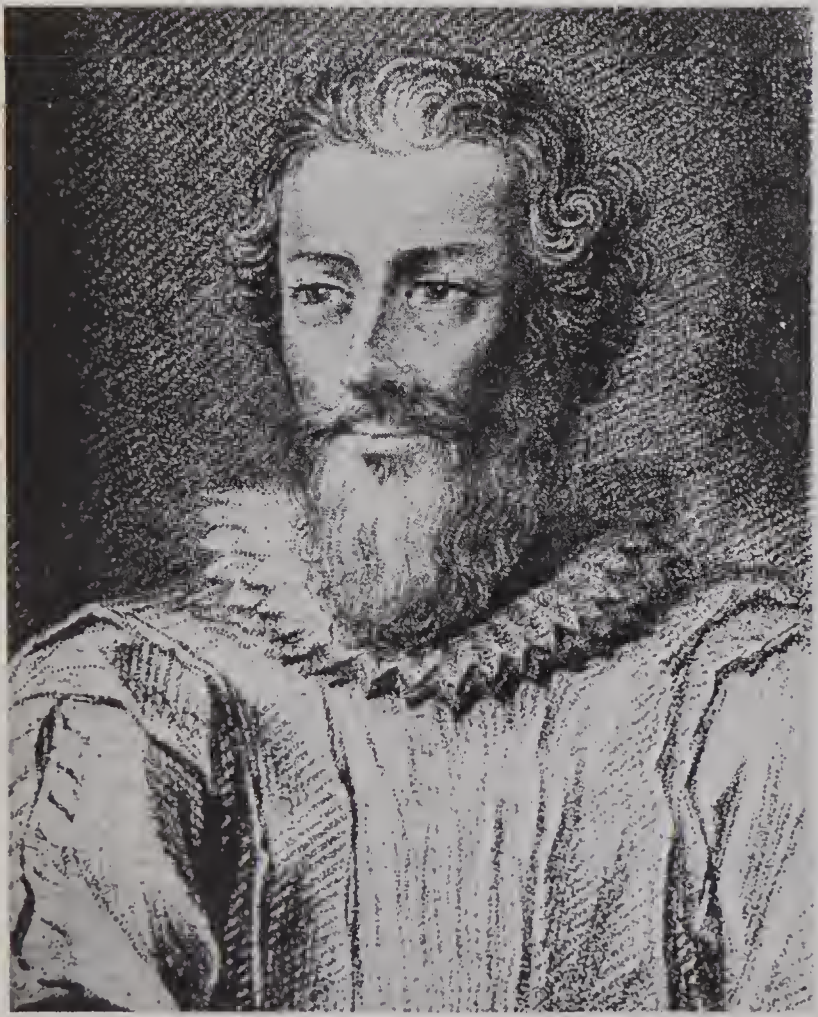
The Amateur Father of Algebra

The time was the sixteenth century. France and Spain were at war. As in every war, both sides sent their messages in code to hide their plans from the enemy. Obviously, secrecy was important.

But the Spanish secret could not be kept. Not that the Spaniards didn't try. When the French captured a Spanish messenger, they read the message as accurately as any Spaniard could have read it. How could this thing be? The Spaniards knew their codes were baffling. How could a mere Frenchman decipher them? In fact, how could any man, unless he had the key? The conclusion was obvious. Something more than man must be at work. The French must be in league with the Devil. They must be using black magic!

The Spaniards complained to the Pope. But the Pope was too wise to interfere, for it was not the Devil who was breaking the codes; it was a French lawyer named Vieta. Nor was it by magic that he did his work, but by mathematics. For Vieta was a lawyer with a hobby, and his hobby was algebra. Code-breaking was nothing to him but solving equations.

The French king owed Vieta a debt of gratitude. So do generations of algebra students. For Vieta not only broke the Spanish code; he simplified the whole subject of algebra. Before his time, there was practically no use of signs and symbols; everything was done the hard way — in words. Vieta introduced the use of letters as variables (he used vowels for unknown numbers and consonants for known numbers). He used signs of operation — to show whether to add, subtract, multiply, or divide. So great were these and other contributions that Vieta, though only an amateur, is known today as "the father of algebra."



A portrait of Vieta, the French lawyer who used algebra for code-breaking.

Chapter Summary

Inventory of Structure and Method

1. In algebraic expressions, **multiplication** may be indicated by no sign, *ab*, parentheses, 8(7), or a raised dot, $9 \cdot 5$. In a term having no other numerical factor, 1 is listed as the numerical coefficient, as in *ab* which is equal to 1*ab*. An exponent tells how many times another number (the base) is a factor, but a coefficient is, itself, a factor.

2. An **algebraic sentence** represents a condition which relates two expressions. The two expressions are called the **left member** and the **right member** of the equation or inequality. An open sentence, becomes true or false as the variables are replaced by numerals. Solving an open sentence in one variable consists in determining the elements of the replacement set of the variable for which the sentence is true.

3. The steps in *solving problems algebraically* are as follows:

   1 — Choose a variable with an appropriate replacement set and use the variable in representing each described number.

   2 — Form an open sentence by using facts given in the problem.

   3 — Find the solution set of the open sentence.

   4 — Check your answer with the words of the problem.

Vocabulary and Spelling

variable (*p. 36*)
replacement set (domain) (*p. 36*)
value of a variable (*p. 36*)
constant (*p. 36*)
variable (open) expression (*p. 36*)
algebraic expression (*p. 36*)
value of an expression (*p. 37*)
evaluate an expression (*p. 37*)
term (*p. 37*)
factor (*p. 40*)
coefficient (*p. 40*)
exponent (*p. 40*)
base (*p. 40*)
power (*p. 40*)
open sentence (*p. 44*)
algebraic sentence (*p. 45*)
equation (*p. 45*)
left & right members (*p. 45*)
inequality (*p. 45*)
solution set (*p. 45*)
solve (*p. 45*)
root (*p. 45*)
graph of an open sentence (*p. 45*)

Chapter Test

2-1 Evaluate the following expressions if $r = 3$ and $s = \frac{1}{3}$.

1. $8r$ **2.** $6s + r$ **3.** $6(r + s) - 3rs$ **4.** $(r + s)(r - s)$

Evaluate the following expressions if $a = 2$ and $b = 3$.

5. ab^3 **6.** $(ab)^2$

2-2 Give the set of factors of each of these expressions.

7. a **8.** $12mn$ **9.** $\frac{pq}{3}$

Give the missing coefficients, as indicated.

10. $8qrs = (\underline{\ ?\ })rs$ **11.** $\frac{st}{5} = (\underline{\ ?\ })st$

12. Identify the numerical coefficient, the base, and the exponent in $3(x - 1)^2$.

13. Write in mathematical symbols, *h used as a factor three times.*

2-3 The replacement set for x is $\{3, 6, 9, 12\}$. Which of the elements make each of the following open sentences true?

14. $x - 5 = 1$ **15.** $x - 5 > 1$

Let $m = 9$ and $n = 1$ in each of the following open sentences. For each resulting sentence, state whether it is true or it is false.

16. $m - 5n \geq 6$ **17.** $m - (n + 8) = (m - n) + 8$

From $\{1, 2, 3, \ldots, 10\}$ determine the solution set of each sentence.

18. $3t + 6 = 9$ **19.** $3t - 6 > 9$

On the number line, graph each of the following inequalities.

20. $1 \leq y < 4$ **21.** $z > 3$

2-4 For each expression, give two interpretations.

22. $x + 5$ **23.** $2x - 5$

2-5 **24.** Write an algebraic expression for the amount of money saved in one year by Mr. Jones if his weekly income is s dollars, and his average monthly expenses, including taxes, are e dollars.

2-6 **25.** The length of a rectangular swimming pool is 5 feet less than twice its width. If the pool is 35 feet in length, find its width.

Chapter Review

2–1 Evaluating Algebraic Expressions Containing Variables

Pages 35–39

1. A variable is a _?_ which represents each of the elements of a specified set.
2. The set whose elements may be used to replace a variable is called the _?_ set or _?_ of the variable.
3. Evaluate $(5r + s)(5r - s)$, letting $r = 2$ and $s = 1$.
4. An expression written as a _?_ or _?_ is called a term.

2–2 Identifying Factors, Coefficients, and Exponents

Pages 40–44

5. Each factor of a term is called the _?_ of the other factors.
6. The usual way of writing $1n$ is _?_.
7. The meaning of s^2 is _?_ times _?_.
8. An _?_ tells how many times another number is to be used as a factor.
9. In mathematical symbols, c used as a factor 5 times is _?_.
10. In the expression $3a^2$, the numerical coefficient is _?_, the base is _?_, and the exponent is _?_.
11. The exponent of r in $7r$ is _?_.
12. Find the value of e^3 and the value of $3e$ when e is 15.

2–3 Solving Open Sentences

Pages 44–49

Exercises 13–15 refer to the general statement: $1 \cdot n = n$.

13. The variable in the given statement is _?_.
14. The _?_ of the variable is the elements of {all the numbers of arithmetic}.
15. The expression $1 \cdot n$ is the _?_ member of the equation.
16. An open sentence may become a true statement or a false statement depending on the replacement for the _?_.
17. Each of the open sentences $3x \neq 6$, $2p > 8$, and $r - 4 < 10$ is called an _?_.

In Exercises 18–21, write numerical sentences using $y = \{0, 1, 2, 3, 4\}$ *as the replacement set for the variable. State which are true and which false.*

18. $y - 1 = 3$ **19.** $y < 5$ **20.** $3y \neq 9$ **21.** $y > 4$

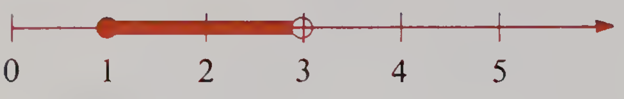
22. The set of numbers which belong to the replacement set of the variable and which make the sentence true is called the $\underline{\ ?\ }$ set of the sentence.

23. A number which satisfies an open sentence is called a $\underline{\ ?\ }$ of the open sentence.

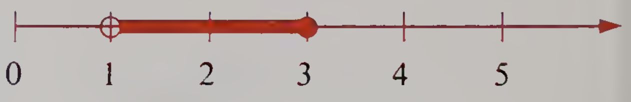
24. Since 4 satisfies the condition expressed in $5x - 1 = 19$, it is a $\underline{\ ?\ }$ of the equation.

25. Of the following graphs, which represents the solution set of the inequality $1 < x \leq 3$?

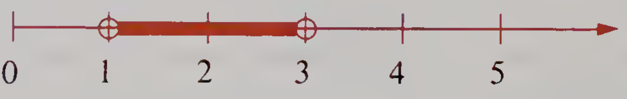
(a)



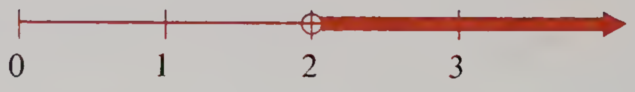
(b)



(c)



26. Of the inequalities (a) $x \neq 2$, (b) $x > 2$, (c) $x \geq 2$, the one whose solution set is represented by the adjoining graph is $\underline{\ ?\ }$.



2-4 Thinking with Variables: From Symbols to Words *Pages 49–51*

Find an interpretation for each of the following algebraic expressions. In each case, identify the replacement set of the variable.

27. $3x$

28. $\frac{x}{3} + 3$

29. $3(x - 1)$

2-5 Thinking with Variables: From Words to Symbols *Pages 51–55*

In Exercises 30–35, translate from words to symbols.

30. 5 less than a

31. $2b$ increased by 3

32. The difference between $3c$ and 3

33. One-third of the sum d and e

34. A line is divided into three equal parts. Using i as the number of inches in one part, represent the length of the entire line.

35. Find an expression for the number of cents Julie received in change from a one-dollar bill after buying n five-cent articles.

2-6 Solving Problems with Open Sentences *Pages 56–60*

36. When you have a problem to solve, first select a $\underline{\ ?\ }$ and use it in representing each described number.

37. After forming an open sentence and finding its solution set, $\underline{\ ?\ }$ each root with the words of the problem.

Write an open sentence expressing the conditions described in Exercises 38–40; then find the solution set.

38. Eighteen times a certain number is 198.

39. In a school cafeteria one week, 1440 bottles of milk were sold. Three times as many bottles of milk as ice cream bars were sold.

40. A class of 25 boys wishes to donate from \$3 to \$5 to a charity. If each boy is to contribute the same amount, a, express as an inequality the amount each boy may give.

Just for Fun

Be a Magician with Numbers

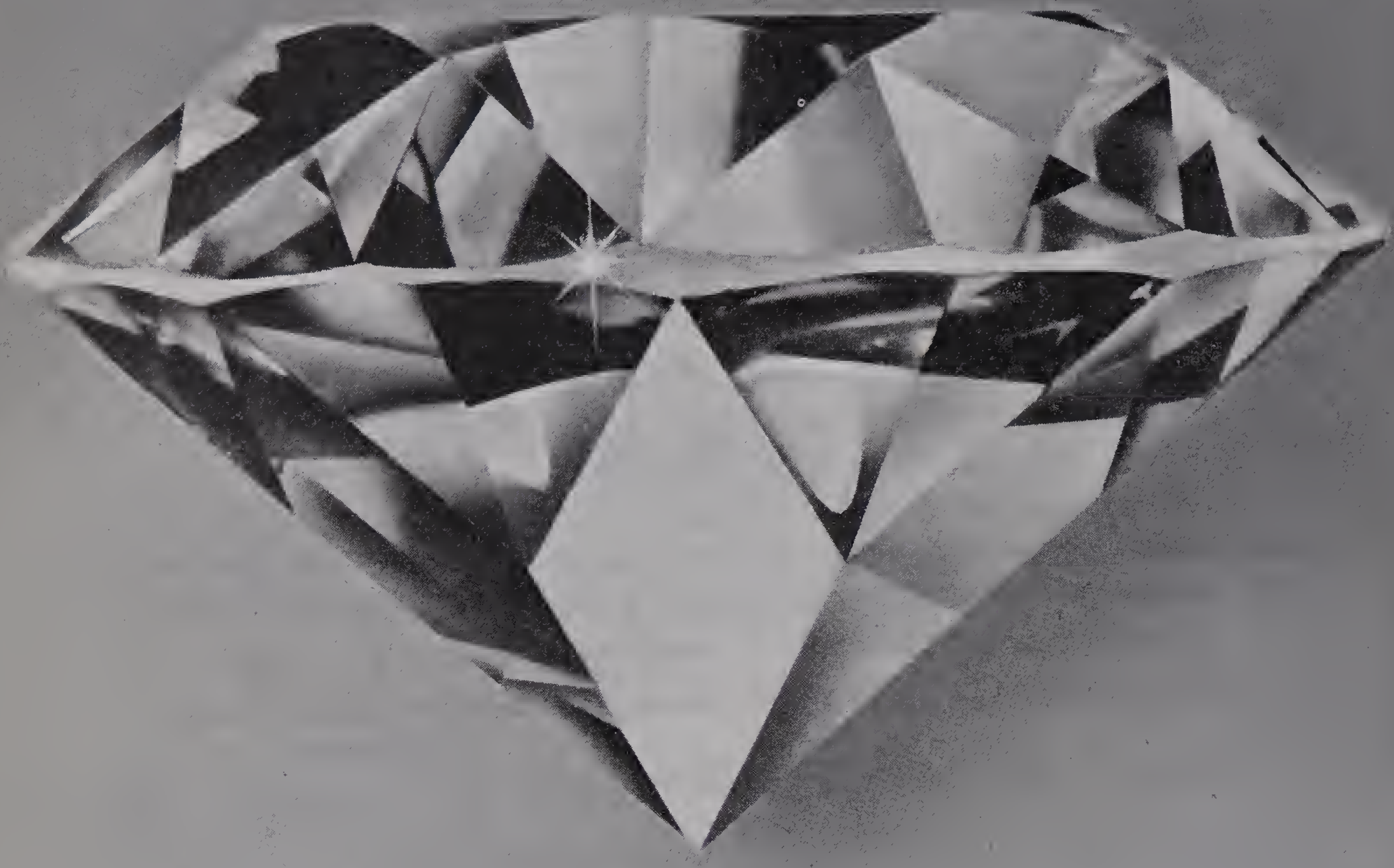
If you practice a bit, you will be able to mystify your family and friends with your seemingly magical knowledge of numbers.

Tell a friend that you can give his age (or any other whole number he might choose) if he will follow a few instructions. Ask him to do the following silently, as you give him directions. If he will then give you the result, you can tell him his age (or the number he chose).

You say	**You think (or write)**	**He thinks** (*if his age is 13*)
Multiply your age by 3.	$3a$	$13 \times 3 = 39$
Add 10 to the result.	$3a + 10$	$39 + 10 = 49$
Subtract twice your age.	$3a + 10 - 2a = a + 10$	$13 \times 2 = 26$ $49 - 26 = 23$
Subtract 6.	$a + 10 - 6 = a + 4$	$23 - 6 = 17$
Tell me the answer.	$a + 4 = 17$ $13 + 4 = 17$ $a = 13$	17
You are 13.		How did you guess it?

Try this with your own age before you try it on anyone else. After a while, you can vary your directions provided you keep track of the steps, using a variable to stand for the number that you want to find. Remember, if your directions say to multiply the number by 5, for example, you must multiply *all your terms* by 5. Or, if the directions say to divide by 3, you must divide *all your terms* by 3.

CHAPTER 3



Axioms, Equations, and Problem Solving

In the diamond you recognize a symbol of value. You also know that diamonds of the same size may have different values. Why is this so? The beautiful pattern you see in the upper photograph is the outward expression of a regular internal structure. This regular pattern gives the diamond (being examined in lower photo) its beauty and hardness, its decorative and practical values.

Mathematics, too, has a regular structure which makes it a source of pleasure to those who understand its internal beauty and discover its many applications. Just as a diamond cutter cannot bring out the hidden beauty of a gem until he understands its internal structure, you cannot make full use of mathematics until you understand its structure.

Because of the importance of structure to both diamonds and mathematics, a diamond is used throughout this book to mark those ideas which form the basic structure of algebra.

See T.M. pg. 10 (1).

IDENTIFYING AND USING NUMBER AXIOMS

3–1 Axioms of Equality

You learned to perform many operations with numbers because you abided by certain rules. These rules, which are statements accepted as true, are called **assumptions**, **axioms**, or **postulates**. Though some of these assumptions may seem simple, you must be able to understand and use these rules in solving complicated problems.

The first fundamental assumptions that you will meet are the *axioms of equality* which govern your work with equations. Perhaps the simplest of all axioms is the ***reflexive property of equality***, which says that any number is equal to itself. Strictly, we should qualify the statements by referring to numbers of arithmetic. See T.M. pg. 11 (2).

For any number a, $a = a$.

T.M. 11 (3). The ***symmetric property of equality*** states that an equality is reversible.

For any numbers, a and b, if $a = b$, then $b = a$.

The use of the diamond symbol and a "structure box" to identify statements concerned with the fundamental structure of algebra is another study device which focuses the student's attention on basic concepts.

The *transitive property of equality* makes it possible for you to identify two numbers as equal if each of them equals a third number.

For any numbers, a, b, and c, if $a = b$, and $b = c$, then $a = c$.

Be sure that students see the significance of the phrase "for any number a."
See T.M. pg. 11 (4) for alternate phrases.

ORAL EXERCISES

In ex. 1-8 S stands for symmetric, T for transitive, R for reflexive

In each exercise, name the property of equality which is illustrated.

SAMPLE 1. If $5 + 6 = 14 - 3$, then $14 - 3 = 5 + 6$

What you say: The symmetric property.

SAMPLE 2. Given $5 + 6 = 11$, and $11 = 14 - 3$, then $5 + 6 = 14 - 3$

What you say: The transitive property.

1. Given that $17 - 2 = 15$; therefore, $15 = 17 - 2$. S
2. Given that $4 + 6 = 10$ and $10 = 5 + 5$; therefore, $4 + 6 = 5 + 5$. T
3. $r + s = r + s$. R
4. Given that $8\frac{2}{3} + 4\frac{1}{3} = 12\frac{3}{3}$ and that $12\frac{3}{3} = 13$; therefore, $8\frac{2}{3} + 4\frac{1}{3} = 13$. T

Name the property of equality which is illustrated in each of the successive conclusions in the following examples.

5. Given that $5(3 + 4) = 5(7)$, that $5(7) = 35$, and that $35 = 15 + 20$; therefore, $5(3 + 4) = 35$ and $5(3 + 4) = 15 + 20$. a. T; b. T
6. Given that $5(1 + 0) = 5$, and that $5 + 0 = 5$; therefore, $5 = 5 + 0$ and $5(1 + 0) = 5 + 0$. a. S; b. T
7. Given that $17 - 6\frac{3}{4} = 16\frac{4}{4} - 6\frac{3}{4}$ and that $10\frac{1}{4} = 16\frac{4}{4} - 6\frac{3}{4}$; therefore, $16\frac{4}{4} - 6\frac{3}{4} = 10\frac{1}{4}$ and $17 - 6\frac{3}{4} = 10\frac{1}{4}$. a. S; b. T
8. Given that $\frac{7}{3} \div \frac{5}{8} = \frac{7}{3} \times \frac{8}{5}$, that $\frac{7}{3} \times \frac{8}{5} = \frac{56}{15}$, and that $\frac{56}{15} = 3\frac{11}{15}$; therefore, $\frac{7}{3} \times \frac{8}{5} = 3\frac{11}{15}$ and $\frac{7}{3} \div \frac{5}{8} = 3\frac{11}{15}$. a. T; b. T

3–2 The Closure Properties

When you add two whole numbers, is the result *always* a whole number? To try every example would be an endless task. After checking a large number of varied examples:

$$138 + 51 = 189; \quad 174 + 236 = 410;$$

and so on, you would probably *assume* that the answer is yes.

Point out that addition and multiplication are *binary* operations. Each is always performed on two numbers.

Is this true also for multiplication? Again you try many examples:

$$5 \times 37 = 185; \quad 23 \times 48 = 1104;$$

and so on. Again you will, no doubt, assume that the product of two whole numbers is always a whole number.

Any set S is said to be **closed under an operation** performed on its elements, provided that each result of the operation is an element of S. This is known as the *closure property* of a set under an operation. Calculations in arithmetic are based on the often unstated assumption that the set of numbers is closed under addition and multiplication.

The ***closure property for addition*** is stated:

For every number a and every number b, the sum $a + b$ is a unique number (one and only one number).

The assumption of the closure properties allows adding and multiplying unrestrictedly in {nos. of arithmetic}.

The ***closure property for multiplication*** is stated:

For every number a and every number b, the product ab is a unique number.

Be sure that "unique" and "one and only one" are understood. See 3-2 T.M. pg. 11.

Closure under any operation depends on both the particular operation and the domain of numbers used. For example, the set of odd numbers is closed under multiplication but not under addition ($3 \cdot 5 = 15$, $3 + 5 = 8$). On the other hand, under division the set of whole numbers is not closed, but the set of arithmetic numbers other than 0 is.

An operation on elements of a specified set may not be possible unless that set is closed under the operation. For example, if you try to subtract any number of arithmetic from a smaller number, you know of no arithmetic number which could be the result. The set of numbers of arithmetic is not closed under subtraction.

Important also is the assumption that an indicated sum or product of numbers does not depend on the particular names designating the numbers.

$$3(2 + 5) = 3 \cdot 7 \quad \text{and} \quad 4 \cdot 99 = 4(100 - 1)$$

because

$$2 + 5 = 7 \quad \text{and} \quad 99 = 100 - 1.$$

These examples illustrate the **substitution principle**:

For any numbers a and b, if $a = b$, then a and b may be substituted for each other.

The substitution principle is essentially equivalent to the assumption that the arithmetic operations are well-defined.

You used the substitution principle often in arithmetic.

Add: $13 + 7 + 4 + 12$

$$\underbrace{13 + 7} + 4 + 12$$

$$\underbrace{20 \quad + 4} + 12$$

$$\underbrace{24 \quad + 12}$$

$$36$$

Note: Each red numeral was substituted for an expression which it equaled: $13 + 7 = 20$; $20 + 4 = 24$; and $24 + 12 = 36$. Implicit here is a chain of equalities leading to the conclusion, 13+7+4+12=36. In building the chain you use the transitive property of equality, the substitution principle, and the associative property of addition.

ORAL EXERCISES

Which of the following sets are closed under the specified operations? Why?

SAMPLE 1. The even numbers, $\{0, 2, 4, 6, \ldots\}$, multiplication

What you say: Closed, because the product of two even numbers is even.

SAMPLE 2. $\{0, 1\}$, addition

C = closed; NC = not closed

What you say: Not closed, because $1 + 1 = 2$ and $2 \notin \{0, 1\}$.

1. $\{0, 1, 2, 3\}$, addition NC; $1+3=4$ and $4 \notin \{0,1,2,3\}$.

2. $\{0, 1\}$, multiplication C; $0 \cdot 0 = 0$, $0 \cdot 1 = 0$, $1 \cdot 0 = 0$, and $1 \cdot 1 = 1$

3. $\{1\}$, multiplication C; $1 \cdot 1 = 1$

4. $\{2\}$, subtraction NC; $2-2=0$ and $0 \notin \{2\}$

5. $\{0, 2\}$, subtraction NC; $0-2 \neq 0$ or 2

6. $\{\frac{1}{2}, 1, 2\}$, division NC; $\frac{1}{2} \div 2 = \frac{1}{4}$ and $\frac{1}{4} \notin \{\frac{1}{2}, 1, 2\}$.

7. $\{0, 1\}$, division NC; $1 \div 0$ not defined.

8. $\{0, 1, 2, 3, 4, \ldots\}$, subtraction NC; result of subtracting any member o set fro a smaller element $\notin \{0,1,2,3,4,\ldots\}$

9. $\{0, 2, 4, 6, 8, \ldots\}$, addition C; sum o 2 even nos. is even

10. $\{1, 3, 5, 7, \ldots\}$, multiplication C; prod of 2 odd nos. is odd

11. $\{1, 3, 5, 7, 9, \ldots\}$, addition NC; $1+3=$ and $4 \notin \{1,3,5,7,9,\ldots$

12. $\{3, 6, 9, 12, \ldots\}$, addition C; sum of a two multiples of 3 is always multipl of 3.

13. $\{1, \frac{1}{2}, 2, \frac{1}{4}, 4, \frac{1}{8}, 8, \ldots\}$, division

14. $\{1, 3, 5, \ldots\}$, multiplying by 5 C; product of 5 and any odd no. is odd and $\therefore \in \{1,3,5,\ldots\}$.

13. C; quotient of any two members of set is always a memb of set.

WRITTEN EXERCISES

Which of the following sets are closed under each of the operations of addition, multiplication, subtraction, and division? When the set is not closed, give an example which shows this.

SAMPLE. {fractions from 0 to 1}

Solution: Addition — not closed, as $\frac{7}{8} + \frac{1}{2}$ is not in the set
Subtraction — not closed, as $\frac{1}{4} - \frac{1}{2}$ is not in the set
Division — not closed, as $\frac{1}{2} \div \frac{1}{4}$ is not in the set
Multiplication — closed

A

1. $\{0\}$
2. $\{1\}$
3. $\{2\}$
4. $\{3\}$
5. $\{0, 1, 2, 3\}$
6. $\{0, 1\}$
7. $\{1, 2\}$
8. {multiples of 5}
9. {numbers between 0 and 2}
10. $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$
11. {nonzero numbers of arithmetic}
12. {all fractions of arithmetic which are not whole numbers}

3–3 Commutative and Associative Properties of Arithmetic Numbers

To help students see the point of this, ask them to compute a sum "physically."

See 3-3 T.M. pg. 11.

You know that

$$6 + 3 = 3 + 6; \qquad 7 + 1 = 1 + 7; \qquad 9 + 2 = 2 + 9.$$

In arithmetic you assume that when you add two numbers, you get the same sum no matter what order you use in adding them. This ***commutative*** (ka-**mu**-ta-tiv) ***property of addition*** may be stated:

For every number a, and every number b, $a + b = b + a$.

Likewise, $6 \times 3 = 3 \times 6$ and $6 \cdot n = n \cdot 6$. When you multiply numbers, you obtain the same product, regardless of the order of the factors. The ***commutative property of multiplication*** is written:

For every number a, and every number b, $ab = ba$.

Notice that subtraction and division do not have the commutative property. For example, $6 - 3 \neq 3 - 6$ and $6 \div 3 \neq 3 \div 6$.

To find the sum of $252 + 60 + 40$, you probably first add 60 and 40, obtaining 100, and then add 252 to the result, getting 352. But if you add 252 and 60, and to that sum add 40, you obtain the same total. That is,

$$252 + (60 + 40) = (252 + 60) + 40.$$

Thus, you are free to choose any adjacent pair in addition, for the answer is the same. This ***associative*** (a-**so**-she-ay′tiv) ***property of addition*** states that: Point out that the associative property says nothing about order.

For every number a, every number b, and every number c,

$$a + (b + c) = (a + b) + c.$$

The ***associative property of multiplication*** is:

For every number a, every number b, and every number c, $a(bc) = (ab)c$.

Are subtraction and division associative? No, because

$24 - (6 - 2) \neq (24 - 6) - 2$ and $24 \div (6 \div 2) \neq (24 \div 6) \div 2$.

The commutative (order) and associative (grouping) properties permit you to omit parentheses in a sum because the numbers may be added in any *groups* of two and in any *order*.

$$\begin{aligned} \mathbf{101 + 33 + 46 + 67 + 14 + 99} &= \mathbf{101 + 99 + 33 + 67 + 46 + 14} \\ &= \mathbf{200 + 100 + 60} \\ &= \mathbf{360} \end{aligned}$$

Point out that the associative property allows you to refer meaningfully to the sum, $a + b + c$, of three nos., although addition is a binary operation. Thus $a + b + c$, means either $a + (b + c)$ or $(a + b) + c$.

ORAL EXERCISES

Name the property illustrated in each of the following true sentences. Every variable has the set of the numbers of arithmetic as its replacement set.

SAMPLE. $3 \times (7 \times 9) = (7 \times 9) \times 3$

What you say: Commutative property of multiplication.

1. $6 + 2 = 2 + 6$ Comm. prop. of add.

2. $(12 + 4) + 5 = 12 + (4 + 5)$ Assoc. prop. of add.

3. $\frac{1}{2} \cdot 6 = 6 \cdot \frac{1}{2}$ Comm. prop. of mult.

4. $8 \times (0 \times 4) = (8 \times 0) \times 4$ Assoc. prop. of mult.

5. For each a, $5(6a) = 30a$ Assoc. prop. of mult. and subst. principle

6. For each m, $m \times 3 = 3 \times m$ Comm. prop.o mult.

7. $\frac{1}{2} \times (2 + 9) = (2 + 9) \times \frac{1}{2}$ Comm. prop. of mult.

8. $9 + 0 = 0 + 9$ Comm. prop. of add.

9. For each x, $7 + 9 + x = 16 + x$ Assoc. prop. of add. and subst. principle

In exercises such as 10, 11, 14, 17, 18, and 19, using the associative property simplifies computation.

10. $17.99 + 15 + 1.01 = 17.99 + 1.01 + 15$ Comm. prop. of add.

11. $(58 + 11) + 139 = 58 + (11 + 139)$ Assoc. prop. of add.

12. For each z, $(32 + 17z) + 33z = 32 + (17z + 33z)$ Assoc. prop. of a

13. For each z, $11 + (4 + z) = 15 + z$ Assoc. prop. of add. and subst. principl

14. $\frac{4}{3} \times (9 \times 7) = (\frac{4}{3} \times 9) \times 7$ Assoc. prop. of mult.

15. For each u and w, $5u + (3u + w) = (5u + 3u) + w$ Assoc. prop. of ad

16. For each r, $(r + 3)19 = 19(r + 3)$ Comm. prop. of mult.

17. $(11 \times 17\frac{1}{2}) \times 2 = 11 \times (17\frac{1}{2} \times 2)$ Assoc. prop. of mult.

18. $25 \times (4 \times 93) = (25 \times 4) \times 93$ Assoc. prop. of mult.

19. $\frac{2}{7} \times (\frac{7}{2} \times 16) = (\frac{2}{7} \times \frac{7}{2}) \times 16$ Assoc. prop. of mult.

20. For each a and each b, $7 \times (4a) \times b = 28ab$ Assoc. prop. of mult. and subst. principle

Assign several of these exercises; they are simple preparations for later proofs.

Name the property that justifies each lettered step of these chains of equality. A check (✓) shows that the step is justified by the substitution principle.

21. $17 + (38 + 3) = 17 + (3 + 38)$ (a) Comm. prop. of add.
$= (17 + 3) + 38$ (b) Assoc. prop. of add.
$= 20 + 38$ (✓)
$= 58$ (✓)

22. $4 \times (59 \times 25) = 4 \times (25 \times 59)$ (a) Comm. prop. of mult.
$= (4 \times 25) \times 59$ (b) Assoc. prop. of mult.
$= 100 \times 59$ (✓)
$= 5900$ (✓)

23. For each m,
$5 + (m + 3) = 5 + (3 + m)$ (a) Comm. prop. of add.
$= (5 + 3) + m$ (b) Assoc. prop. of add.
$= 8 + m$ (✓)

24. For each n,
$5 \times (n \times 3) = 5 \times (3 \times n)$ (a) Comm. prop. of mult.
$= (5 \times 3) \times n$ (b) Assoc. prop. of mult.
$= 15n$ (✓)

25. For each r, s, and t,
$r(st) = r(ts)$ (a) Comm. prop. of mult.
$= (rt)s$ (b) Assoc. prop. of mult.
$= (tr)s$ (c) Comm. prop. of mult.

26. For each r, s, and t,
$r + (s + t) = r + (t + s)$ (a) Comm. prop. of add.
$= (r + t) + s$ (b) Assoc. prop. of add.
$= (t + r) + s$ (c) Comm. prop. of add.

3–4 The Distributive Property; Special Properties of 1 and 0

Dave works 3 hours Friday evening and 7 hours Saturday at 95 cents an hour. Since he works 3 + 7 hours, his weekly pay in cents is

$$95(3 + 7) = 95 \times 10 = 950.$$

His weekly pay also is the sum of his Friday pay and his Saturday pay:

$$95 \times 3 + 95 \times 7 = 285 + 665 = 950.$$

A few more examples will suggest the distributive property to your students; e.g. 54·9 + 54·1, 62·5 + 62·95, and 12·1/3 + 12·2/3.

Either way you get the same result; that is,

$$95(3 + 7) = 95 \times 3 + 95 \times 7.$$

Note that 95, the coefficient (multiplier) of the sum $(3 + 7)$, is *distributed* as a multiplier of each term of $(3 + 7)$. The property shown in this example is called the ***distributive*** (dis-**trib**-u-tiv) ***property of multiplication with respect to addition*** and is stated:

For every number a, every number b, and every number c,

$$a(b + c) = ab + ac \quad \text{or} \quad ab + ac = a(b + c).$$

The following show how the distributive property is used:

a. $28\left(\frac{1}{2} + \frac{1}{7}\right) = 28 \times \frac{1}{2} + 28 \times \frac{1}{7} = 14 + 4 = 18$

b. $9 \times 4\frac{7}{9} = 9\left(4 + \frac{7}{9}\right) = 9 \times 4 + 9 \times \frac{7}{9} = 36 + 7 = 43$

c. $\dfrac{15}{4} + \dfrac{9}{4} = \dfrac{15 + 9}{4} = 6$, or

$$\frac{1}{4}(15) + \frac{1}{4}(9) = \frac{1}{4}(15 + 9) = \frac{1}{4}(24) = 6$$

You can readily show that the following sentences involving subtraction and multiplication are true.

$$14\left(\frac{3}{2} - 1\right) = 14 \times \frac{3}{2} - 14 \times 1 = 7$$

$$20 \cdot 8 - 20 \cdot 5 = 20(8 - 5) = 60$$

The common error of writing $2(3 \cdot 4) = (2 \cdot 3)(2 \cdot 4)$ is equivalent to assuming a distributive property of multiplication with respect to multiplication.

See T.M. pg. 11 (5).

These two sentences illustrate the ***distributive property of multiplication with respect to subtraction:***

Point out that in general, $a(b \times c) \neq (a \times b)(a \times c)$. Similarly, $\frac{b \times c}{a} \neq \frac{b}{a} \times \frac{c}{a}$, $(b \neq 0,\ c \neq 0)$.

For each a and each b and each c for which $b - c$ is a number,

$$a(b - c) = ab - ac, \text{ or } ab - ac = a(b - c).$$

Often you will use these properties to simplify variable expressions. For example, to show that $5x + 3x = 8x$ for each number x:

$5x + 3x$	$= x5 + x3$	**Commutative property of multiplication**
	$= x(5 + 3)$	**Distributive property**
	$= x8$	**Substitution principle**
	$= 8x$	**Commutative property of multiplication**

Similarly,

$$7ab - 4ab = (7 - 4)ab = 3ab.$$

The distributive property enables you to write the sum ($5x + 3x = 8x$) or the difference ($7ab - 3ab = 4ab$) of similar terms as a single term. Terms such as 7 and 9, $5x$ and $3x$, or $7ab$ and $4ab$ are called *similar terms* or *like terms*. **Similar terms** are numerical terms or variable terms whose variable factors are the same. Caution! See T.M. pg. 12 (6).

Terms such as $8x$ and $3ab$ are *unlike terms*, because their variable factors are different. Also, the terms 9 and $7b$ are unlike terms. Hence, expressions such as $8x + 3ab$ and $7b - 9$ cannot be written in simpler form.

$1 \times 3 = 3 \times 1 = 3$ illustrates the ***multiplicative property of 1*** (**mul**-ti-pli-kay′tiv): one times any given number equals the given number itself.

For each number a, $1 \cdot a = a \cdot 1 = a$.

Since the given number and the product are always *identical* when you use the multiplicative property of 1, do you see why 1 is the **multiplicative identity element**?

Likewise, $0 + 3 = 3 + 0 = 3$ illustrates that the **additive identity element** is 0. The ***additive property of 0*** states that when 0 is added to any given number, the sum is the given number itself, or simply:

For each number a, $0 + a = a + 0 = a$.

The ***multiplicative property of 0***, shown in $0 \times 3 = 3 \times 0 = 0$, states that when one of the factors of a product is 0, the product itself is 0.

For each number a, $0 \cdot a = a \cdot 0 = 0$.

At this stage students are not ready for this proof.

This multiplicative property of 0 affects the use of 0 as a divisor. The statement $\frac{6}{3} = 2$ means that $6 = 3 \times 2$. Likewise, $\frac{a}{0} = b$ should mean that $a = 0 \times b$. If $a \neq 0$, *no* value of b can make the latter statement true, since $0 \times b = 0$ for each b. If $a = 0$, *every* value of b makes the statement true for the same reason. Thus, the fraction $\frac{a}{0}$ either has *no* value or is *indefinite* in value. A consequence of the multiplicative property of 0 is that *you may not divide by 0*

Students should understand that they cannot define $\frac{a}{0}$ without either contradicting the multiplicative prop. of zero or giving up the requirement that a quotient be unique.

Discussion of several of these helps students see that everything they do with nos. depends on the basic properties.

ORAL EXERCISES

Name the property of numbers which justifies each step in the following exercises.

Comm. prop. of mult. **1.** $7 \times (4 \times \frac{1}{7}) = 7 \times (\frac{1}{7} \times 4)$

Assoc. prop. of mult. $= (7 \times \frac{1}{7}) \times 4$

Subst. principle $= 1 \times 4$

Mult. prop. of 1 $= 4$

Comm. prop. of mult. **2.** $(\frac{3}{4} - \frac{1}{6})48 = 48(\frac{3}{4} - \frac{1}{6})$

Distrib. prop. $= 48(\frac{3}{4}) - 48(\frac{1}{6})$

Subst. principle $= 36 - 8$

Subst. principle $= 28$

Subst. principle **3.** $973(101) = 973(100 + 1)$

Distrib. prop. $= 973(100) + 973(1)$

Subst. principle and mult. prop. of 1 $= 97{,}300 + 973$

Subst. principle $= 98{,}273$

4. $24 \times 5\frac{1}{2} = 24(5 + \frac{1}{2})$ Subst. principle

Distrib. prop. $= 24(5) + 24(\frac{1}{2})$

Subst. principle $= 120 + 12$

Subst. principle $= 132$

5. For each r, s, and t,

$(r + s)t = t(r + s)$ Comm. prop. of mult.

Distrib. prop. $= tr + ts$

Comm. prop. of mult. $= rt + st$

6. For each r,

$\frac{3r + 15}{3} = \frac{1}{3}(3r + 15)$ Subst. principle

Distrib. prop. $= \frac{1}{3}(3r) + \frac{1}{3}(15)$

Assoc. prop. of mult. $= (\frac{1}{3} \cdot 3)\, r + 5$ and subst. principle

Subst. principle $= 1 \cdot r + 5$

Mult. prop. of 1 $= r + 5$

7. For each x,

Comm. prop. of add. $x + (7 + 6x) = x + (6x + 7)$

Assoc. prop. of add. $= (x + 6x) + 7$

Mult. prop. of 1 $= (1 \cdot x + 6x) + 7$

Distrib. prop. $= (1 + 6)x + 7$

Subst. principle $= 7x + 7$

Mult. prop. of 1 and distrib. prop. $= 7(x + 1)$

Read each expression with similar terms combined.

In exercises like 13 & 14, you extend the distributive prop.: $a(b + c + d) = ab + ac + ad$. See T.M. pg. 12 (7).

SAMPLE. $8z + 3 - 3z$ *What you say:* $5z + 3$

8. $3x + 5x$ $8x$
9. $5a + 6a$ $11a$
10. $45s - 15s$ $30s$
11. $56y - 16y$ $40y$
12. $5x + 4x + x$ $10x$
13. $8n + 2n + n$ $11n$
14. $6m + 5m + 7$ $11m+7$
15. $9y + y - 3n$ $10y-3n$
16. $10y + n - 5y$ $5y+n$
17. $x + 4x - 5x$ 0
18. $8p + 2p - 10p$ 0
19. $2(a + 1) - 2$ $2a$
20. $3 + 3(b - 1)$ $3b$
21. $5(2t + 3) + t$ $11t+15$
22. $4(4s + 5) + 3s$ $19s+20$
23. $7k + 3(2 - k)$ $4k+6$
24. $8v + 5(7 - v)$ $3v+35$
25. $6v + 7(3 + m)$ $6v+21+7$
26. $x - x + x - x$ 0
27. $9y + 10y + 7 - 4$ $19y+3$
28. $3x + 5y - 6u - 4$ $3x+5y-6$ -4
29. $3y + 12 + y + 8$ $4y+20$

In these and the Written Exercises you need not ask students to justify each simplifying step unless they give erroneous answers.

WRITTEN EXERCISES

Simplify each of the following expressions by combining similar terms.

SAMPLE. $8x + 3y + 2x - 3y$

Solution: $(8x + 2x) + (3y - 3y) = 10x + 0$

$= 10x$

Ⓐ **1.** $17x + 39x$

2. $48y + 37y$

3. $100a - 35a$

4. $75b - 52b$

5. $13a + a + 16a$

6. $15x - x - 7x$

7. $5x + 7x + 6$

8. $9n + 6n + 3$

9. $2s + 3r + s + r$

10. $3x + y + x + y$

11. $2a + 3b + 4c - d$

12. $6u - 5v - 3w - 4$

13. $15n + 13 - 5n + 8$

14. $18z + 27 + 8z - 17$

15. $5a + 6b - a + b - 4a$

16. $3x + x + 5 + 24 + 1$

17. $16 + 16a + 5b - 15 - b$

18. $19 + s + 6t - t - 17$

19. $5ab + 7a + 6ab - a$

20. $7xy + 3x - 2xy + 9x$

Ⓑ **21.** $3(a + b) + 7(a + b)$

22. $9(m + n) + 7(2m + n)$

23. $8(3 + a) + 4(3 + a)$

24. $2(a + 5) + 3(a - 2)$

25. $4(d + 5) + 3(2d - 1)$

26. $7(6e + 3) + 5(4e - 3)$

27. $10(3w + v) + 6(3w - v)$

28. $9(2r + 8t) + 4(2r - 8t)$

29. $3(a + 2b + 1) + 2(a - 1)$

30. $4(r + 3s + 2) + 7(s - 1)$

Ⓒ **31.** $3[8a + 5(3 - a)] - 17$

32. $19 + 2[4b + 3(5b - 2)]$

33. $6(2r + s) + 2[5r + 3(4s - r + 1)]$

34. $5[4(2m + n + 3) - 6m - 1] + 2(5m - n)$

Determine the value of each numerical expression. Whenever you can, use the properties of numbers to simplify the calculation.

35. $28 \times 37 + 22 \times 37$

36. $77 \times 19 + 23 \times 19$

37. $50(\frac{4}{5} - \frac{1}{2})$

38. $779(11) - 779$

39. $4 \times (0 \times 25)$

40. $(68 \times 32) \times 0$

41. $7\frac{1}{2} \times 9 + 7\frac{1}{2} \times 1$

42. $\frac{1}{2} \times 57 + 9\frac{1}{2} \times 57$

43. 326×1001

44. $30 \times 4\frac{7}{10}$

45. $3 \times 29 + 4 \times 29 + 3 \times 29$

46. $7 \times 46 - 5 \times 46 - 2 \times 46$

TRANSFORMING EQUATIONS WITH EQUALITY PROPERTIES

3–5 Addition and Subtraction Properties of Equality

Certain properties of equality can be proved from the properties of equalities given in Section 3–1. A knowledge of these properties will help you to solve complicated equations more readily. Consider the following illustration.

	Man *A*		Man *B*
Two men receive equal salaries.	$6000	=	$6000
Each gets the same $500 raise.	$6000 + $500	=	$6000 + $500
Their salaries change, but the new salary of man *A* is equal to the new salary of man *B*.	$6500	=	$6500

This example of the ***addition property of equality*** shows that if the same number is added to equal numbers, the sums are equal:

For each a, each b, and each c, if $a = b$, then $a + c = b + c$.

This new property of equality follows from facts already learned. The reasoning leading from the assumption $a = b$ to the conclusion $a + c = b + c$ is shown in the following sequence of statements, each justified by the indicated reason:

This is an important
in the proof.

$a + c$ is a number	**Closure property of addition**
$a + c = a + c$	**Reflexive property of equality**
$a = b$	**Given**
$a + c = b + c$	**Substitution principle**

This form of logical reasoning, from known facts and assumptions to conclusions, is called a **proof**. See T.M. pg. 12 (8), for proof of the subtraction propert
equalit

Can you prove the ***subtraction property of equality:*** if the same number is subtracted from equal numbers, the differences are equal, provided the indicated subtraction is possible?

For each a, each b, and each c for which $a - c$ is a number, if $a = b$, then $a - c = b - c$.

The addition and subtraction properties of equality are used to solve equations. To see how to use them, first notice:

$$100 - 70 + 70 = 100 \quad \text{and} \quad n - 6 + 6 = n$$

In general, $a - c + c = a$ and c is the number you add to $a - c$ to produce a. To undo a subtraction, you add.

Similarly, $8 + 5 - 5 = 8$ and $y + 3 - 3 = y$. In general, $a + c - c = a$, and c is the number you subtract from $a + c$ to obtain a. To undo an addition you subtract.

Because the operations of adding and subtracting the same number are opposite in effect, they are called **inverse operations**. Can you think of another pair of inverse operations?

Study the following sequence of equations:

(1) $n - 6 = 19$

(2) $n - 6 + 6 = 19 + 6$

(3) $n = 25$

Because you obtain the third equation by adding 6 to each member of the first equation, while you can obtain the *first* by subtracting 6 from each member of the third equation, the addition and subtraction properties of equality imply that these equations have the same solution set, {25}, and are **equivalent equations**. This process which uses addition to transform an equation such as (1) into a simple equivalent equation such as (3) is called ***transformation by addition***.

To *check* the value found, substitute it for the variable in equation (1). If the resulting sentence is true, this value satisfies the equation and is a root. Because numerical errors may occur in transforming an equation, you should check all values found. A successful check *suggests* that the computation is correct, but does not prove it. See T.M. pg. 12 (9) for justification of checking.

Check: $n - 6 = 19$ ← **original equation**

$25 - 6 \overset{?}{=} 19$ ← **The question mark above**
the equals sign means,
"Is this statement true?"

$19 = 19$ ✓ ← **The check (✓) means "Yes, it is."**

∴ The solution set is {25}, Answer.

EXAMPLE 1. $x - 2 = 5$

Solution:

$$\begin{aligned} x - 2 &= 5 \\ x - 2 + 2 &= 5 + 2 \\ x &= 7 \end{aligned}$$

Check:

$$\begin{aligned} 7 - 2 &\stackrel{?}{=} 5 \\ 5 &= 5 \ \checkmark \end{aligned}$$

$\therefore$ The solution set is $\{7\}$, Answer.

EXAMPLE 2. $m - 8 = 0$

Solution:

$$\begin{aligned} m - 8 &= 0 \\ m - 8 + 8 &= 0 + 8 \\ m &= 8 \end{aligned}$$

Check:

$$\begin{aligned} 8 - 8 &\stackrel{?}{=} 0 \\ 0 &= 0 \ \checkmark \end{aligned}$$

$\therefore$ The solution set is $\{8\}$, Answer.

EXAMPLE 3. $y + 3 = 14$

Solution:

$$\begin{aligned} y + 3 &= 14 \\ y + 3 - 3 &= 14 - 3 \\ y &= 11 \end{aligned}$$

Check:

$$\begin{aligned} 11 + 3 &\stackrel{?}{=} 14 \\ 14 &= 14 \ \checkmark \end{aligned}$$

$\therefore$ The solution set is $\{11\}$, Answer.

EXAMPLE 4. $9 = a + 2$

Solution:

$$\begin{aligned} 9 &= a + 2 \\ 9 - 2 &= a + 2 - 2 \\ 7 &= a \end{aligned}$$

Check:

$$\begin{aligned} 9 &\stackrel{?}{=} 7 + 2 \\ 9 &= 9 \ \checkmark \end{aligned}$$

$\therefore$ The solution set is $\{7\}$, Answer.

Examples 3 and 4 illustrate ***transformation by subtraction***, a process leading to equivalent equations because of the subtraction and addition properties of equality.

ORAL EXERCISES

Solve each equation, first stating what number must be subtracted from each member of the equation or what number must be added to each member of the equation. A = add; S = subtract

1. $x + 2 = 6$ S2; $x = 4$
2. $n + 5 = 7$ S5; $n = 2$
3. $x - 1 = 8$ A1; $x = 9$
4. $y - 6 = 7$ A6; $y = 13$
5. $k + 7 = 11$ S7; $k = 4$
6. $r + \frac{3}{5} = \frac{7}{5}$ S$\frac{3}{5}$; $r = \frac{4}{5}$
7. $n - 9 = 11$ A9; $n = 20$
8. $h - 18 = 6$ A18; $h = 24$
9. $k + \frac{2}{3} = \frac{5}{3}$ S$\frac{2}{3}$; $k = 1$
10. $m + 3 = 3$ S3; $m = 0$
11. $a - 40 = 8$ A 40; $a = 48$
12. $p - 8 = 5$ A8; $p = 13$
13. $n - .8 = 1.1$ A .8; $n = 1.9$
14. $b - .5 = 3.2$ A .5; $b = 3.7$
15. $x + 4 = 4$ S4; $x = 0$
16. $t + .2 = .7$ S.2; $t = .5$
17. $18 = x - 6$ A 6; $24 = x$
18. $65 = n - 5$ A5; $70 =$
19. $75 = 60 + x$ S60; $15 = x$
20. $14 = 7 + n$ S7; $7 = n$
21. $\frac{2}{7} = w - \frac{5}{7}$ A$\frac{5}{7}$; $1 = w$
22. $\frac{2}{5} = z - \frac{3}{5}$ A$\frac{3}{5}$; $1 = z$
23. $\frac{11}{9} = \frac{2}{9} + n$ S$\frac{2}{9}$; $1 = n$
24. $\frac{17}{11} = s + \frac{5}{11}$ S$\frac{5}{11}$; $\frac{12}{11} =$

WRITTEN EXERCISES

If a student can solve an equation by inspection, do not force him to transform it.

Solve each equation by using addition or subtraction.

A

1. $x + 32 = 81$
2. $y + 17 = 45$
3. $27 + R = 105$
4. $91 + S = 91$
5. $39 = t + 39$
6. $73 = u + 73$
7. $v - 47 = 47$
8. $w - 37 = 73$
9. $175 = z + 82$
10. $314 = 165 + a$
11. $b - 17 = 0$
12. $c - 51 = 0$
13. $d + 3.2 = 7.8$
14. $f + 2.7 = 3.1$
15. $h - .78 = 9.2$
16. $k - .37 = 4.1$
17. $\frac{3}{5} + m = \frac{18}{5}$
18. $\frac{2}{7} + n = \frac{16}{7}$
19. $\frac{2}{3} = Q - \frac{5}{3}$
20. $\frac{3}{10} = T - \frac{13}{10}$
21. $m - 750 = 5$
22. $0.04 + p = 1$
23. $a - 0.85 = 0.15$
24. $b - 5 = \frac{3}{5}$
25. $h - \frac{2}{3} = 3$
26. $d - 2.5 = 3.95$
27. $y - \frac{3}{8} = 2$
28. $p - \frac{5}{6} = 3\frac{1}{6}$
29. $.8 = n - 75.7$
30. $\frac{3}{4} = .75 + r$

3–6 Division and Multiplication Properties of Equality

The ***division property of equality*** states that when equal numbers are divided by the same number, the quotients are equal (recall that 0 may not be used as a divisor):

For each a, each b, and each nonzero c, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$.

The ***multiplicative property of equality*** says that when equal numbers are multiplied by the same number, the products are equal:

For each a, each b, and each c, if $a = b$, then $a \cdot c = b \cdot c$.

These properties permit you to apply ***transformation by division*** and ***by multiplication*** to an equation without changing its solution set. See 3-6 T.M. pg. 12.

EXAMPLE 1. $6k = 84$

Solution:

$$\begin{aligned} 6k &= 84 \\ \frac{6k}{6} &= \frac{84}{6} \leftarrow \\ k &= 14 \end{aligned}$$

Since $6k$ shows k multiplied by 6, you use the inverse operation, division; that is, divide each member by 6, the coefficient of the variable.

Check:

$$\begin{aligned} 6(14) &\overset{?}{=} 84 \\ 84 &= 84 \checkmark \end{aligned}$$

$\therefore$ The solution set is $\{14\}$, Answer.

EXAMPLE 2. $\frac{n}{4} = 6$

Solution:

$$\frac{n}{4} = 6$$

$$4 \cdot \frac{n}{4} = 4 \cdot 6$$

$$n = 24$$

Check:

$$\frac{24}{4} \stackrel{?}{=} 6$$

$$6 = 6 \checkmark$$

$\therefore$ **The solution set is $\{24\}$, Answer.**

EXAMPLE 3. $\frac{2x}{5} = 86$

Solution:

Ask the class to suggest another sequence of transformation (divide by 2, then multiply by 5).

$$\frac{2x}{5} = 86$$

$$5 \cdot \frac{2x}{5} = 5 \cdot 86$$

$$2x = 430$$

$$\frac{2x}{2} = \frac{430}{2}$$

$$x = 215$$

Check:

$$\frac{2(215)}{5} \stackrel{?}{=} 86$$

$$86 = 86 \checkmark$$

$\therefore$ **The solution set is $\{215\}$, Answer.**

You know that you may not divide by zero, but do you see why you may not use zero as a multiplier in transforming an equation? Consider the equation $\frac{n}{4} = 6$ whose solution set is $\{24\}$. If you multiplied by 0, the equation $0 \cdot \frac{n}{4} = 0 \cdot 6$ would be satisfied by any value of n; its solution set would be the set of all arithmetic numbers. Thus, the original equation $\frac{n}{4} = 6$, and the derived equation $0 \cdot \frac{n}{6} = 0 \cdot 6$ would have different solution sets; they would not be equivalent equations. Therefore, *multiplication by zero* cannot be used because it does not produce an equivalent equation. Emphasize this point. It will be most important in fractional equations.

WRITTEN EXERCISES

Solve each of the following.

Ⓐ

1. $\frac{n}{19} = 6$
2. $\frac{r}{27} = 8$
3. $17s = 187$
4. $13t = 169$
5. $1.5n = 3$
6. $.7 = .7m$
7. $\frac{f}{1.2} = 15$
8. $\frac{x}{1.8} = 1.9$
9. $\frac{5}{3} = \frac{1}{12}x$
10. $\frac{3}{8} = \frac{1}{24}y$
11. $16 = 12t$
12. $20 = 16z$
13. $.8 = \frac{y}{5}$
14. $1.6 = \frac{h}{1.5}$
15. $\frac{2}{3} = 6c$
16. $12d = \frac{3}{4}$
17. $.36(y) = 4.8$
18. $.14(y) = 14$

B

19. $\frac{k}{3.2} = 0$

20. $\frac{m}{.25} = 0$

21. $\frac{n}{.6} = .25$

22. $\frac{p}{.03} = .2$

23. $4.5(y) = 12.8$

24. $3.2(t) = 6.7$

25. $3.4 = y \div 10$

26. $.75 = x \div 15$

27. $\frac{2}{3}y = 6$

28. $\frac{5}{4}w = 20$

29. $\frac{1}{2}r = 3\frac{1}{2}$

30. $\frac{1}{3}v = 4\frac{2}{3}$

C

31. $4 + 8y = 52$

32. $.5c + 8 = 23$

33. $9r - 9 = 27$

34. $6x - 3 = 27$

35. $2n - 5 = 8.5$

36. $29 = 7h + 7.79$

Assign at least one or two of these proofs. When a proof is discussed, ask a student to state the conclusion as a number property. For the Sample: If m, n, r, and s are numbers such that $m = n$ and $r = s$, then $m + r = n + s$.

Supply the reasons for each step in these proofs. Assume that $m = n$ and $r = s$ and that $m + r$, mr, $\frac{m}{r}$, and $m - r$ represent numbers.

		Step		*Reason*
SAMPLE.	**a.**	$m + r = m + r$	**a.**	Reflexive property
	b.	$m = n$	**b.**	Given
	c.	$m + r = n + r$	**c.**	Substitution principle
	d.	$r = s$	**d.**	Given
	e.	$\therefore m + r = n + s$	**e.**	Substitution principle

37. **a.** $mr = mr$

b. $m = n$

c. $\therefore mr = nr$

38. Assume $r \neq 0$.

a. $\frac{m}{r} = \frac{m}{r}$

b. $m = n$

c. $\therefore \frac{m}{r} = \frac{n}{r}$

39. **a.** $m - r = m - r$

b. $m = n$

c. $\therefore m - r = n - r$

40. **a.** $mr = mr$

b. $m = n$

c. $mr = nr$

d. $r = s$

e. $\therefore mr = ns$

41. Assume $r \neq 0$.

a. $\frac{m}{r} = \frac{m}{r}$

b. $m = n$

c. $\frac{m}{r} = \frac{n}{r}$

d. $r = s$

e. $\therefore \frac{m}{r} = \frac{n}{s}$

42. **a.** $m - r = m - r$

b. $m = n$

c. $m - r = n - r$

d. $r = s$

e. $\therefore m - r = n - s$

3–7 Combining Terms and Using Transformation Principles

To analyze the solution of an equation, study the steps used in building it to its present form. The following example shows that in the solution, you *undo* the *operations* used in forming the equation, but in *reverse order*. The two parts outline the proof that $v = 5$ and $2v + 7 = 17$ are equivalent.

Building an Equation		*Solving the Equation*	
$v = 5$	Given	$2v + 7 = 17$	Given
$2 \cdot v = 2 \cdot 5$	Multiply by 2	$2v + 7 - 7 = 17 - 7$	Subtract 7
$2v = 10$		$2v = 10$	
$2v + 7 = 10 + 7$	Add 7	$\frac{2v}{2} = \frac{10}{2}$	Divide by 2
$2v + 7 = 17$		$v = 5$	

You may use the following as a guide in solving equations:
This is simply a guide. Able students may combine steps or vary their order.

1. Combine any similar terms in either member of the equation.
2. If there are still indicated additions or subtractions, use inverse operations to undo them.
3. If there are any indicated multiplications or divisions in the variable term, use the inverse operations to find the value of the variable.
4. Check by substituting the value in the given equation to see whether it satisfies that equation.

EXAMPLE 1 Solve $7x + 3x - 4 = 12 + 39$

Solution:

$$\begin{aligned} \underbrace{7x + 3x} - 4 &= 12 + 39 \\ 10x - 4 &= 51 \\ 10x - 4 + 4 &= 51 + 4 \\ 10x &= 55 \\ \frac{10x}{10} &= \frac{55}{10} \\ x &= 5.5 \end{aligned}$$

Check:

$$\begin{aligned} 7(5.5) + 3(5.5) - 4 &\overset{?}{=} 12 + 39 \\ 38.5 + 16.5 - 4 &\overset{?}{=} 51 \\ 51 &= 51 \quad \checkmark \end{aligned}$$

$\therefore$ **The solution set is $\{5.5\}$, Answer.**

EXAMPLE 2 **Eighteen-carat gold contains three times as much pure gold as copper. How much of each metal is there in 19.6 grains (gr.) of 18-carat gold?**

Solution:

1 Choose a symbol and use it in representing each described number.	**Let** p = **the number of grains of copper.** **Then** $3p$ = **the number of grains of pure gold.**
2 Form an open sentence by using the facts given in the problem.	$\underbrace{\text{copper gr.}}_{p} \quad \underbrace{\text{added to}}_{+} \quad \underbrace{\text{pure gold gr.}}_{3p} \quad \underbrace{\text{gives}}_{=} \quad \underbrace{\text{total gr.}}_{19.6}$
3 Find the solution set	$p + 3p = 19.6$ $4p = 19.6$ $p = 4.9$ **gr. copper** $3p = 3(4.9) = 14.7$ **gr. pure gold**
4 Use the words of the problem to *check* that *all* stated conditions are satisfied by the values you have found. Is there three times as much gold as copper? Is the total weight 19.6 gr.?	$14.7 \overset{?}{=} 3(4.9)$ **Yes, because** $14.7 = 14.7$ ✓ **Yes, because** $14.7 + 4.9 = 19.6$ ✓ ∴ **4.9 gr. copper, 14.7 gr. pure gold, Answer.**

EXAMPLE 3 **Solve** $n - \frac{3}{5}n + 6 = 27 - 9$

Solution:

$$
\begin{aligned}
\underbrace{n - \tfrac{3}{5}n}_{} + 6 &= 27 - 9 \\
\tfrac{2}{5}n \quad + 6 &= 18 \\
\tfrac{2}{5}n + 6 - 6 &= 18 - 6 \\
\tfrac{2}{5}n &= 12 \\
5 \cdot \tfrac{2}{5}n &= 5 \cdot 12 \\
2n &= 60 \\
\frac{2n}{2} &= \frac{60}{2} \\
n &= 30
\end{aligned}
$$

(*continued on page 88*)

Check:

$$30 - \tfrac{3}{5}\cdot 30 + 6 \stackrel{?}{=} 18$$

$$30 - 18 + 6 \stackrel{?}{=} 18$$

$$18 = 18 \quad \checkmark$$

∴ The solution set is $\{30\}$, **Answer.**

WRITTEN EXERCISES

Solve the following equations.

When a pupil's solution is incorrect, ask him to retrace his steps to determine whether he has erred numerically or misused a number property.

Ⓐ

1. $3x + 5x = 34 - 10$

2. $7n - 3n = 40 + 16$

3. $28 = 8x + 10x + 10$

4. $45 = 7x + 8x + 30$

5. $x - \frac{1}{2}x - 10 = 0$

6. $\frac{6}{5}v - v - 5 = 0$

7. $13a + 2a + 5a + 85 = 95$

8. $8b + 2b + 7b + 173 = 221$

9. $5.6x + 2.4x + 176 = 176$

10. $8.3y + 2.7y + 154 = 154$

11. $57 = 8y + 25$

12. $127 = 45t + 37$

13. $0 = 17h - 102$

14. $0 = 19n - 57$

15. $26 = 14 + 4n$

16. $36 = 17 + 6r$

17. $\frac{3}{4}p + \frac{1}{8} = 2$

18. $\frac{5}{8}m + \frac{3}{4} = 7$

19. $\frac{2}{3}z - \frac{2}{3} = 12$

20. $\frac{3}{4}k - \frac{3}{4} = 24$

21. $11.3n + 4.2n - 129.5 = 10$

22. $15.8x + .5x - 130.4 = 16.3$

23. $23.6x - x = 45.2$

24. $4.5a - a = 70$

25. $\frac{1}{4}z + \frac{1}{4}z = 3.4$

26. $\dfrac{k}{2} - \dfrac{k}{6} = 2.9$

27. $\frac{17}{6}m - \frac{5}{6}m = 1$

28. $\frac{3}{5}b + \frac{12}{5}b = 8$

Ⓑ

29. $3(x + 5) - 2x = 51 - 25$

30. $67 - 48 = 7(y - 3) - 6y$

31. $\dfrac{3n}{2} + \dfrac{5n}{3} - \dfrac{13n}{6} - \dfrac{2}{3} = \dfrac{5}{6}$

32. $\dfrac{3k}{4} - \dfrac{11k}{20} + \dfrac{4k}{5} - \dfrac{3}{4} = \dfrac{17}{20}$

33. $.3t + .2 = 1.7$

34. $.7t - .5 = 2.3$

35. $11.4 + 2n + 4n = 11.4$

36. $4.4 + 7x - x = 10.4$

37. $1.5r - r - 2.4 = 17.4$

38. $3h - 7.2 - .5h = 32.8$

39. $4a + 3 - \frac{1}{2}a = 10$

40. $4z + \frac{1}{3} - \frac{2}{3}z = 7$

Ⓒ

41. $z + 4 + 2z + 5 + 4z = 51$

42. $5a + 6 - 2a + 2 - a = 56$

43. $11 + 4b - 2b = 73 - 6$

44. $25 + 6a + a + 7 = 32$

45. $7n + 5 - 3 - n = 8$

46. $25s - 15 - 5s = 0$

47. $z + 7 + 3 - z = 10$

48. $2t + 5 - t + 1 - t = 6$

49. $y + 5 - y + 1 = 4$

50. $5x + 7 - 3x - 2x = 8$

The problems are assorted so that pupils will not seek to fit them into special patterns.

PROBLEMS

Solve these problems with variables, using the four steps identified on page 57. Make a sketch whenever you can do so.

A

1. The sum of twice a number and 16 is 86. Find the number.

2. The sum of three times a number and 17 is 98. Find the number.

3. If six times a number is diminished by 5, the result is 67. Find the number.

4. Find a number such that 17 less than twice the number is 109.

5. A tennis court 78 feet long is 6 feet longer than twice its width. What is the width?

6. A badminton court 44 feet long is 4 feet longer than twice its width. What is the width?

7. When sirloin steak was priced at 21 cents more per pound than round steak, Mrs. Carney bought 4 pounds of round steak and 2 pounds of sirloin steak, for $4.50. Find the cost per pound of the sirloin.

8. Mary sold 19 more women's sport coats at a special sale than half the number Pat sold. The two sold 157 coats. How many did each sell?

Find the numbers described in Problems 9–14.

9. Three less than 4 times the number is 325.

10. Twenty-one more than six times the number is 177.

11. Twice the number is increased by 17. The result is 27.6.

12. If four times the number is diminished by 13, the result is 31.8.

13. Five more than the sum of 3 times a number and 7 times the number is 385.

14. If the sum of 3 times a number and twice the number is decreased by 15, the result is 165.

15. The sum of two numbers is 78. If three times the smaller is increased by the larger, the result is 124. Find the smaller number.

16. The sum of two numbers is 121. When the larger number is added to 4 times the smaller, the sum is 235. Find the larger number.

17. Bob is twice as old as Emma; Kent is 16 years older than Emma. The sum of their ages is 60 years. Find the age of each.

18. A rectangular house lot is twice as long as it is wide. The sum of its four sides is 222 feet. Find the dimensions of the lot.

19. The length of a rectangular house is three times its width. The distance around the house is 192 feet. Make a sketch, and find the length and width of the house.

20. A farmer's rectangular hen yard is three times as long as it is wide. It is enclosed by 72 feet of chicken wire. What are its dimensions?

21. When hydrogen and oxygen unite to form water, the weight of the oxygen is eight times that of the hydrogen. How many grams of oxygen are in 126 grams of water?

22. Two boys publish a neighborhood news sheet. The boy owning the press is to receive twice as much of the profits as the other. How much will each receive when a week's profits are $1.05?

23. A girl bought a jacket and a skirt for $15. The jacket cost 1.5 times as much as the skirt. How much did each cost?

24. A quart of kerosene weighs 0.8 as much as a quart of water. If the combined weight of 1 quart of water and 1 quart of kerosene is 3.6 pounds, find the weight of (a) the water (b) the kerosene.

25. Helen had a hamburger sandwich and a glass of milk which totaled 495 calories. The milk contained half as many calories as the sandwich. Determine the number of calories in the (a) milk (b) sandwich.

26. The total weight of a space capsule put into orbit was 840 pounds. If the regulating devices in the capsule weighed twice as much as the container, while the recording equipment weighed half as much as the container, find the weight of each of the three parts.

B **27.** Information in numerical form is stored in the memory units of an electronic computer. The storage capacity of Computer *B* is twice that of Computer *A*, while Computer *C* has a capacity four times as great as that of Computer *B*. If the total storage capacity of the three machines is 22,000 words, find the capacity of each computer.

28. A certain type of concrete contains twice as much sand as cement, and two and a half times as much gravel as sand. How many kilograms of cement are needed to make 300 kilograms of the dry concrete mixture?

29. The longest side of a triangle is twice the shortest side. The third side of the triangle is 1 inch shorter than the longest side. If the perimeter of the triangle is 149 inches, how long is each side?

30. In a day, Machine *A* produces one and a half times as many cartons as Machine *B*. Machine *C* produces 100 more cartons than Machine *A*. If the total production is 5020, how many cartons does each produce?

31. Ted has 4 times as many dimes as nickels and half as many pennies as dimes. If he has $4.70, how many coins of each kind has he?

32. In a den, there are three lamps of equal size, a radio which uses one-third the number of watts used by a lamp, and a heater which uses fifteen times as many watts as a lamp. When all are in use, the total electrical power is 1.1 kilowatts. How many watts does the radio use? (1 *kilo*watt = *1000* watts)

3–8 Equations Having the Variable in Both Members

An equation such as $3t = 2t + 16$ differs from the open sentences you have met, since the variable appears in both members of the equation. Fortunately, transformation by addition or subtraction allows you to add to or to subtract from each member any product of a number and the variable, without changing the solution set.

Later you will extend this transformation by allowing any polynomial to be added to or subtracted from each member of the equation without changing the solution set.

EXAMPLE 1. Solve $3t = 2t + 16$

Solution:

$$\begin{aligned} 3t &= 2t + 16 \\ 3t - 2t &= 2t + 16 - 2t \\ t &= 16 \end{aligned}$$

Check:

$$\begin{aligned} 3(16) &\overset{?}{=} 2(16) + 16 \\ 48 &\overset{?}{=} 32 + 16 \\ 48 &= 48 \quad \checkmark \end{aligned}$$

$\therefore$ The solution set is $\{16\}$, Answer.

EXAMPLE 2. Solve $x + 4x - 8 = 6 + 2x + 1$

Solution:

$$\begin{aligned} \underbrace{x + 4x}_{} - 8 &= 6 + 2x + 1 \\ 5x - 8 &= 7 + 2x \\ 5x - 8 - 2x &= 7 + 2x - 2x \\ 3x - 8 &= 7 \\ 3x - 8 + 8 &= 7 + 8 \\ 3x &= 15 \\ \frac{3x}{3} &= \frac{15}{3} \\ x &= 5 \end{aligned}$$

Check:

$$\begin{aligned} 5 + 4 \cdot 5 - 8 &\overset{?}{=} 6 + 2 \cdot 5 + 1 \\ 5 + 20 - 8 &\overset{?}{=} 6 + 10 + 1 \\ 17 &= 17 \quad \checkmark \end{aligned}$$

$\therefore$ The solution set is $\{5\}$, Answer.

EXAMPLE 3. Solve $1 + 3(2s + 4) = 15 + 6s$

Solution:

$$\begin{aligned} 1 + 3(2s + 4) &= 15 + 6s \\ 1 + 3 \cdot 2s + 3 \cdot 4 &= 15 + 6s \\ 1 + 6s + 12 &= 15 + 6s \\ 6s + 13 &= 15 + 6s \\ 6s + 13 - 6s &= 15 + 6s - 6s \\ 13 &= 15 \end{aligned}$$

Check: For every replacement of the variable s, the equation will be converted into a *false* statement; thus, the equation has *no root*.

∴ The solution set is $\emptyset$, Answer.

Equations without roots have occurred earlier.

EXAMPLE 4. Solve $9d + 3(5 + 2d) = 15(d + 1)$

Solution:

$$\begin{aligned} 9d + 3(5 + 2d) &= 15(d + 1) \\ 9d + 3 \cdot 5 + 3 \cdot 2d &= 15 \cdot d + 15 \cdot 1 \\ 9d + 15 + 6d &= 15d + 15 \\ 15d + 15 &= 15d + 15 \end{aligned}$$

∴ The solution set is {all numbers}, Answer.

Any replacement for d converts this equation into a true statement. Such an equation is called an **identity.**

You may have introduced the term "identity" earlier, or you may prefer not to mention it.

EXAMPLE 5. Len and Vito together collected 98 pounds of waste paper in the clean-up drive. Vito announced, "Four pounds more than the amount I collected is exactly twice the amount Len collected." How much did each collect?

Solution:

1 Let p = number of pounds Vito collected

Then $98 - p$ = number of pounds Len collected

2

$$\underbrace{4}_{4}\quad \underbrace{\text{more than}}_{+}\quad \underbrace{\text{Vito's share}}_{p}\quad \underbrace{\text{is}}_{=}\quad \underbrace{\text{twice}}_{2}\quad \underbrace{\text{Len's share}}_{(98 - p)}$$

3

$$\begin{aligned}
4 + p &= 196 - 2p \\
4 + p + 2p &= 196 - 2p + 2p \\
4 + 3p &= 196 \\
4 + 3p - 4 &= 196 - 4 \\
3p &= 192 \\
p &= 64 \\
98 - p &= 34
\end{aligned}$$

4 The "check" is left for you. Show that Vito collected 64 lb., and Len, 34 lb.

ORAL EXERCISES

Solve each equation, first telling what term must be added to or subtracted from each member of the equation. A = add; S = subtract

1. $2x = 3 + x$ S x; x = 3
2. $3a = 8 + 2a$ S 2a; a = 8
3. $c = 9 - 2c$ A 2c; c = 3
4. $4m = 10 - m$ A m; m = 2
5. $3y = 8 - y$ A y; y = 2
6. $5r = 2r + 30$ S 2r; r = 10
7. $2x + 5 = 7x$ S 2x; 1 = x
8. $3z = 2z + 1$ S 2z; z = 1
9. $1.5x = .5x + 4$ S .5x; x = 4
10. $2.9t = 12 - .1t$ A .1t; t = 4
11. $6x - 9 = 5x$ A 9 and S 5x; x = 9
12. $4v - 10 = 3v$ A 10 and S 3v; v = 10
13. $7s = 7 - 7s$ A 7s; s = $\frac{1}{2}$
14. $18 = \frac{3}{4}r - \frac{1}{2}r$ 72 = r
15. $x = -5 + 3x$ A 5 and S x; $\frac{5}{2}$ = x
16. $11b = 6 + 5b$ S 5b; b = 1
17. $4 + 2a = 14$ S 4; a = 5
18. $5z = 2z + 9$ S 2z; z = 3

WRITTEN EXERCISES

Solve each equation.

A

1. $7v = 45 + 2v$
2. $9u = 6u + 39$
3. $6l = 18 - 3l$
4. $5b = 28 - 2b$
5. $11x = 8 + 5x$
6. $4y = 8 + 2y$
7. $12 - 3r = 3r$
8. $7h - 35 = 0$
9. $9x - 24 = 3x$
10. $5 + 2b = 7b$
11. $8a = 5a + 18$
12. $4x + 18 = 10x$
13. $7y - 9 = 2y$
14. $b + 28 = 6b$
15. $5x - 6 = 3x$
16. $12 - 3h = 9$
17. $5x - 5 = 13$
18. $12 - y = 5y$

B

19. $4z + 2 = 2z + 8$
20. $5w + 2 = w + 7$
21. $7c - 7 = 15 - c$
22. $12a - 3 = 4 - 2a$
23. $4r + 2 = 2r + 2$
24. $7f - 1 = 29 + f$
25. $8u + 2 = 13 - 3u$
26. $16 + 4y = 10y - 20$

27. $19r + 4 = 19 + 14r$

28. $5x + 1 = 4x + 2$

29. $24x - 24 = 12 + 2x$

30. $12n + 8 = 18 - 6n$

31. $5y + 10 = 6 + 6y$

32. $10w + 6 = 504 + 8w$

C

33. $6 + 4(2 - t) = 3t$

34. $7m + 5(3 - m) = 19$

35. $12\left(\frac{x}{3} - \frac{1}{2}\right) = x + 21$

36. $\frac{4}{5}(a + 10) = a$

37. $3t + 4 = 3(t + 2)$

38. $5 - b = b + 5$

39. $10 + 18x - 2 = 2x + 12 + 4x$

40. $9y + 3 - 2y = 12 - 6y + 4$

41. $4(b + 1) + 9 = 2(3b - 4) + b$

42. $5(2y + 3) - 4y = 3(2 + y) + 39$

43. $\frac{2}{5}(x + 1) + \frac{3}{5}(x + 1) = 14$

44. $2(t + 4) - 3 = \frac{1}{2}(10 + 4t)$

PROBLEMS

Use variables to solve these problems. Make a sketch when possible.

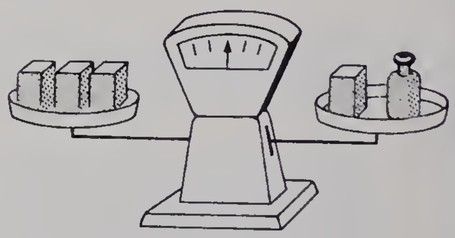
A

1. Paul said: "I am thinking of a number which equals its double decreased by 1. What is the number?"

2. Roger asked: "Can you tell me the number I am thinking of? When I multiply it by 2 and then add 3, I get the same answer as when I multiply it by 3 and then subtract 2."

3. Dividing a certain number by 2 yields the same result as subtracting 15 from 3 times the number. Find the number.

4. Tom has a set of blocks, each having the same weight. He found that if he put three of the blocks in one pan of a beam balance, and put one block together with a 7-pound weight in the other pan, then the pans would balance each other. How much did each block weigh?



5. Harry earns three times as much per week as does Tom, while Dick earns $80 a week more than Tom. If Dick and Harry have the same salary, how much does each of the three men earn?

6. The sum of two numbers is 46. Five times the smaller number is 6 more than twice the larger. Find the numbers.

7. Bill is twice as old as Mary. If he is also exactly 10 years older than Mary was last year, how old are Bill and Mary?

8. Divide 73 into two parts, such that twice the larger number is 4 less than three times the smaller.

9. Sam challenged: "Tell me my number. When I subtract 3 from it and then multiply the result by 2, I get the same result as when I divide my number by 2 and then add 18 to the quotient."

10. A deluxe ball-point pen costs $1 more than the standard model. Peter bought 3 standard pens and 5 deluxe pens. His bill totaled $6.20. What is the cost of each model of the pen?

11. The length of one rectangle is 2 feet more than the length of a second rectangle. The width of the first rectangle is 3 feet; the width of the second is 7 feet. If the total area of the two rectangles is 121 square feet, determine the length of each rectangle.

12. The sum of the length and width of a rectangle is 42 inches. Twice the length is 1 inch less than 3 times the width. Find the dimensions of the rectangle.

13. A purchase of 50 stamps, some costing 7 cents each and the rest 4 cents each, cost $2.15. How many of each kind were bought?

14. The base of a triangle has the same length as a side of a square. The second side of the triangle is 2 inches longer than the base, and the third side is 6 inches longer than the base. If the perimeter of the triangle equals that of the square, find the longest side of the triangle.

15. A company has 4 large buses and 5 small ones. Each large bus has 12 seats more than a small bus. The total seating capacity of all the buses is 336. How many seats are in a large bus?

16. Ten less than the sum of 4 times a number and 2 times the number is 260 more than 4 times the number. Find the number.

B

17. Twice the sum of half a number and 3 times the number is 27 more than half of 5 times the number. Find the number.

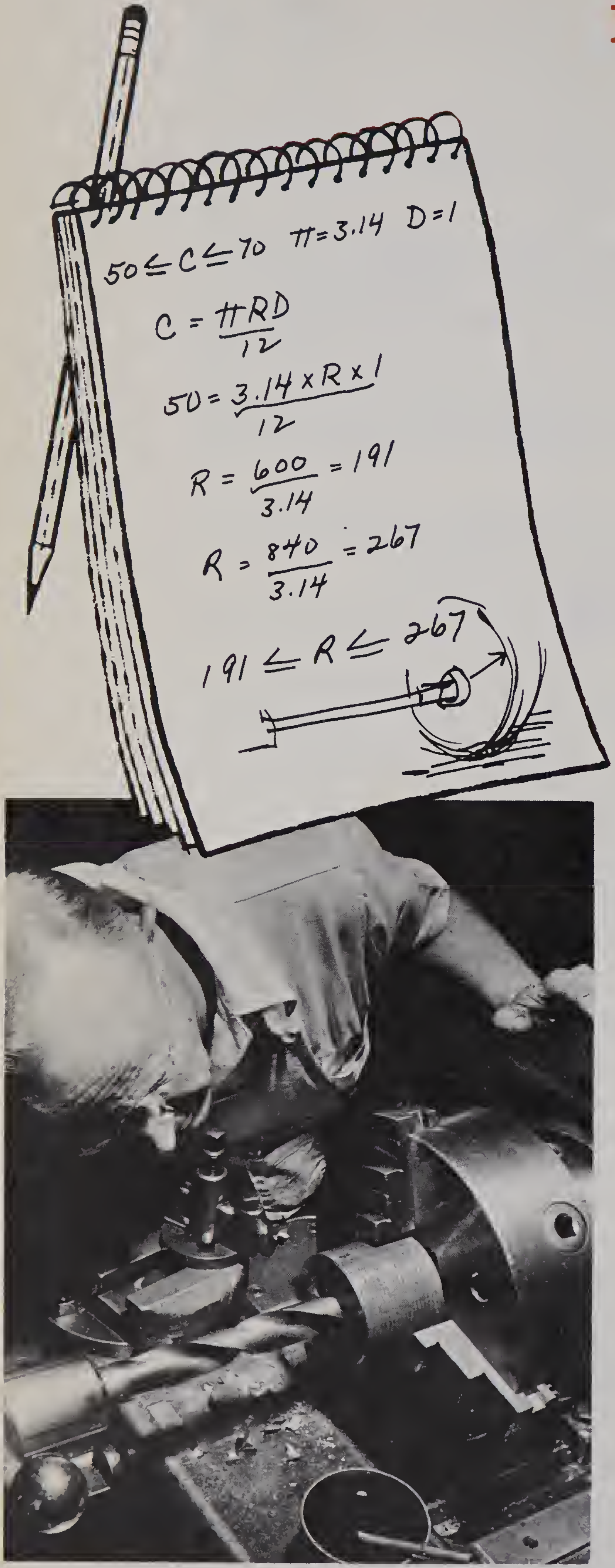
18. The temperature in Omaha was twice the temperature in Juneau. If the temperature in Juneau were increased by 15° and the temperature in Omaha decreased by 15°, the temperatures would have been the same. What were the temperatures in those cities?

19. A baseball bat costs half a dollar more than a ball. When the team bought 2 bats and 5 balls, it paid $11.50. Find the cost of a bat.

20. Nine pounds of potatoes cost the same as 6 pounds of apples. At the same time 1 pound of potatoes costs twice as much as a pound of onions, while 1 pound of apples costs 8 cents more than a pound of onions. Determine the cost of one pound of each commodity.

21. "I am 3 times as old as Dave is," said Frank to Kay. "On the other hand, I am 15 years older than Joe, while Dave is 1 year younger than Joe. How old are Joe, Dave, and I?"

Machinists and Mathematics



Machinists are skilled mechanics who cut, file, and grind metals to shape them into specified sizes and forms. Working closely with engineers and scientists, machinists must be able to understand blueprints and mathematical equations in order to make tools and parts to exact specifications.

The example illustrated is typical of the machinists' problems involving the use of mathematics. The photograph shows a machinist using a boring mill to rough cut a collar for a steel shaft. In such an operation the rotation speed of the lathe, which turns the cylinder to be cut against the stationary drill, must be carefully set. Cutting too rapidly would burn the steel, while cutting too slowly would be inefficient and might produce a jagged edge.

To determine the proper lathe speed, the machinist performs the calculation illustrated on the work pad. The number of revolutions per minute (*R*) is given by the equation

$C = \frac{\pi RD}{12}$, in terms of the optimum cutting speed (C) in feet per minute for the materials being used and the diameter (*D*) in inches of the drill.

In this example, the machinist is using a drill made of tool steel to cut a machine-steel cylinder. The optimum cutting speed (C) for this combination of metals is 50–70 ft. per minute. The diameter (*D*) of the drill is one inch. By first substituting 50 for C, the machinist finds that the lathe speed should be at least 191 revolutions per minute (r. p. m.). Then, substituting 70 for C, he determines that the speed of the lathe should not exceed 267 r. p. m. As shown, the speed should be set somewhere between 191 and 267.

Chapter Summary

Inventory of Structure and Method

1. The use of the equality sign is governed by these assumptions:

   The **reflexive property:** Any number is equal to itself.

   The **symmetric property:** The members of an equation may be interchanged.

   The **transitive property:** If two numerical expressions are both equal to a third expression, the two are equal to each other.

   The **axiom of closure** states that the sum and product of every pair of arithmetic numbers are unique arithmetic numbers.

2. Arithmetic numbers have properties involving operations. The **commutative property for addition:** For each a and each b, $a + b = b + a$. **The commutative property for multiplication:** For each a and each b, $ab = ba$. The commutative properties for addition and multiplication show that you get the same sum or same product of two numbers, no matter what order you use.

   The **associative property for addition:** $a + (b + c) = (a + b) + c$ for each a, b, and c. The **associative property for multiplication:** For every a, b, and c, $a(bc) = (ab)c$. The associative properties for addition and multiplication show that you get the same sum or same product of three numbers, no matter which adjacent numbers are grouped.

   The **distributive property for multiplication with respect to addition or subtraction:** For each a, each b, and each c, $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$. These properties are used in simplifying expressions, especially in combining **similar terms.**

   The numbers 1 and 0 have special properties. The **multiplicative property of 1 (multiplicative identity):** For every a, $1 \cdot a = a \cdot 1 = a$. The **additive property of 0 (additive identity):** For every a, $0 + a = a + 0 = a$. The **multiplicative property of 0:** For every a, $0 \cdot a = a \cdot 0 = 0$. As a result of the multiplicative property of zero, *you may not divide* by zero.

   The **substitution principle** states: If two numerical expressions are equal, one may be substituted for the other.

3. Properties of equality involving operations can be derived from previous axioms and properties. In the following four properties, a, b, and c represent any three numbers.

   Addition property of equality: if $a = b$, then $a + c = b + c$. **Subtraction property of equality:** if $a = b$, then $a - c = b - c$, provided $a - c$ represents an element in the set. **Multiplicative property of equality:** if $a = b$, then $ac = bc$. **Division property of equality:** if $a = b$, then

$\frac{a}{c} = \frac{b}{c}$, if c is not zero. Thus, if equal numbers are added to equal numbers, or subtracted from equal numbers, or multiplied by equal numbers, or divided by equal nonzero numbers, the results are equal.

4. You solve an equation by transforming it into an **equivalent equation,** repeating this process until you derive an equation which shows the solution set. You may transform an equation **by addition, by subtraction, by multiplication by a nonzero number,** and **by division by a nonzero number.** Each time you transform an equation, you undo an indicated operation by using the **inverse operation.** To **check** the tentative value, you **substitute** it for the variable in the **original equation** and see whether it satisfies the equation by evaluating each member separately. If the result is the same for each member, the equation is true for the value substituted, and this value is a root of the equation, or an element in its solution set. Suggested steps in solving an equation are as follows: (a) Simplify by carrying out indicated multiplications where possible; combine similar terms in each member. (b) Undo any indicated addition or subtraction by using the inverse operation. (c) Undo any indicated multiplication or division by using the inverse operation. (d) Check the tentative answer in the original equation.

Vocabulary and Spelling

assumption (*p. 69*)
axiom (*p. 69*)
postulate (*p. 69*)
reflexive property (*p. 69*)
symmetric property (*p. 69*)
transitive property (*p. 70*)
closure (*p. 71*)
substitution principle (*p. 71*)
commutative properties (*p. 73*)
associative properties (*p. 73*)
distributive property (*p. 76*)
similar terms (*p. 77*)
unlike terms (*p. 77*)
multiplicative properties (*p. 77*)
identity element (*p. 77*)
additive property (*p. 77*)
proof (*p. 80*)
equivalent equations (*p. 81*)
transformation (*p. 81*)
check (*p. 81*)
inverse operation (*p. 81*)
identity (*p. 92*)

Chapter Test

3–1 Which property of equality does each of these examples illustrate?

1. $\frac{217}{5} = 43\frac{2}{5}$

2. If $x = y$, then $y = x$.

3–2 State whether the given set is closed under the indicated operation.

3. $\{1, 2\}$, multiplication

4. $\{1, 3, 5, 7, 9, \ldots\}$, addition of any three elements

3–3 In the chains of equalities of Exercises 5–7, give a property of numbers that justifies each lettered step. For each n,

5.

$$\begin{aligned} 4 + (n + 7) &= 4 + (7 + n) && \text{(a)} \\ &= (4 + 7) + n && \text{(b)} \\ &= 11 + n && \end{aligned}$$

6.

$$\begin{aligned} 4 \times (n \times 7) &= 4 \times (7 \times n) && \text{(a)} \\ &= (4 \times 7) \times n && \text{(b)} \\ &= 28n && \end{aligned}$$

3–4 For each r,

7.

$$\begin{aligned} 5(r + 1) + 3(r + 1) &= 5 \times r + 5 \times 1 \\ &\qquad + 3 \times r + 3 \times 1 && \text{(a)} \\ &= 5 \times r + 5 + 3 \times r + 3 && \text{(b)} \\ &= 5 \times r + 3 \times r + 5 + 3 && \text{(c)} \\ &= (5 + 3)r + 5 + 3 && \text{(d)} \\ &= 8r + 8 && \\ &= 8(r + 1) && \text{(e)} \end{aligned}$$

8. Simplify and combine like terms: $3x + 4(x + 2) - 5x - 8$.

9. Find the value of $\frac{3}{7}(39) + 9\frac{4}{7}(39)$.

3–5 Solve: **10.** $s - 73 = 91$ **11.** $42 = t + 18$

3–6 Solve: **12.** $7y = 28$ **13.** $\frac{3}{2}v = 48$

3–7 **14.** State whether the indicated value of the variable is a root of the equation. Show work. $\frac{2}{5}m - 4 = 3 - .3m$; $m = 10$

15. In a Junior Achievement enterprise, Ben invests three times as much as Amy. At the end of the year, the total net profit is \$46.28. Find Ben's proportionate share of the profit.

16. Solve: $3d + 2(6d - 5) = 5$

17. A Bonrite pen and pencil set costs \$2.78, the pen costing \$.80 more than the pencil. Find the cost of each.

3–8 **18.** Solve: $3(2h + 7) - 5 = 2(5h - 4) + 4h$

19. The length of a rectangle is 3 inches less than the length of a smaller rectangle. The larger rectangle is 9 inches wide; the smaller is 4 inches wide. If the area of the larger rectangle is 48 square inches more than the area of the smaller, find the length of the larger rectangle.

Chapter Review

3–1 Axioms of Equality *Pages 69–70*

1. $a + b = a + b$ illustrates the __?__ property of equality.

2. If $b = d$, then $d = b$ illustrates the __?__ property.

3. $23 - 15 = 8$, $2 \cdot 4 = 8$, $23 - 15 = 2 \cdot 4$ illustrates the __?__ property.

3–2 The Closure Properties *Pages 70–73*

4. The sum of two numbers of arithmetic will be a __?__.

5. A set R is closed under multiplication if the product of any two of its elements is an __?__ of R.

6. The set of numbers of arithmetic is closed under __?__ and __?__.

State whether each given set is closed under the indicated operation.

7. $\{3, 6, 9, 12, \ldots\}$, division **8.** $\{6, 4, 2, 0\}$, subtraction

3–3 Commutative and Associative Properties of Arithmetic Numbers *Pages 73–75*

9. You know that $4n + 1 = 1 + 4n$ because of the __?__ property for __?__.

10. For any numbers m and n, $mn =$ __?__ because multiplication is __?__.

11. You know that $8 \div 4 \neq 4 \div 8$ because division does not have the __?__ property.

12. $(8 + 5) + 4 = 8 + 9$ because addition is __?__.

13. $(157 \times 4) \times 25 = 157 \times 100$ because multiplication is __?__.

14. $16 - (8 - 2) \neq (16 - 8) - 2$ because subtraction is not __?__.

3–4 The Distributive Property; Special Properties of 1 and 0 *Pages 75–79*

15. $5(x - y) = 5x - 5y$ because of the __?__ property for multiplication with respect to __?__.

16. When you combine $4w + w$ to get $5w$, you are using the __?__ property.

17. The expression $4 \cdot 1 \cdot 1 = 4$ because of the __?__ property of 1.

18. $0 \cdot 5 = 0$ because of the __?__ property of __?__.

19. When the product of several factors is 0, at least one of the factors must be __?__.

20. Zero, or any expression whose value is __?__, may never be used as a __?__.

In the following chain of equality, give a property of numbers that justifies each lettered step. For each n,

21.
$$\begin{aligned}
6n - 6n + 3 \times (4 \times \tfrac{1}{3}) &= n(6 - 6) + 3 \times (4 \times \tfrac{1}{3}) && \text{(a)}\\
&= n \cdot 0 + 3 \times (\tfrac{1}{3} \times 4) && \text{(b)}\\
&= 0 + 3 \times (\tfrac{1}{3} \times 4) && \text{(c)}\\
&= 0 + (3 \times \tfrac{1}{3}) \times 4 && \text{(d)}\\
&= 0 + 1 \times 4 && \text{(e)}\\
&= 0 + 4 && \text{(f)}\\
&= 4 && \text{(g)}
\end{aligned}$$

Simplify by combining similar terms.

22. $7r + 3(2r + 1) - 13r - 2$ **23.** $2k + 3[5 + 2(k - 1)]$

3-5 Addition and Subtraction Properties of Equality *Pages 80–83*

24. The properties of equality involving operations are not axioms because they can be __?__.

25. Equations having the same solution set are __?__ equations.

26. The expression on one side of an equation is called a __?__ of the equation.

27. Solve $0 = h - 6$. **28.** Solve $n + 8 = 8$.

29. Check a root by __?__ it for the variable in the given equation.

30. In checking, evaluate each __?__ of the equation separately.

31. Check an answer to a problem by showing that it satisfies the __?__ of the problem.

3-6 Division and Multiplication Properties of Equality *Pages 83–85*

32. To find x, __?__ each member of the equation $5x = 3$ by __?__.

33. The inverse of subtraction is __?__; the inverse of __?__ is division.

34. You may not multiply each member of an equation by 0 because the resulting equation will not be __?__ to the original equation.

35. Solve $12k = 3$. **36.** Solve $18 = \frac{2}{3}m$.

3–7 Combining Terms and Using Transformation Principles

Pages 86–91

37. Solve $9t - 3t = 9 + 3$. **38.** Solve $1 + y = \frac{4}{3}$.

39. The sum of a number n and 6 times that number may be represented by $n +$ _?_, or _?_.

Solve.

40. Three times a number decreased by half the number gives 10. Find the original number.

41. Undo indicated _?_ and _?_ before considering multiplications and divisions.

42. $27 = 7.2a - 5 + .8a$ **43.** $38 + 7b - 26 + 5b = 16\frac{4}{5}$

44. A man is 2 years younger than three times his daughter's age. Their ages total 50 years. Find the age of each.

3–8 Equations Having the Variable in Both Members

Pages 91–95

45. When an equation has the variable in each member, transform it into an equation containing the variable in only _?_ member.

Solve.

46. $2n + 7 = 3(2n - 1) + 6$ **47.** $\frac{2}{3}(9r - 4) = r + 3(r + \frac{2}{3})$

48. A rectangle is 6 inches wide. If a rectangular strip 4 inches long were cut from the end, the area of the remaining rectangle would be $\frac{3}{4}$ of the original area. Find the dimensions of the original rectangle.

49. At a County Fair, 200 ice cream cones were sold in one day, some at 15 cents each, the rest at 10 cents. If the proceeds from the sale of cones were $23.75, how many of each were sold?

Cumulative Review: Chapters 1–3

State whether each of the following sets is (a) finite, or (b) infinite.

1. {points on a circumference}

2. {all the trees in the world}

3. {odd numbers between 3 and 5}

4. {1970, 1980, 1990, . . .}

If the set below is given by a roster, specify it by a rule; if given by a rule, specify the set by a roster.

5. {the states in the U.S.A. whose names begin with the letter K}

6. $\{5, 11, 17, 23, \ldots 47\}$ **7.** $\{1, 3, 9, 27, 81\}$

Draw the graph of each set described below.

8. {the numbers between 2 and 4.5, inclusive}

9. {the numbers between 1 and $2\frac{1}{2}$}

10. {the numbers greater than or equal to 3}

11. Simplify $8 \div 8 \times 2$. **12.** Simplify $3[9 - 2(1 + 1.2)]$.

Given the sets: $A = \{0, 2, 6, 10, 20\}$ and $B = \{1, 2, 3, 9, 12\}$, find the set specified as follows:

13. {the elements in A or B, or in both A and B}

14. The subset of B containing all multiples of 3

15. The set of those elements which are in both A and B

In each case, give the property which makes the conclusion true.

16. Given: $b = m; m = 6$
Conclusion: $b = 6$

18. Given: $3 + x = 5x - 1$
Conclusion: $5x - 1 = 3 + x$

17. $5(s - 2) = 5s - 10$

19. $5(t + 2) = (t + 2)5$

In the following chain of equalities, give the property that justifies each lettered step. For each n,

20.
$$\begin{aligned}\{2(5n + 1)\} - 10n &= \{10n + 2\} - 10n &&\text{(a)}\\ &= \{2 + 10n\} - 10n &&\text{(b)}\\ &= 2 + \{10n - 10n\} &&\text{(c)}\\ &= 2 + n(10 - 10) &&\text{(d)}\\ &= 2 + n \cdot 0 &&\text{(e)}\\ &= 2 + 0 &&\text{(f)}\\ &= 2 &&\text{(g)}\end{aligned}$$

Simplify and combine like terms.

21. $7n + 3[4n + 5(2 - n) - 3]$ **22.** $\frac{11}{9}(d - 4) - \frac{2}{9}(d - 4)$

Graph the solution set of each sentence.

23. $2x - 1 \neq 5$ **24.** $3r \geq 12$

State whether each of the given sets is closed under the indicated operation.

25. $\{0, 2, 4, 6, \ldots\}$, multiplication by 3 **26.** $\{4, 3, 2, 1\}$, subtraction

If $a = 1$, $b = 3\frac{1}{2}$, $c = 4$, and $d = 0$, find the value of each expression.

27. $4(b + c) - 3d(a + b)$

28. $\dfrac{2b + c}{2b - c}$

29. $\dfrac{3a^4 + 3c^2 - 2}{4b^2}$

30. $16(c - b)^3$

Express in algebraic symbols.

31. Twice a number n, decreased by 3

32. One-fifth of the sum of a and b

33. The larger of two numbers when it is 5 more than the smaller, s

34. The larger of two numbers when their sum is 50 and the smaller is x

35. The square of the sum of two numbers (R and r)

36. Twice the sum of the squares of two numbers (R and r)

If the replacement set for n is $U = \{0, 1, 2, 3\}$, find the subset of U whose elements make each of the following sentences true.

37. $2n - 1 = 5$

38. $2(n - 1) = 5$

39. $3n > 0$

40. $4n + 1 \geq 6$

41. $n(5n) = n \cdot 5 \cdot n \cdot n$

42. $n(n - 1) = n \times n - n$

Write an algebraic expression for each of the following.

43. **a.** A whole number that is 5 more than a given number, t

b. Give the replacement set for t.

44. The third side of a triangle whose perimeter is 27, and whose second side is twice the first, a

Solve each equation.

45. $8m - 13 = 67 + 3m$

46. $7s - 4 - s = 31 + 2s - 5$

47. $\frac{3}{4}(5y - 4) = 13 + \frac{1}{4}(6y + 8)$

48. $\frac{2}{3}w + 1 + \frac{5}{3}w = \frac{13}{3}w - 1$

49. $3(2t + 5) - 7 = 9t - 4$

50. $12 + 3(2k - 1) = 2(3k + 1)$

Solve each problem.

51. Find a number such that the difference of twice the number and two-thirds of the number is 68.

52. The sum of two numbers is 20. Four times the larger is 1 less than five times the smaller. Find the numbers.

53. For Mother's Day, Tony bought three kerchiefs. For Father's Day, he bought two ties. If a tie cost 50 cents more than a kerchief and Tony spent a total of \$5.95, how much did a tie cost?

Extra for Experts

The Algebra of Logic and Sets

Consider the following compound sentences:

$$3 < x < 5$$
$$x \leq 4$$

Each of these may be broken into two simple sentences:

$3 < x < 5$ means $3 < x$ *and* $x < 5$
$x \leq 4$ means $x < 4$ *or* $x = 4$

Do you see the essential difference in the nature of the limitations implied by these two different compound sentences? The one containing *and* is called a **conjunction**; for it to be true, x must satisfy both simple statements. The one containing *or* is called a **disjunction**; it is true if x satisfies at least one of the simple statements.

If you limit the domain of the variable to $\{0, 1, 2, 3, 4, 5\}$ and consider the solution sets (truth sets) of each simple sentence and that of the compound sentence, you will discover a relationship between conjunctions of sentences and intersections of sets, and between disjunctions of sentences and unions of sets.

Sentence	*Truth Set*	*Sentence*	*Truth Set*
$3 < x$	$\{4, 5\} = A$	$x < 4$	$\{0, 1, 2, 3\} = C$
$x < 5$	$\{0, 1, 2, 3, 4\} = B$	$x = 4$	$\{4\} = D$
$3 < x < 5$	$\{4\} = A \cap B$	$x \leq 4$	$\{0, 1, 2, 3, 4\} = C \cup D$

A special notation is sometimes used to write compound sentences. The sign of conjunction is $\wedge$ (a capital A without the bar) and is read *and*. The sign of disjunction is $\vee$ and is read *or*. The above examples would be written thus:

$$3 < x \wedge x < 5$$
$$x < 4 \vee x = 4$$

The symbol for conjunction resembles that for intersection, and the truth set of a conjunction of simple sentences is the intersection of their truth sets. Further, the symbol for disjunction resembles that for union, and the truth set of a disjunction of simple sentences is the union of their truth sets.

Can you see that the truth of a compound sentence is determined by the truth of its component simple sentences? Relationships between them can be displayed in the form of a truth table. If you will consider p and q as standing for simple sentences, the truth table for the disjunction and the conjunction of two simple sentences takes this form.

If p represents $6 < 7$ and q represents $7 < 8$, both p and q are true, their conjunction is true, and their disjunction is true. This fits the first line of the truth table.

TRUTH TABLE

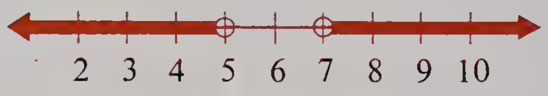
p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

If p represents $7 = 5 + 2$ and q represents $6 = 3(3)$, p is true, q is false, their conjunction is false, and their disjunction is true. This fits the second line of the truth table. Can you illustrate the third and fourth lines?

Questions

Solve these compound sentences, and graph each truth set.

SAMPLE 1. $(x < 5) \vee (x > 7)$



Solution: {all numbers less than 5} ∪ {all numbers greater than 7} = {all numbers except those between 5 and 7, inclusive}

1. $(x = 1) \vee (x > 1)$

2. $(x = 1) \wedge (x > 1)$

3. $(x > 2) \vee (x < 4)$

4. $(x > 2) \wedge (x < 4)$

5. $(2x > 0) \vee (x \in \{\text{whole no.}\})$

6. $(2x > 0) \wedge (x \in \{\text{whole no.}\})$

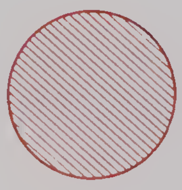
Let a, b, and c represent the sentences: x is a member of set A; x is a member of set B; and x is a member of set C, respectively. By drawing Venn diagrams verify the properties of conjunction and disjunction for these sentences.

SAMPLE 2. Disjunction is distributive over conjunction; that is, $c \vee (b \wedge a)$ has the same truth set as $(c \vee b) \wedge (c \vee a)$.

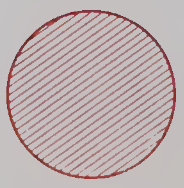
Solution: Rewrite the expressions in terms of their truth sets:

$$C \cup (B \cap A) \overset{?}{=} (C \cup B) \cap (C \cup A)$$

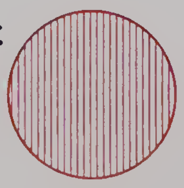
Mark C:



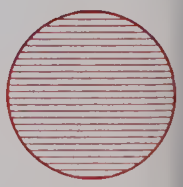
$B \cap A$:

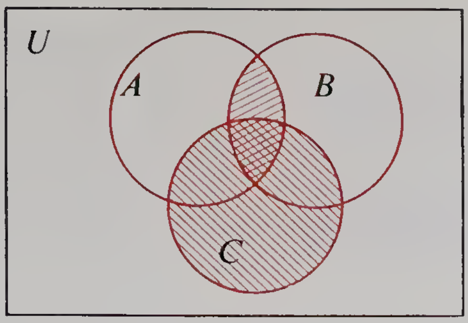


$C \cup B$:



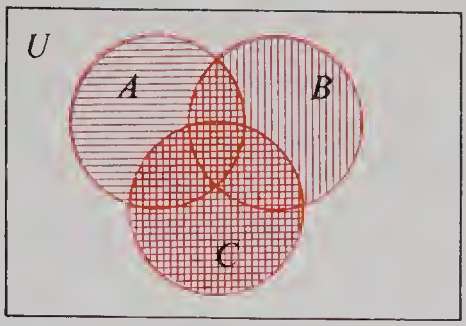
$C \cup A$:





$C \cup (B \cap A)$:

Entire region marked



$(C \cup B) \cap (C \cup A)$

Region marked: ▦

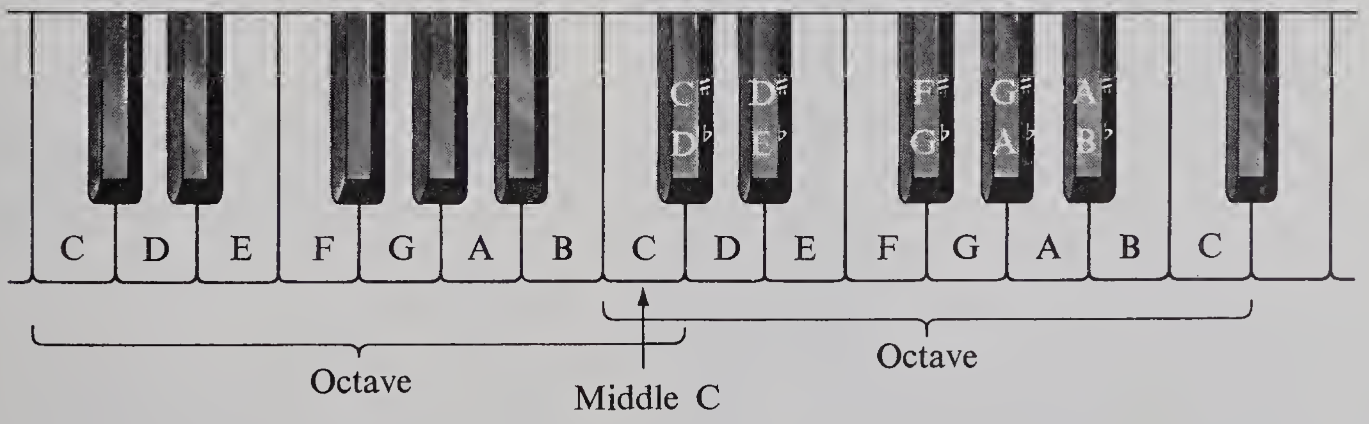
These two regions are the same.

7. Conjunction is distributive over disjunction; that is, $c \wedge (b \vee a)$ has the same truth set as $(c \wedge b) \vee (c \wedge a)$.
8. Disjunction is associative; that is, $c \vee (b \vee a)$ has the same truth set as $(c \vee b) \vee a$.
9. Conjunction is associative; that is, $c \wedge (b \wedge a)$ has the same truth set as $(c \wedge b) \wedge a$.
10. The truth set of $(a \vee b) \wedge (a \wedge b)$ is the same as that of $(a \wedge b)$.

Just for Fun

Some Musical Mathematics

The piano keyboard (figure below) consists of eighty-eight keys. For every seven white keys, there are five black keys.

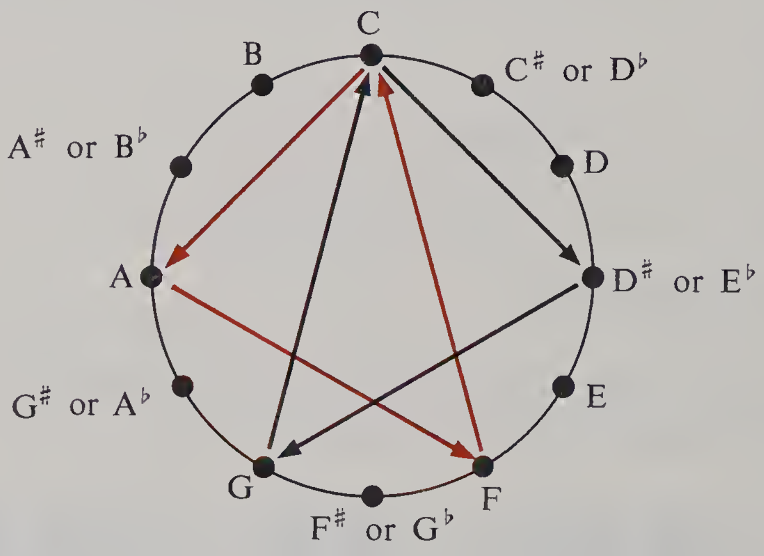


Vibrating strings produce the tones of a piano. The A next above middle C (which is the key used to tune a piano) vibrates 440 times per second; the A one octave higher (to the right) vibrates 880 times per second; and the A one octave lower vibrates 220 times per second. Notice the relation between these numbers: 2 × 220 = 440 and 2 × 440 = 880. By making use of it, you can calculate the number of vibrations per second of any A on the keyboard.

The next note to A is the black key for A-sharp (A♯) or B-flat (B♭). It vibrates 1.06 times as rapidly as A. And the next white key, B, vibrates 1.06 times as rapidly as A♯. A similar relation exists for each key if the piano is so tuned; for example, F vibrates 1.06 times as rapidly as E. Why would this be the correct number to use? How can you now calculate the number of vibrations of any key on the piano?

You know that some combinations of notes are harmonious and that some are discordant. Harmony and discord are related to the vibrations of the notes that are sounded together. To find some pleasing combinations without using a musical instrument, first draw a circle and divide its circumference into twelve parts. Each division represents one of the twelve notes from C to B. Label the divisions to correspond to these notes. (See figure below.)

Place your pencil on point C of your "tone circle." Go three divisions counterclockwise to A. Then go four more divisions counterclockwise to F. It is now five divisions back to C. The notes C, A, F, C make a harmonious chord when you strike them together. So do any other notes determined by going counterclockwise according to the pattern 3:4:5. Find some of these chords by using your tone circle, and then try them on a piano. Chords found by the scheme just used are *major chords*.



Harmonious chords may also be identified by proceeding in a clockwise direction according to the same 3:4:5 pattern. Thus C, E♭, G, C is a pleasing combination of tones, as you will find if you sound it on a piano. It is a *minor chord*. So is any other chord found in the same way.

Identify some minor chords; then listen to their harmony. Next sound some major chords. Do minor chords in general have a different effect on you from that of major chords? You can account for the difference if you have calculated how many times each note vibrates. Compare the numbers of vibrations of the notes in a chord. You will find that the relation of these numbers to one another is the same in every major chord you identified from your tone circle. It is the same in every minor chord you identified. But it is different in major chords from what it is in minor chords.

Murder by a Mob

Hypatia was beautiful. She was also a brilliant mathematician. Her lectures at the University were attended by the most learned men of her era.

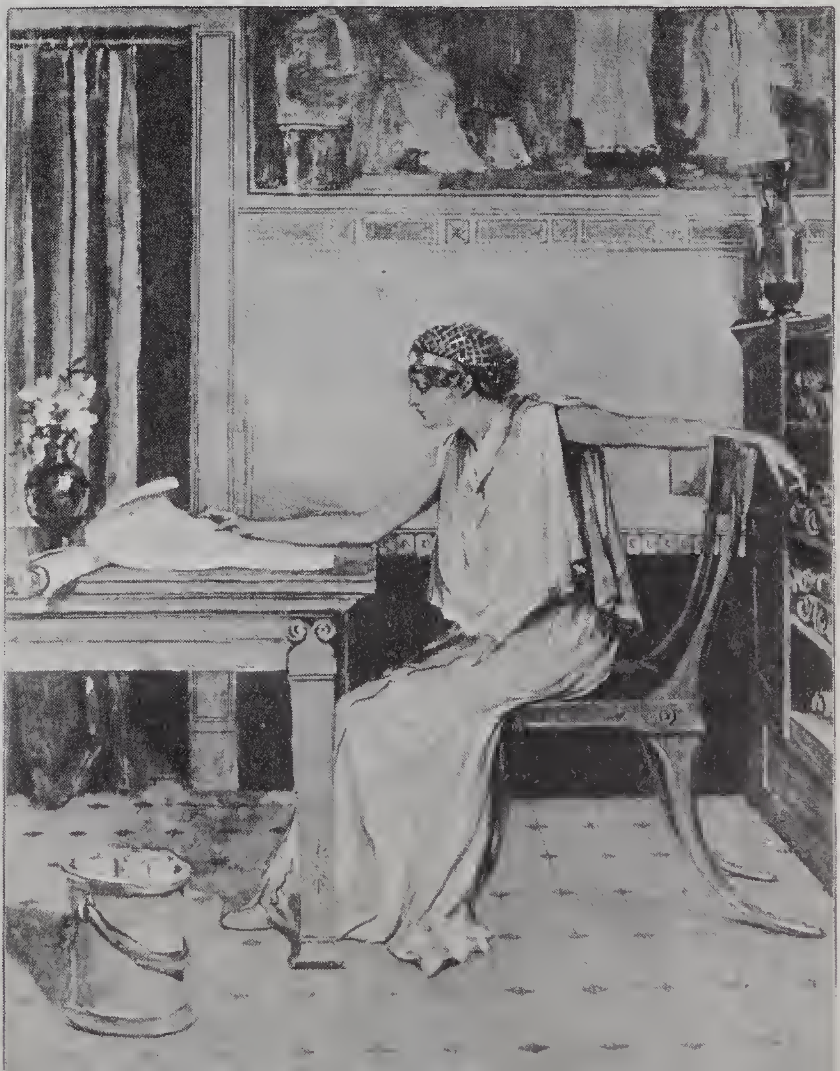
Until Hypatia's time, few Greek scholars were interested in algebra. Most of them preferred the study of geometry. But one, named Diophantus, had written on algebra. Hypatia studied his work and wrote discussions and explanations of it. Perhaps it was she who told the only story known of Diophantus' life; it dates back to her time, and clearly it is a tale told by a mathematician:

> God granted him childhood for a sixth part of his life, and youth for a twelfth part. He lit him the light of wedlock after a seventh part more, and five years after his marriage he granted him a son. Alas, late-born child! After attaining the measure of half his father's life, cruel Fate overtook him [that is, he died], thus leaving to Diophantus during the last four years of his life only such consolation as mathematics can offer.

Hypatia lived in troubled times — 400 A.D. Her home was in the city of Alexandria, a part of the crumbling Roman Empire. Alexandria was inhabited by Romans, Greeks, Egyptians; by Christians, Jews, pagans; by slaves, freedmen, and freeborn. And great numbers in each group were unlearned and ignorant, easily stirred to mob violence.

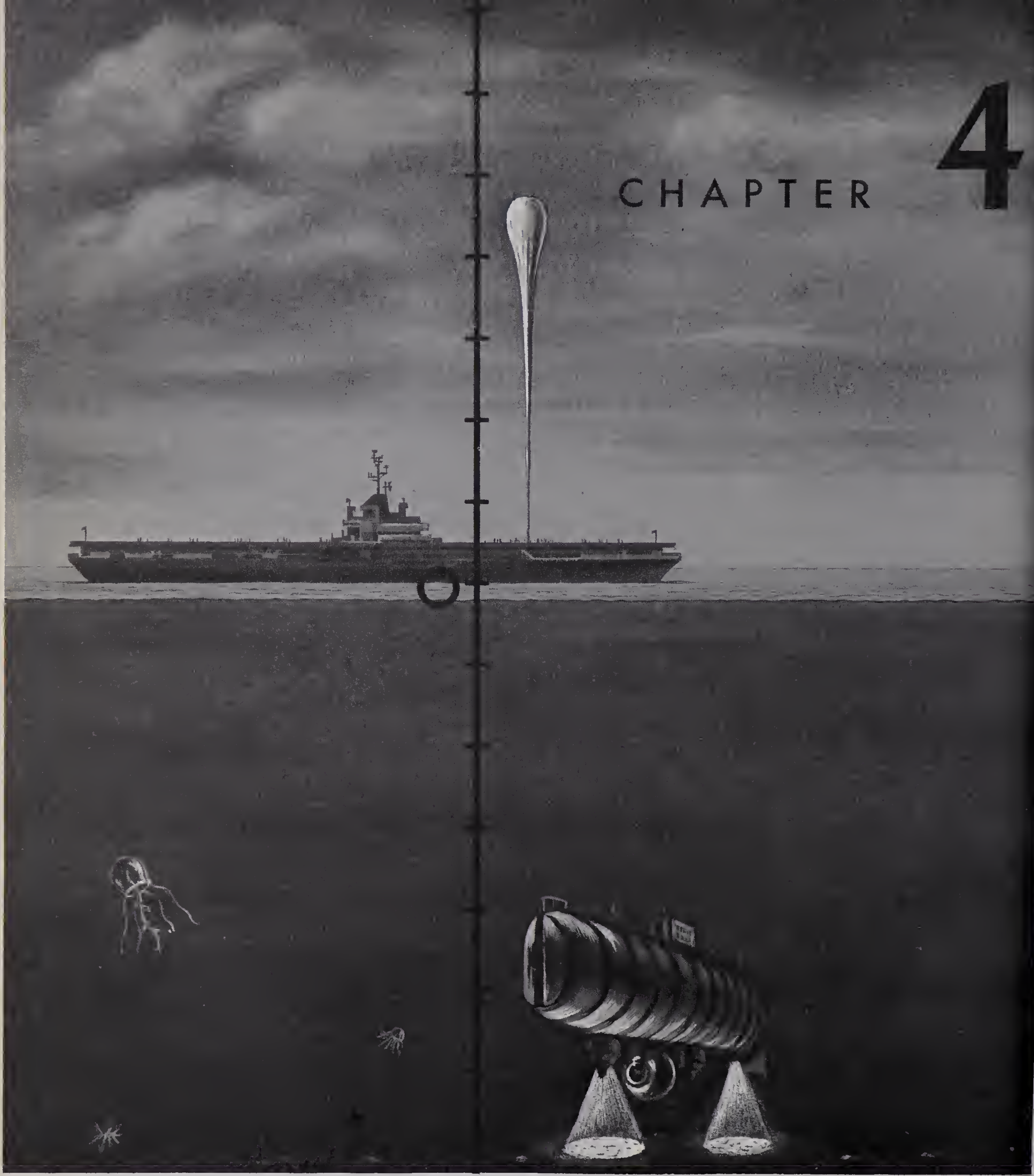
One day trouble started; how, no one knows. There were riots in the streets. Hypatia was probably unaware of danger when she ventured out that day. Why should anyone hate a lecturer on algebra?

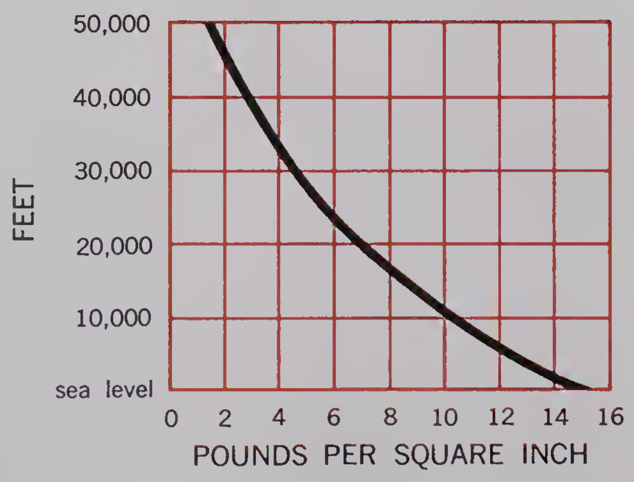
The rioting mob knew nothing of algebra and cared less. But they knew that Hypatia was different from most persons — a woman who taught men at the University. When she appeared, the rioters had found a victim for their fury. They tore her from her chariot; they slashed her to death with the sharpened oyster shells that were their weapons. So, on a day in March, 415, the first woman mathematician of history was murdered by a mob.

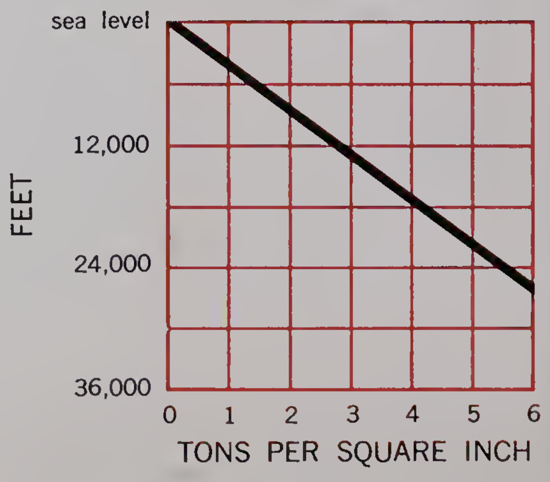


The "first lady" of algebra.

CHAPTER 4




ATMOSPHERIC PRESSURE


OCEAN PRESSURE

The Negative Numbers

Can you write a headline for this picture? As you analyze the picture, you may observe that it implies opposite directions.

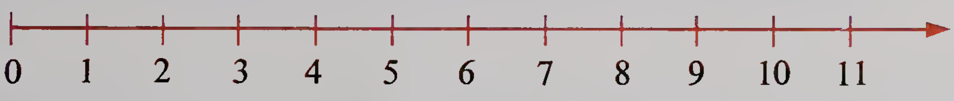
A "dip" in the stockmarket is the opposite of a "rise." A temperature "below zero" is the opposite of a temperature "above zero." A "deficit" is the opposite of a "surplus." "Below the sea" is the opposite of "above the sea" (graphs at left). A "loss" is the opposite of a "gain." "B.C." is the opposite of "A.D." Can you think of other opposites to add to this list? Mathematically, all of these concepts are treated by a single device, the subject of this chapter.

Call attention to familiar situations which suggest negative numbers. See T.M. pg. 13 (1).

EXTENDING THE NUMBER LINE

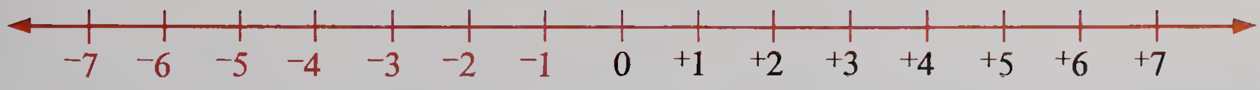
4–1 Directed Numbers

The number line you have studied so far extends only to the right of the point labeled 0, as in Figure 4–1.



· *Figure 4–1* ·

Have you ever seen a thermometer which read "below zero" or played a game in which you were "in the hole" or heard a businessman speak of being "in the red"? These examples suggest uses for a *two-way scale*, such as that in Figure 4–2.



· *Figure 4–2* ·

Do you notice the small $^{+}$ and $^{-}$ signs written above and to the left of the numerals? Since no two points are to have the same label, you cannot write merely the numerals from 0 on, to the left, without some method of distinguishing them from those to the right. You can, no doubt, devise other schemes, but this is a convenient notation.

Emphasize that the negative numbers are *new numbers.* Avoid calling -2 "minus 2"; let "minus" refer to subtraction.

.The + and − signs are the symbols used to indicate addition and subtraction, but here they indicate the direction of the point from the 0-point, not an operation to be performed. In this use they are called **signs of direction**. Actually, you could have left the right side of the scale as it was. Attaching the small $^{+}$ signs to the old numerals serves to emphasize their position, but they still represent the familiar numbers of arithmetic. Each numeral with the small $^{-}$ sign, however, represents a new number — a *negative number*. You should read $^{+}2$ as positive 2 and $^{-}2$ as negative 2. You also can read $+2$ as "two" and write 2. See T.M. pg. 13 (2).

You speak of the direction from 0 to the positive numbers as the positive direction (to the right) and of the direction from 0 to the negative numbers as the negative direction (to the left). The distance between a number and 0 is called the *magnitude* of the number, regardless of its direction. Thus $^{+}2$ and $^{-}2$ have the same magnitude 2.

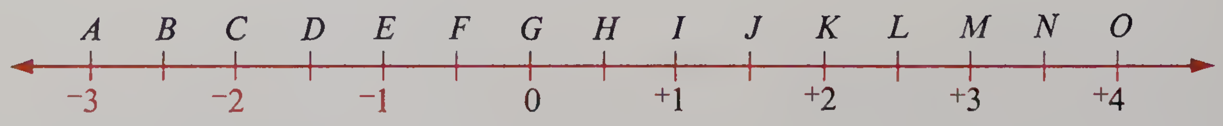
Here we introduce absolute value very informally. Formal treatment occurs in Section 4-5.

Because their numerals involve signs of direction, positive and negative numbers are known as **signed numbers** or **directed numbers**. For convenience, 0 is included in the set of directed numbers. Here are some examples illustrating the uses of directed numbers. Call them *real numbers*, if you wish. See T.M. pg. 13 (3).

1. A profit of \$3 could be described as $\$^{+}3$, and a loss of \$2 as $\$^{-}2$.
2. If latitudes north of the equator are taken as positive, the latitude of the South Pole is $^{-}90°$.
3. If counterclockwise rotations are taken as positive, the minute hand of a clock rotates through an angle of $^{-}30°$ in 5 minutes.
4. Suppose the thermometer reads just freezing, that is, $^{+}32°$ (Fahrenheit). If the temperature drops 40°, we say the change in temperature is $^{-}40°$, and the resulting temperature is $^{-}8°$ (8° below zero).

ORAL EXERCISES

In Exercises 1–10, tell what letter on the number scale corresponds to each number. Take as positive directions *up, north, east, right, counterclockwise.*



SAMPLE. A gain of \$3

What you say: A gain of \$3 corresponds to $^{+}3$, ∴ point M.

1. A gain of \$2 $^{+}2$; K
2. A loss of \$2 $^{-}2$; C
$^{+}3$; M 3. Three degrees above zero
$^{-}3$; A 4. Three degrees below zero
5. A credit of \$3.50 $^{+}3.5$; N
6. A point 2 miles west of a starting point $^{-}2$; C
7. The level of the Atlantic Ocean 0; G
8. A point 3 miles south of a base line $^{-}3$; A
9. $2\frac{1}{2}$° north latitude $^{+}2.5$; L
10. The latitude of the equator 0; G

Express each of the following by means of a directed number.

11. Ten degrees below freezing, Centigrade scale (Freezing is 0°C.) $^{-}10$°
12. Ten degrees below freezing, Fahrenheit scale (Freezing is $^{+}32$°F.) $^{+}22$°
13. A gain of six dollars $^{+}6$
14. A loss of seven dollars $^{-}7$
15. Nine steps to the left $^{-}9$
16. Eight steps to the right $^{+}8$
17. A debt of \$300 $^{-}300$
18. Receipts of \$17 $^{+}17$
6200 19. A mountain height of 6200 feet
20. An ocean depth of 330 feet $^{-}330$
21. A counterclockwise angle of 15° $^{+}15$
22. A latitude of 54° north $^{+}54$
23. 12° north latitude $^{+}12$
24. A five-yard gain in football $^{+}5$
25. A walk of three miles east $^{+}3$
26. A longitude of 15° east $^{+}15$
27. A deposit of \$75 $^{+}75$
28. A withdrawal of \$25 $^{-}25$

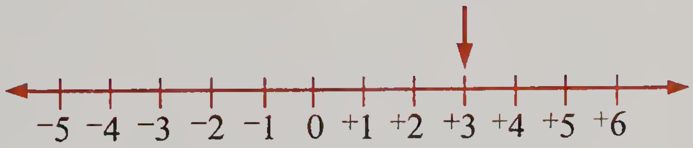
For each of the following, give another number having the same magnitude.

29. $^{+}69$ $^{-}69$
30. $^{+}3.2$ $^{-}3.2$
31. $^{-}\frac{1}{4}$ $^{+}\frac{1}{4}$
32. $^{-}\frac{1}{2}$ $^{+}\frac{1}{2}$

WRITTEN EXERCISES

Draw a horizontal number scale. In Exercises 1–8, mark with an arrow the point associated with the given number.

SAMPLE. $^{+}3$ *Solution:*



Ⓐ 1. $^{+}8$
2. $^{-}7$
3. $^{+}10$
4. $^{+}2.5$
5. $^{-}9$
6. $^{-}3$
7. $^{+}7.5$
8. $^{-}6\frac{1}{2}$

Give the coordinate of each point at which you would arrive.
These exercises anticipate the addition of directed numbers.

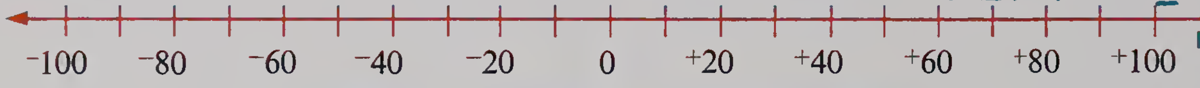
9. Start at zero on the number scale; go a distance of 5 units in the positive direction, then another 5 units in the positive direction.
10. Start at zero; go a distance of 3 units in the positive direction, then 2 units in the positive direction.
11. Start at zero; go a distance of 2 units in the negative direction, then a distance of 1 unit in the negative direction.
12. Start at zero; go 2 units in the negative direction, then 3 units in the negative direction.

4–2 Comparing Numbers

The sentence $3 < 5$, and the statements, 3 **is less than** 5 and 3 **is to the left of** five describe the same order relationship. Can you extend this type of thinking to negative and positive numbers?

Which is the higher temperature, $^{-}2°$ or $^{-}10°$? The answer, of course, is $^{-}2°$. On the other hand, a temperature of $^{+}10°$ is higher than a temperature of $^{+}2°$. Similarly, although $^{+}50°$ is less than $^{+}100°$ $(50 < 100)$, $^{-}50$ is greater than $^{-}100$, $(^{-}50 > {}^{-}100)$.

Notice that any number on the number line, whether positive, negative, or zero, is greater than any number to its left, and less than any number to its right. Thus, directed numbers are ordered. Students readily extend the meaning of "is less than" to directed numbers. $<$, $\leq$, $>$, and $\geq$ have the expe meaning. See 4-2 T.M. pg.



ORAL EXERCISES

Replace each question mark with $<$ or $>$ to make a true sentence.

1. $^{+}7$ _?_ $^{+}9$ $^{+}7 < {}^{+}9$
2. $^{-}3$ _?_ $^{-}5$ $^{-}3 > {}^{-}5$
3. $^{-}4$ _?_ $^{+}3$ $^{-}4 < {}^{+}3$
4. $^{+}2$ _?_ $^{+}1$ $^{+}2 > {}^{+}1$
5. 0 _?_ $^{+}2$ $0 < {}^{+}2$
6. 0 _?_ $^{+}10$ $0 < {}^{+}10$
7. $^{-}10$ _?_ 0 $^{-}10 < 0$
8. $^{-}1$ _?_ $^{+}1$ $^{-}1 < {}^{+}1$
9. $^{-}.1$ _?_ $^{-}.2$ $^{-}.1 > {}^{-}.2$
10. $^{-}.5$ _?_ $^{-}.1$ $^{-}.5 < {}^{-}.1$
11. $^{+}.01$ _?_ $^{+}.1$ $^{+}.01 < {}^{+}.1$
12. $^{+}1.05$ _?_ $^{+}1.6$ $^{+}1.05 < {}^{+}1.$

Which of the following sentences are true? Which are false?

13. $^{-}1 < {}^{-}\frac{1}{2}$ T
14. $^{-}15 < {}^{-}5$ T
15. $^{+}7 \geq {}^{-}20$ T
16. $^{-}\frac{2}{3} < {}^{-}\frac{4}{3}$ F
17. $^{+}\frac{1}{9} \leq {}^{-}\frac{5}{9}$ F
18. $^{-}3 \leq {}^{+}2$ T
19. $^{-}10 \neq {}^{+}10$ T
20. $^{+}8 > {}^{-}10$ T
21. $^{-}\pi \neq {}^{-}3.1$ T

WRITTEN EXERCISES

Let the replacement set for the variable in each of the following open sentences be $\{^{-}6, {}^{-}4, {}^{-}1, 0, {}^{+}2, {}^{+}4\}$. In each case give the solution set.

A

1. $w < {}^{-}2$
2. $y \geq {}^{-}5$
3. $^{-}1 \leq z$
4. $^{-}3 \geq t$
5. $0 \neq u$
6. $^{-}1 < k \leq {}^{+}3$
7. $^{-}6 \leq v < {}^{-}1$
8. $r > {}^{+}1$
9. $x \leq x$
10. $q > q$
11. $t \geq 0$
12. $v < 0$

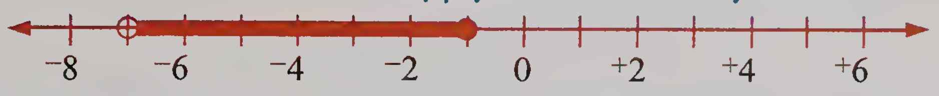
The various substitutions should be made for each variable. Pupils should see that variables like w may stand for a *negative* number, as well as for zero or a positive number.

Given the domain of each variable as the set of directed numbers, graph the solution set of each of the following open sentences.

The conventions (see page 16) employed in picturing graphs on the arithmetic number line apply to the two-way scale.

SAMPLE. ${}^{-}7 < a \leq {}^{-}1$

Solution:



B

13. $p \geq {}^{-}1$

14. $q < {}^{+}6$

15. ${}^{+}4 \leq t$

16. ${}^{-}3 \geq m$

17. ${}^{-}2 \leq c \leq {}^{+}2$

18. ${}^{-}5 < d < {}^{+}5$

19. $t \leq {}^{+}2$

20. $s < {}^{+}5$

21. ${}^{-}5 \leq w \leq {}^{-}1$

22. ${}^{-}6 < f < {}^{+}4$

23. $z \neq {}^{+}1$

24. $r \neq {}^{+}3$

25. $e = 0$

26. $0 = m$

Exercises 32-35 anticipate Section 5-3 (Optional).

C

27. $k = k$

28. $t > t$

29. $h \leq h$

30. $x = {}^{-}5$

31. $y < 0$

32. $m < {}^{-}3$ or $m > {}^{+}2$

33. $x > 0$ or $x < {}^{-}2$

34. $t > {}^{+}1$ and $t < 0$

35. $k < {}^{-}2$ and $k > {}^{+}2$

PROBLEMS

Translate each of the following English sentences into an open sentence. In each case specify the replacement set for the variable.

SAMPLE. To the nearest penny, the price of a quart of milk has never been higher than 30 cents.

Solution: Let c = the price of a quart of milk. $0 < c \leq {}^{+}30$;

$c \in \{{}^{+}1, {}^{+}2, {}^{+}3, \ldots, {}^{+}30\}$

A

1. Phil's assets exceed $10,000.

2. Mary has less than $50 in her bank account.

3. Ron hit more than twice as many home runs as did Joe, who hit 5 home runs.

4. The winter temperature at the North Pole is less than -10° Fahrenheit.

5. The altitude of the land surface of the earth varies from 1292 feet below sea level to 29,028 feet above sea level.

6. Latitude in the equatorial zone ranges between $23\frac{1}{2}^\circ$ above and $23\frac{1}{2}^\circ$ below the equator.

7. Over the past 5 years Marjorie's weight has neither increased nor decreased more than 4 pounds.
8. To the nearest eighth of a dollar, the change in the price of a stock since 1950 has never been more than \$10 in either direction.
9. The temperature in Death Valley has been as high as 56.7°C above zero and as low as 9.4°C below zero.
10. In a certain game Roger never scores more than 15 points and has never "gone into the hole" for more than 10 points.

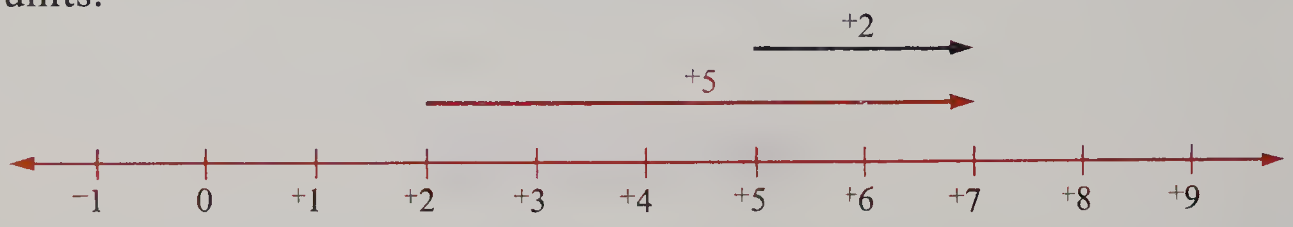
OPERATING WITH DIRECTED NUMBERS

4–3 Addition on the Number Line

See T.M. pg. 13 (4).

To use directed numbers effectively, you must agree on rules for operating with them. Begin by looking at the addition of positive numbers in a new way — in terms of *displacements* (changes of position) on the number line.

To add $^{+}5$ to $^{+}2$, start at $^{+}2$ and move to the right a distance of 5 units.

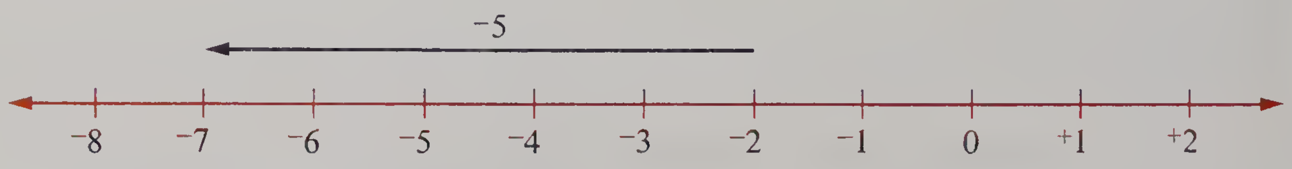


· *Figure 4–3* ·

This displacement, indicated by the red arrow in Figure 4–3, brings you 7 units to the right of zero, to $^{+}7$. The addition may be written as $^{+}2 + {}^{+}5 = {}^{+}7$.

You know that $^{+}5 + {}^{+}2$ should give the same result. To use the method of displacement, you would start at $^{+}5$, then move 2 units to the right, again arriving at $^{+}7$ (indicated by the black arrow). What property of addition is illustrated by this? Consider the following examples.

To add $^{-}5$ to $^{-}2$, start at $^{-}2$, then move 5 units to the left (Figure 4–4).

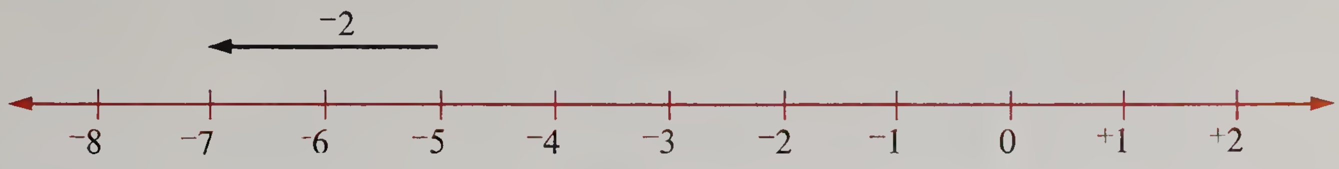


· *Figure 4–4* ·

This brings you 7 units to the left of zero, that is, to $^{-}7$. You have

$$^{-}2 + {}^{-}5 = {}^{-}7.$$

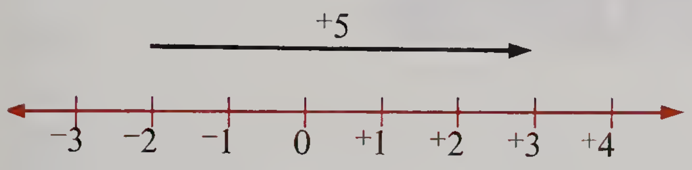
To add $^{-}2$ to $^{-}5$, start at $^{-}5$, and go 2 units to the left, again arriving at $^{-}7$ (Figure 4–5).



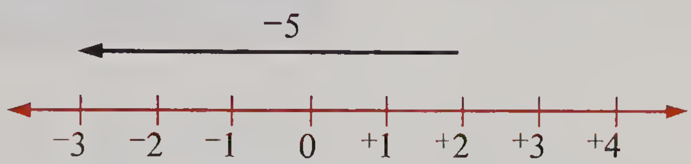
· *Figure 4–5* ·

That is, $^{-}5 + {}^{-}2 = {}^{-}7$, the same as $^{-}2 + {}^{-}5$.

To find $^{-}2 + {}^{+}5$, start at $^{-}2$ and move 5 units to the right. The result is $^{+}3$, shown in Figure 4–6. Thus, $^{-}2 + {}^{+}5 = {}^{+}3$. Show that $^{+}5 + {}^{-}2$ gives the same result.



· *Figure 4–6* ·



· *Figure 4–7* ·

Similarly, as shown in Figure 4–7, $^{+}2 + {}^{-}5 = {}^{-}3$. Show that $^{-}5 + {}^{+}2$ also gives $^{-}3$.

Can you visualize $^{+}2 + {}^{-}2$ on the number line? Do you see that your final position is at zero? $^{+}2 + {}^{-}2 = 0$. Interpreting "adding 0" as no displacement, you see also that $^{+}2 + 0 = 0 + {}^{+}2 = {}^{+}2$, and that $^{-}2 + 0 = 0 + {}^{-}2 = {}^{-}2$. Because adding 0 to any number gives the identical number as the sum, zero is called the **identity element for addition**.

Any two numbers on the number line, positive, negative, or zero, can be added this way, and the result will be a number on the number line. The rule may be stated:

> To add two numbers on the number line, start at the position of the first number, then move a distance equal to the magnitude of the second number in the direction associated with the number; the number at the resulting position is the required sum.

This kind of addition is commutative and associative. (Verify this.)

Pupils should verify this in two or three specific examples. See T.M. pg. 14 (5).

ORAL EXERCISES

Find each of the following sums.

1. $^{+}6 + {}^{+}7$ +13

2. $^{+}7 + {}^{+}6$ +13

3. $^{-}5 + {}^{-}6$ −11

4. $^{-}6 + {}^{-}5$ −11

5. $^{+}12$ tickets
$^{+}22$ tickets
+34 tickets

6. $^{+}5$ grams
$^{+}6$ grams
+11 grams

7. $^{-}5$ feet
$^{-}4$ feet $^{-}9$ feet

8. $^{-}3$ yards
$^{-}11$ yards $^{-}14$ yards

9. $^{-}7 + 0$ $^{-}7$

10. $0 + ^{-}8$ $^{-}8$

11. $^{+}7 + ^{-}2$ $^{+}5$

12. $^{+}9 + ^{-}6$ $^{+}3$

13. $^{-}7 + ^{+}7$ 0

14. $^{-}9 + ^{+}6$ $^{-}3$

15. $0 + \frac{^{+}3}{2}$ $^{+}\frac{3}{2}$

16. $\frac{^{+}3}{2} + 0$ $^{+}\frac{3}{2}$

17. $\frac{^{+}7}{5} + \frac{^{-}7}{5}$ 0

18. $^{-}5 + ^{+}5$ 0

19. $^{+}4 + ^{-}9$ $^{-}5$

20. $^{-}11 + ^{+}3$ $^{-}8$

21. $^{-}12$ hours
$^{+}15$ hours $^{+}3$ hours

22. $^{+}11$ dollars
$^{-}11$ dollars 0 dolla

23. $^{-}3$ miles
$^{+}3$ miles 0 miles

24. $^{+}4$ inches
$^{-}4$ inches 0 inch

Hereafter, encourage pupils to visualize the number line mentally, referring to an actual picture only when confused. Although able students may devise formal rules of operation without using the line, not have the whole class formulate such rules.

WRITTEN EXERCISES

(a) Add each of the following, (b) as Exercises 15–28, use the associative property to regroup the addends in Exercises 1–14, and then compute each sum again.

SAMPLE 1. Add $(^{+}9 + ^{-}7) + ^{-}3$

Solution: $(^{+}9 + ^{-}7) + ^{-}3 = ^{+}2 + ^{-}3 = ^{-}1$

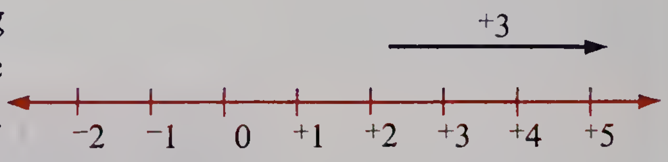
Ⓐ

1. $(^{+}4 + ^{+}54) + ^{+}22$
2. $(^{+}3 + ^{+}22) + ^{+}16$
3. $^{-}43 + (^{-}25 + ^{-}21)$
4. $^{-}34 + (^{-}24 + ^{-}62)$
5. $(^{-}7° + ^{+}8°) + 0°$
6. $^{+}5° + (0° + ^{-}15°)$
7. $^{-}1.4 + (^{-}3.0 + ^{+}4.4)$
8. $(^{-}14 + ^{-}30) + ^{+}44$
9. $(^{+}.8 + ^{-}3.8) + ^{-}20$
10. $^{+}8 + (^{-}38 + ^{-}20)$
11. $^{+}\frac{5}{2} + (^{+}\frac{1}{2} + ^{-}3)$
12. $^{+}6 + (^{-}\frac{14}{3} + ^{-}\frac{4}{3})$
13. $(^{-}3 + ^{-}4) + (^{+}6 + ^{-}6)$
14. $(^{+}8 + ^{-}3) + (^{-}14 + ^{+}14)$

Solve each of the following equations, given that the replacement set of the variable is the set of directed numbers. You may use the number line as an aid.

SAMPLE 2. $^{+}2 + y = ^{+}5$

Solution: To arrive at $^{+}5$ starting from $^{+}2$, go a distance of 3 units to the right.

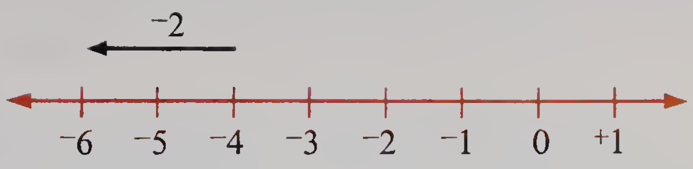


$^{+}2 + y = ^{+}5$
$^{+}2 + ^{+}3 = ^{+}5$
$^{+}5 = ^{+}5$ ✓

∴ The solution set is $\{^{+}3\}$, **Answer.**

SAMPLE 3. $^{-}6 = r + {}^{-}4$

Solution: To arrive at $^{-}6$ starting from $^{-}4$, go 2 to the left.



$^{-}6 = r + {}^{-}4$

$^{-}6 = {}^{-}2 + {}^{-}4$

$^{-}6 = {}^{-}6$ ✓ $\therefore$ The solution set is $\{{}^{-}2\}$, **Answer.**

B **29.** $^{+}1 + x = {}^{+}9$
30. $^{+}4 + y = {}^{+}7$
31. $^{-}2 + z = {}^{-}5$
32. $t + {}^{-}1 = {}^{-}6$
33. $k + {}^{+}3 = {}^{-}10$
34. $^{-}6 + n = {}^{+}8$
35. $m + {}^{-}5 = {}^{-}14$
36. $^{+}12 = v + {}^{-}6$
37. $^{-}100 = {}^{+}9 + h$
38. $^{+}29 + p = {}^{+}25$
39. $^{+}3 + u = {}^{+}3$
40. $w + {}^{-}5 = {}^{-}5$
41. $t + {}^{+}5 = {}^{+}11$
42. $^{+}3 + z = {}^{+}8$
43. $^{-}6 + u = {}^{-}1$
44. $g + {}^{-}4 = {}^{-}3$
45. $b + b = 0$
46. $k + k = {}^{+}6$

PROBLEMS

First write the answer to each of the following problems as the indicated sum of directed numbers; then do the addition.

A **1.** An airplane is flying east. It has a speed of 200 miles per hour in still air and is pushed by a tail wind of 45 miles per hour. Find its speed.

2. An airplane ascends to 8000 feet on its take-off. Later the pilot descends 2000 feet and levels off. At what altitude does he level off?

3. The mate on a Mississippi River boat takes soundings as follows: 7 fathoms, 3 fathoms, 5 fathoms. Represent by signed numbers (a) each sounding with respect to river level, and (b) the change between each two successive soundings.

4. In descending from the stratosphere, a pilot came down 24,000 feet, leveled off, and later descended another 7000 feet. After flying at this altitude for a time, he descended 8000 feet to land. Express the total change in altitude of the plane.

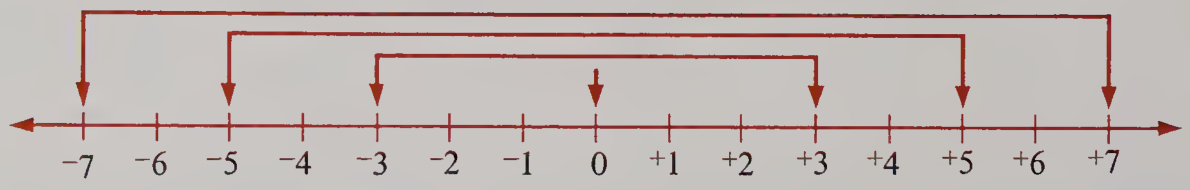
5. During a cold snap, the temperature dropped three degrees in an hour. The next hour it dropped two degrees more, and the third hour it fell another three degrees, but the fourth hour it rose one-half degree. Express the total change in temperature.

6. Mr. Jordan owed \$3.56, \$2.34, \$34.43, \$23.25. Two friends owed him \$25 and \$35. What was Mr. Jordan's net financial status?

7. An elevator descended two floors, stopped for passengers, descended three floors, made another stop, and then went to the lobby, ten floors below. Express the total change in position of the elevator.
8. John Byer owned stock which rose 2 dollars per share the first day, then $\frac{1}{8}$ dollar on each of three succeeding days, and $1\frac{1}{2}$ dollars on the fifth. What was the net change in price of the stock in that time?
9. A stock increased in value during the week of March 13, as follows: Monday, $\frac{1}{8}$ dollar; Tuesday, $\frac{1}{4}$ dollar; Wednesday, $\frac{1}{8}$ dollar; Thursday, $1\frac{1}{8}$ dollars; and Friday, $\frac{3}{4}$ dollar. Express the net change in value.
10. Al Herrick's stock fell $\frac{1}{2}$ dollar per share the first day he owned it. It continued to fall $\frac{1}{8}$ dollar for each of the next two days. Express the change in value of a share of this stock during these three days.
11. One week the price of a stock changed as follows: Monday, down $1\frac{1}{8}$ dollars; Tuesday, down $\frac{3}{8}$ dollar; Wednesday, unchanged; Thursday, down $1\frac{3}{4}$ dollars; Friday, up $2\frac{1}{8}$ dollars. Find the net change.
12. To test its equipment, a submarine was submerged 20 meters; then 15 meters twice; then 5 meters three times; and finally 3 meters each of 8 times. Represent the position of the submarine at its lowest depth.

4–4 The Opposite of a Directed Number

You can see that each number on the number line may be paired with another number which is the same distance from zero but in the opposite direction.



Furthermore, adding two such paired numbers on the number line gives 0. This suggests defining the *opposite* (additive inverse) *of a* directed *number a* as the directed number whose sum with a is 0. The symbol $-a$ (note the lowered position of the minus sign) denotes the opposite of a or the additive inverse of a.

$-(^{+}7) = {}^{-}7$ read **the opposite of** $^{+}7$ is $^{-}7$ → $^{+}7 + {}^{-}7 = 0$
$-(^{+}5) = {}^{-}5$ read **the opposite of** $^{+}5$ is $^{-}5$ → $^{+}5 + {}^{-}5 = 0$
$-(0) = 0$ read **the opposite of** 0 is 0 → $0 + 0 = 0$
$-(^{-}3) = {}^{+}3$ read **the opposite of** $^{-}3$ is $^{+}3$ → $^{-}3 + {}^{+}3 = 0$

The → is translated as *because* or *meaning*.
An important assumption about directed numbers is:

$-a$ is also called "the negative of a." If you introduce that term, watch for students who think that the negative of a number must be a negative number.

For every directed number a there is a unique number $-a$ such that $a + (-a) = (-a) + a = 0$.

Several facts about a number and its opposite follow from this assumption:

Emphasize that both a and -a can represent zero, or a positive number or a negative number.

1. If a is positive, $-a$ is negative; if a is negative, $-a$ is positive; if a is 0, $-a$ is 0.
2. The opposite of $-a$ is a, that is, $-(-a) = a$.

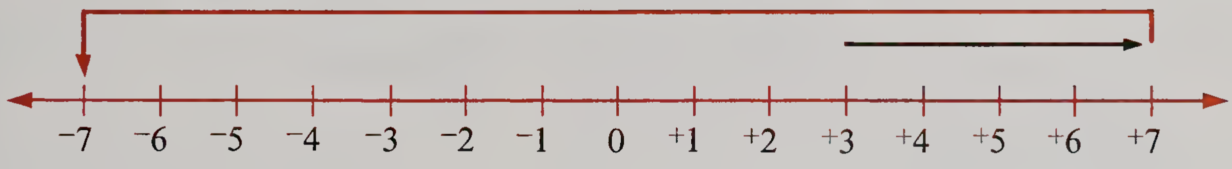
All these relationships help in simplifying expressions.

EXAMPLE 1. **Show on the number lines that** *a.* $-(^+3 + {}^+4) = {}^-7$ **and**

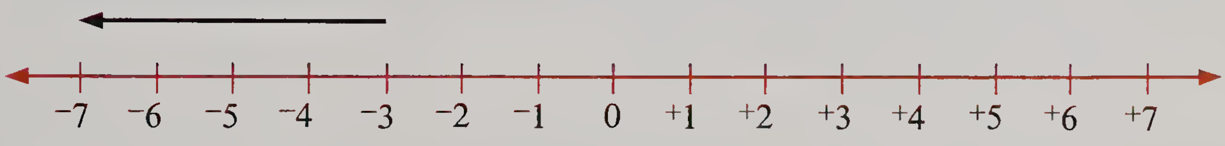
b. $-\,{}^+3 + (-\,{}^+4) = {}^-7$; **therefore,**

c. $-\,(^+3 + {}^+4) = -\,{}^+3 + (-\,{}^+4)$.

Solution: **a.** **Add $^+3$ and $^+4$, and then find the opposite of this result, $^-7$.**



b. **Add the opposite of $^+3$ and the opposite of $^+4$; that is, add $^-3$ and $^-4$. This is $^-7$.**



c. **By the transitive property of equality, as both expressions equal $^-7$, they equal each other.**

Example 1 illustrates a very important property of opposites:

This property is assumed, although it can be proved from other postulates for directed numbers.

The opposite (additive inverse) of the sum of two numbers is the sum of their opposites:

$$-(a + b) = -a + (-b).$$

Try problems similar to Example 1 until you see clearly that this statement is true for any numbers a and b, positive, negative, or zero.

Assuming that every directed number has an opposite is a device that would have enabled us to invent the negative numbers without having to think of them as partners of points on the number line. This way, $^-3$ is simply the number whose sum with $^+3$ is 0, $^-3 = -\,{}^+3$. By agreement (page 112), $^+3 = 3$. **As a result, hereafter we will simplify notation by dropping the small $^+$ and $^-$ signs. Thus, write 3 rather than $^+3$, and -3 for $^-3$ or $-^+3$. You now read -3 either as** ***negative*** **3 or as** ***the opposite of 3.*** We drop one of the two symbols for "negative 3," $^-3$. See T.M. pg. 14 (6).

Using addition on the number line, you may find the following sums:

EXAMPLE 2. $-(-2) + (-5) = 2 + (-5) = -3$

EXAMPLE 3. $[-(-2) + 3] + (-4) = [2 + 3] + (-4) = 1$

ORAL EXERCISES

Name the additive inverse of each of the following directed numbers.

SAMPLE. -12 *What you say:* Twelve or positive twelve.

1. 5 -5
2. $\frac{1}{3}$ $-\frac{1}{3}$
3. -2 2 or $+2$ (positive two)
4. -6 6 or $+6$
5. $-\frac{3}{2}$ $\frac{3}{2}$ or $+\frac{3}{2}$
6. $-1\frac{1}{2}$ $1\frac{1}{2}$ or $+1\frac{1}{2}$
7. $2 + 9$ -11
8. $-(3 + 7)$ 10 or $+10$
9. 0 0
10. -0 0
11. $7 + (-8)$ 1 or $+1$
12. $(-9) + 4$ 5 or $+$
13. $-[5 + (-11)]$ -6
14. $-[2 + (-4)]$ -2
15. $6 + (-5) + [-(-4)]$ -5
16. $-[-6 + 8 - (-5)]$ 7 or $+7$

Give two expressions for the additive inverse of each of the following.
Students may use the number line as an aid.

SAMPLE. $(-6) + 7$ *What you say:* $6 + (-7)$ or -1.

17. $-8 + (-4)$ $8+4$ or 12
18. $-10 + (5)$ $10+(-5)$ or 5
19. $-[6 + (-6)]$ $[6+(-6)]$ or 0
20. $-[9 + (-8)]$ $[9+(-8)]$ or 1
21. $-(-8) + [-(9)]$ $(-8)+9$ or 1
22. $(-8) + [-(-7)]$ $8+(-7)$ or 1
23. $-[-(7)]$ $[-(7)]$ or -7
24. $-[-(3 + 2)]$ $[-(3+2)]$ or -5
25. $-[-(-8)]$ $[-(-8)]$ or 8
26. $-[-(-7 + 9)]$ $[-(-7+9)]$ o
27. $9 + (-5) + (-8)$ $(-9)+5+8$
28. $-4 + (7) + [-(-9)]$ $4+(-7)+(-9)$ or $-$

WRITTEN EXERCISES

Determine the value of the following expressions.

Advise students to think of the nu line when in doubt. Numerals lik $-(-4)$ should be simplif before the sum is computed.

A

1. $-(-4) + (-6)$
2. $-(-3) + [-(-6)]$
3. $-(5 + 4)$
4. $-[-3 + 8]$
5. $-(.7 + 5)$
6. $-[.9 + (-5)]$
7. $-(-5) + (-1)$
8. $-[4 + (-6)]$

9. $-8 + (-3) + 6.6$

10. $-4 + (-0) + (-2.4)$

11. $15\frac{1}{2} + (-3\frac{1}{4}) + [-(6\frac{7}{8})]$

12. $-5\frac{2}{3} + 7\frac{3}{5} + (-2\frac{5}{6})$

If $\{-3, -2, -1, 0, 1, 2, 3\}$ is the replacement set for the variables, find the solution set of the following open sentences by substitution. Encourage your better students to solve some of Exercises 13-27 without actually making the substitutions.

SAMPLE. $-t < \frac{1}{2}$ Be sure a variable expression like -x is read "the opposite of x."

Solution: $-(3) < \frac{1}{2}$; $-(2) < \frac{1}{2}$; $-(1) < \frac{1}{2}$; $-(0) < \frac{1}{2}$

$\therefore$ The solution set for $-t < \frac{1}{2}$ is $\{3, 2, 1, 0\}$.

B

13. $-x = 2$

14. $-3 = -t$

15. $-y = 0$

16. $-r = 1$

17. $-2 = -v$

18. $-a = 3$

19. $t + (-2) = 1$

20. $u + 1 = -1$

21. $-t < 1$

22. $-x > 1$

23. $-z \leq -2$

24. $-b \leq -1$

25. $0 < -m < 3$

26. $-3 \geq -n \geq 3$

27. $-2 \leq -q \leq 2$

If $a = \frac{5}{3}$, $b = \frac{1}{5}$, $c = .25$, and $d = 1.5$, evaluate the following expressions.

28. $a + (-b) + c$

29. $-a + b + (-c)$

30. $-[b + c + (-d)]$

31. $-b + (-c) + d$

32. $a + b + (-c) + (-d)$

33. $-a + b + c + d$

34. $-[-a + c + (-d)]$

35. $a + -[(-c) + d]$

36. $b + (-a) + (-d)$

37. $b + [-(a + d)]$

38. $[a + b] + [-(c + d)]$

39. $[-a + b] + [c + d]$

4–5 Absolute Value

Although the members of a pair of opposite numbers, such as 2 and -2, -7 and 7, 12 and -12, are equally distant from 0, the positive member of such a pair is the greater. The greater of any directed number, other than 0, and its opposite is called the **absolute value** of either number. The absolute value of a number is denoted by placing the number between a pair of vertical bars $|\ \ |$. The **absolute value of 0** is 0. It is obvious that the absolute value of any number and that of its opposite are equal. For example,

Also called *numerical value* or *magnitude*.

Do not allow pupils to think of it as a number "without its sign."

$$|-6| = |6| = 6$$

which is read: The absolute value of negative 6 equals the absolute value of positive 6, equals positive 6.

ORAL EXERCISES

Tell which of these statements are true and which are false.

1. $|9| = 9$ T **5.** $|8| > |-8|$ F **9.** $-3 < |-5|$ T
2. $|-345| = -345$ F **6.** $|-234| > |234|$ F **10.** $|17| < 17$ F
3. $|-56| = 56$ T **7.** $|32| > |-56|$ F **11.** $|-27| < |19|$ F
4. $|12\frac{1}{2}| = -12\frac{1}{2}$ F **8.** $|-4.9| > |-3.4|$ T **12.** $|18.9| < |-19.8|$ T

Evaluate these expressions.

13. $|3| + |16|$ 19 **17.** $-|-7|$ −7 **21.** $-(2|-5|)$ −10
14. $|-4| + |-8|$ 12 **18.** $|-(-7)|$ 7 **22.** $-(-|2|\,|-5|)$ 10
15. $|21| + |-19|$ 40 **19.** $-|5| + |7.1|$ 2.1 **23.** $-(|-15| - |-12|)$ −3
16. $|-13| + |12|$ 25 **20.** $-|-7| + |7|$ 0 **24.** $-(|-23| + |15|)$ −38

WRITTEN EXERCISES

Give the solution set of each of these equations.

Ⓐ **1.** $|q| = 0$ **2.** $|m| = 4$ **3.** $|n| + |-2| = 7$ **4.** $|s| + |4| = 3$

Graph the solution set of each inequality.

5. $|x| < 3$ **6.** $|y| > 4$ **7.** $1 < |x| < 3$ **8.** $-2 < |y| < 1$

Describe the solution sets of these open sentences.

Be sure to include Exercises 13 ar
14. In discussing them, you get the usual definition of absolute value. See 4-5 T.M. pg

9. $|y| < 0$ **11.** $|y| < -y$ **13.** $y = |y|$ **15.** $|y| > 0$
10. $|y| \leq 0$ **12.** $|y| < y$ **14.** $-y = |y|$ **16.** $|y| \geq 0$

Evaluate, if $a = 2.3$, $b = -16$, $c = -5\frac{1}{2}$, and $d = 3\frac{2}{5}$.

Ⓑ **17.** $|a| + |b| + |c|$ **20.** $a + (-b) + |c| + d$ **23.** $a + d + |c|$
18. $-(a + |c|) + d$ **21.** $|a|\,|b|\,|c|$ **24.** $|b + c| + d$
19. $-a + b + |c + d|$ **22.** $|b|\,|c|\,|d|$ **25.** $a + |b + c|$

4–6 Adding Directed Numbers

See T.M. pg. 14 (7).

To treat addition of directed numbers without the number scale, you must list the addition properties assumed for the directed numbers.

We have assumed that there is one, and only one, opposite for each a.

For all members a, b, and c of the set of directed numbers:

1. The closure property: the sum $a + b$ is a unique directed number.
2. The commutative property: $a + b = b + a$.
3. The associative property: $(a + b) + c = a + (b + c)$.
4. The additive property of zero: $0 + a = a + 0 = a$.
5. The property of opposites: $-a$ is a unique directed number such that $a + (-a) = (-a) + a = 0$.
6. The property of the opposite of a sum: $-(a + b) = (-a) + (-b)$.

You will find the sixth property particularly useful in making such substitutions for negative numbers as these:

$$-7 = -(4 + 3) = -4 + (-3); \qquad -6\tfrac{2}{3} = -(6 + \tfrac{2}{3}) = -6 + (-\tfrac{2}{3}).$$

The following examples indicate several ways in which you can use the properties of directed numbers and of equality to perform addition.

EXAMPLE 1. **Add:** $-6 + (-5)$

Solution:

$-6 + (-5) = -(6 + 5)$ **Property of the opposite of a sum**

$= -11$ **Substitution principle**

EXAMPLE 2. **Add:** $7 + (-3)$

See T.M. pg. 14 (8).

Solution:

$7 + (-3)$

$= (4 + 3) + (-3)$ Subst. Princ.

$= 4 + [3 + (-3)]$ Assoc. Prop.

$= 4 + 0$ Prop. of Opposites

$= 4$ Add. Property of 0

EXAMPLE 3. **Add:** $-7 + 4$

Solution: $-7 + 4 = -[3 + 4] + 4$ Property of the Opposite of a Sum.

$= [-3 + (-4)] + 4$

$= -3 + (-4 + 4)$

$= -3 + 0$

$= -3$

Can you supply the properties which justify each step in Examples 2 and 3?

From the above examples and properties you can deduce the following *rules for addition:* Experience should suggest these rules. See T.M. pg. 15 (9).

1. If a and b are each positive, $a + b = |a| + |b|$.
   EXAMPLE. $4 + 3 = 7$
2. If a and b are each negative, $a + b = -(|a| + |b|)$.
   EXAMPLE. $(-3) + (-4) = -(3 + 4) = -7$

3. If a is positive and b is negative and $|a| \geq |b|$, $a + b = |a| - |b|$.
EXAMPLE. $7 + (-3) = 7 - 3 = 4$

4. If a is positive and b negative and $|b| \geq |a|$, $a + b = -(|b| - |a|)$.
EXAMPLE. $4 + (-7) = -(7 - 4) = -3$

EXAMPLE 4. **Add:** $5 + (-13) + 11 + (-6) + (-8)$

Solution: $5 + (-13) = -8$; $-8 + 11 = 3$; $3 + (-6) = -3$; $-3 + (-8) = -11$, **Answer.**

or $[5 + 11] + [(-13) + (-6) + (-8)] = 16 + (-27) = -11$, **Answer.**

Point out the use of the commutative and associative properties.

EXAMPLE 5. **Add:**

$$\begin{array}{r} -115 \\ 171 \\ 235 \\ -313 \\ \hline \end{array}$$

Solution:

$$\begin{array}{rcrcr} \text{Step 1} & & \text{Step 2} & & \text{Step 3} \\ -115 & & 171 & & -428 \\ -313 & & 235 & & 406 \\ \hline -428 & & 406 & & -22 \end{array}$$

-22, **Answer.**

ORAL EXERCISES

Add the following.

1. $\begin{array}{r} 4 \\ 9 \\ \hline \end{array}$ 13

2. $\begin{array}{r} -3 \\ -2 \\ \hline \end{array}$ -5

3. $\begin{array}{r} 8 \\ -6 \\ \hline \end{array}$ 2

4. $\begin{array}{r} 10 \\ -5 \\ \hline \end{array}$ 5

5. $\begin{array}{r} \frac{1}{5} \\ -\frac{4}{5} \\ \hline \end{array}$ $-\frac{3}{5}$

6. $\begin{array}{r} -3 \\ 6 \\ \hline \end{array}$ 3

7. $\begin{array}{r} -2 \\ 9 \\ \hline \end{array}$ 7

8. $\begin{array}{r} -4 \\ -4 \\ \hline \end{array}$ -8

9. $\begin{array}{r} -5 \\ -4 \\ \hline \end{array}$ -9

10. $\begin{array}{r} \frac{5}{7} \\ -\frac{2}{7} \\ \hline \end{array}$ $\frac{3}{7}$

11. $-3.8 + (-2.9)$ -6.7

12. $-6.2 + (-4.8)$ -11

13. $\frac{6}{4} + (-1.5)$ 0

14. $(-\frac{1}{4}) + .2$ -.05 or $-\frac{1}{20}$

15. $3 + (-8) + 8$ 3

16. $(-6) + (-1) + 6$ -1

17. $(-7) + k + (-2) + (-k)$ -9

18. $r + (-12) + (-r) + 9$ -3

Name the properties of addition which justify these sentences.

Comm. prop. **19.** $3 + (-7) = -7 + 3$

Comm. prop **20.** $-14 + (-12) = -12 + (-14)$

Prop. of opposites **21.** $7 + (-7) = 0$

22. $-\frac{1}{3} + \frac{1}{3} = 0$ Prop. of opposite

23. $5 + (-2 + 2) = 5$ Prop. of oppo and prop.

24. $[6 + (-6)] + (-7) = -7$ Prop. of opposites and prop. of

WRITTEN EXERCISES

Add the following. Suggest the use of the associative property.

A

1.	**2.**	**3.**	**4.**	**5.**	**6.**
6	-7	$54\frac{1}{2}$	$43\frac{1}{3}$	-18	-14
10	-1	$-11\frac{3}{4}$	$-45\frac{3}{5}$	24	22
-3	8	16	$-54\frac{2}{3}$	-32	-11
$\underline{-9}$	$\underline{9}$	$\underline{-4\frac{1}{3}}$	$\underline{-12\frac{2}{5}}$	$\underline{17}$	$\underline{35}$

7. $-9 + 6 + 7 + (-3)$

8. $7 + (-5) + (-6) + 2$

9. $24 + (-19) + (-35) + (-6)$

10. $(-55) + (-11) + 65 + 1$

11. $(-3) + 4 + 1 + (-2)$

12. $23 + 3 + (-6) + (-15)$

13. $7 + (-14) + (-8) + 9$

14. $(-3) + 8 + (-5) + 35$

15. $\frac{1}{2} + 2\frac{1}{2} + (-3\frac{1}{2}) + 4\frac{1}{2} + (\frac{1}{2}) + 2\frac{1}{3} + (-3\frac{1}{3}) + \frac{2}{3} + (-\frac{1}{3})$

16. $\frac{1}{3} + (-\frac{2}{3}) + \frac{4}{5} + (-\frac{5}{6}) + \frac{1}{6} + (-\frac{3}{10}) + 2 + (-3\frac{1}{2}) + (-1) + 1$

17. $2.5 + (-3.5) + 4 + (-5) + 2.5 + (-.5) + 3.5 + 2.4 - 2.9 - 3.6 + 2.6$

18. $-.5 + (-10.1) + .4 + (-1.2) + (-3) + (-.7) + (-1.4) + 5$

In each case, write a chain of equations leading to the stated equation. Justify each equation. Call attention to this sample. It shows that when you add $[a + (-b)]$ to b you obtain a. $a + (-b) = a - b$.

SAMPLE. $b + [a + (-b)] = a$

Solution:
$$\begin{aligned} b + [a + (-b)] &= b + [(-b) + a] && \text{Commutative property}\\ &= [b + (-b)] + a && \text{Associative property}\\ &= 0 + a && \text{Property of opposites}\\ &= a && \text{Property of 0} \end{aligned}$$

B

19. $[b + (-4)] + (-b) = -4$

20. $[(-60) + c] + (60) = c$

21. $a + (b + c) = (a + c) + b$

22. $[a + (-b)] + [b + (-a)] = 0$

23. $a + (b + c) = b + (c + a)$

24. $(-m) + (k + m) = k$

25. $(-5) + [(-s) + (s + 5)] = 0$

26. $[a + c] + [(-a) + (-c)] = 0$

See T.M. pg. 15 (10).

Replace the question mark with a numeral to make a true statement.

27. $(?) + 17 = 15$

28. $10 + (?) = 7$

29. $4 + (?) = -1$

30. $-5 + (?) = -4$

31. $9 + (?) = -17$

32. $(?) + 6 = 5$

33. $(?) + 3 = -8$

34. $-5 + (?) = -5$

35. $\frac{2}{3} + (?) = 1$

PROBLEMS

A

1. A merchant's transactions had the following results: a gain of \$35, a gain of \$14.75, a loss of \$26.10, a gain of \$18.15, a loss of \$7.50. Represent his net gain or loss by means of a signed number.

2. A girls' club took in \$13.00 for the semester's dues and paid out \$7.50 for refreshments, \$1.25 for programs, and \$2.00 for a charity project. Their share in the proceeds of the class play was \$6.50. Use a signed number to represent the financial condition of the club.

3. A housewife made the following entries in her household account one day: groceries \$13.68, bakery \$1.09, meat \$4.17, return for bottles \$.37, Joan's allowance \$.75. Represent each item by a directed number and find the sum.

4. To buy graduation prizes, the Parent-Teacher Association needed more than the \$73 in its treasury. The members presented a play, for which they paid a royalty of \$25. Scenery and costumes cost \$18, and the programs, \$15. The sale of tickets amounted to \$185. Program advertisements brought \$64. Find the amount they then had.

5. A submarine submerged 375 feet below sea level fires a rocket which rises 650 feet. How far above sea level does the rocket go?

6. A football player made the following yardage on five plays: 15, -3, 8, -9, -12. What was his total net gain in yards?

7. If $G = \{-4, -1, 0, 1, 3\}$, find the set of all sums of pairs of elements of G. Is G closed under addition?

8. Bob lost 3 pounds the first week on his 900-calorie diet, gained $1\frac{1}{2}$ pounds the second week, gained $\frac{1}{2}$ pound the third week, and lost 4 pounds the fourth week. What was his total gain oı loss?

9. The temperature at noon was 49°F and at 5 P.M. it was 21.5°F. What was the net change in temperature?

4–7 Subtracting Directed Numbers

One evening the thermometer read 11° above zero. The next morning it read 5° below zero. How much had the temperature changed? Do you get a 16° drop? Do you realize that you just subtracted 11 from -5?

Another situation illustrates something else you know about subtraction. If you buy 85 cents' worth of goods and give the clerk a dollar, he may count your change saying, 85, 95 (handing you a dime), one dollar (handing you a nickel). The clerk did a subtraction problem (100–85) by adding. Another way of saying this is that x has the same value in both of these equations:

Discourage pupils from thinking of subtraction as "taking away." See T.M. pg. 15 (11) for geometric interpretation.

$$100 - 85 = x \qquad \text{and} \qquad 85 + x = 100.$$

Similarly, x has the same value in both of these equations:

$$(-5) - 11 = x \qquad \text{and} \qquad 11 + x = -5.$$

Guided by these results, we make this definition: *For all directed numbers a and b, any directed number satisfying the equation $b + x = a$ is called the* **difference** *of a and b, that is, $a - b$.*

Using only this definition, you do a subtraction problem by asking yourself, "What number added to b gives a?" You can find a simple expression for $a - b$ by transforming the equation $b + x = a$:

$$\begin{aligned} b + x &= a \\ x + b &= a \\ x + b + (-b) &= a + (-b) \\ x + 0 &= a + (-b) \\ \therefore x &= a + (-b) \end{aligned}$$

Run through one or two specific examples like $-3 + x = -1$ and $7 + x = 4$ in which you transform the equation by addition.

The last equation evidently has just one root, $a + (-b)$. Checking this root in the original equation, you have:

$$\begin{aligned} b + x &= a \\ b + a + (-b) &\overset{?}{=} a \\ b + (-b) + a &\overset{?}{=} a \\ 0 + a &\overset{?}{=} a \\ a &= a \quad \checkmark \end{aligned}$$

This is equivalent to $x = a + (-b)$ which has exactly one root.

Since the one and only root of $\boxed{b + x = a}$ is $a + (-b)$, it follows that:

Here − means subtraction. Here − means "the opposite of."

$$a - b = a + (-b).$$

To perform a subtraction, replace the subtrahend by its opposite, and add.

See T.M. pg. 15 (12) for geometric interpretation.

Does this rule give a meaningful expression for $a - b$? As every number has an opposite, if you know b then you know $-b$. Also, since $a + (-b)$ is a sum, it represents a definite number. Hence, the rule shows that the set of directed numbers is closed under subtraction.

Using this rule, you always can replace a subtraction by an addition:

Subtraction	Addition	Value	Check
$6 - 2 =$	$6 + (-2) =$	4	$4 + 2 = 6$
$6 - (-2) =$	$6 + 2 =$	8	$8 + (-2) = 6$
$-6 - 2 =$	$-6 + (-2) =$	-8	$-8 + 2 = -6$
$-6 - (-2) =$	$-6 + 2 =$	-4	$-4 + (-2) = -6$
$a - (a + 7) =$	$a + (-a) + (-7) =$	-7	$-7 + (a + 7) = a$

Ask for examples showing that subtraction is not commutative or associative. Has subtraction an identity element? See T.M. pg. 16 (13).

ORAL EXERCISES

In each case, subtract the lower number from the number above it.

1. $\begin{array}{r} 13 \\ \underline{8} \end{array}$ 5

2. $\begin{array}{r} 17 \\ \underline{9} \end{array}$ 8

3. $\begin{array}{r} 9 \\ \underline{17} \end{array}$ -8

4. $\begin{array}{r} 8 \\ \underline{13} \end{array}$ -5

5. $\begin{array}{r} 7 \\ \underline{-3} \end{array}$ 10

6. $\begin{array}{r} 11 \\ \underline{-5} \end{array}$ 16

7. $\begin{array}{r} -18 \\ \underline{17} \end{array}$ -35

8. $\begin{array}{r} -22 \\ \underline{46} \end{array}$ -68

9. $\begin{array}{r} -28 \\ \underline{-3} \end{array}$ -25

10. $\begin{array}{r} -157 \\ \underline{-108} \end{array}$ -49

11. $\begin{array}{r} -12 \\ \underline{-12} \end{array}$ 0

12. $\begin{array}{r} 15 \\ \underline{15} \end{array}$ 0

13. $\begin{array}{l} a + 4 \\ \underline{a } \end{array}$ 4

14. $\begin{array}{r} 5 + t \\ \underline{t} \end{array}$ 5

15. $\begin{array}{r} -2 + m \\ \underline{-8 + m} \end{array}$ 6

16. $\begin{array}{l} r + (-4) \\ \underline{r + 7} \end{array}$ -11

Perform each of the indicated subtractions.

17. $27 - 5$ 22

18. $25 - 29$ -4

19. $1.3 - (-2.0)$ 3.3

20. $2.3 - (-.7)$ 3

21. $(-.12) - (-.2)$ .08

22. $(-.20) - (-.30)$ .10

23. $(-.3) - .5$ -.8

24. $(-.16) - .4$ -.56

25. $(-.5) - (-.3)$ -.2

26. $(-.2) - (-.7)$ .5

27. $0 - .4$ -.4

28. $0 - (-.8)$ .8

29. $-3.9 - 0$ -3.9

30. $-2.4 - 0$ -2.4

31. $8\frac{1}{2} - (-8\frac{1}{2})$ 17

32. $-3\frac{1}{3} - 3\frac{1}{3}$ $-6\frac{2}{3}$

33. $-2\frac{3}{5} - (-2\frac{3}{5})$ 0

34. $25\frac{4}{9} - 25\frac{4}{9}$ 0

35. $100 - 100$ 0

36. $(-7) - (-7)$ 0

37. $(x + 3) - (x + 2)$ 1

38. $(a + 5) - (a + 4)$ 1

39. $h - (h + 1)$ -1

40. $p - (p + 3)$ -3

WRITTEN EXERCISES

Rewrite these subtraction exercises as additions, and then find the sums.

SAMPLE 1. $\begin{array}{r} 89 \\ \underline{99} \end{array}$

Solution: $89 + (-99) = -10$

Check: $-10 + 99 \overset{?}{=} 89$

$89 = 89$ ✓

SAMPLE 2. $[r + 5] - [r - 4]$

Solution:

$$\begin{aligned} [r + 5] - [r - 4] &= r + 5 - [r + (-4)] \\ &= r + 5 + (-r) + 4 \\ &= 9 \end{aligned}$$

Check: $9 + [r - 4] = 9 - 4 + r = r + 5$

A

1. $\begin{array}{r} 29 \\ \underline{-30} \end{array}$

2. $\begin{array}{r} -69 \\ \underline{-72} \end{array}$

3. $\begin{array}{r} -25 \\ \underline{-25} \end{array}$

4. $\begin{array}{r} -80 \\ \underline{20} \end{array}$

5. $\begin{array}{r} t + 2 \\ \underline{t + 5} \end{array}$

6. $\begin{array}{r} s + (-7) \\ \underline{s + (-3)} \end{array}$

7. $\begin{array}{r} a + b - 12 \\ \underline{a + b - 4} \end{array}$

8. $\begin{array}{r} a - k + 14 \\ \underline{a - k - 9} \end{array}$

9. $1.5 - .5$

10. $1.7 - 1.9$

11. $.6 - (-.3)$

12. $-5.7 - (-.7)$

Recall that the convention on order of operations (see page 23) implies that $a - k - 9 = (a - k) - 9 = a + (-k) + (-9)$. Hence, $a - k - 9$ is the sum of a, $-k$, and -9.

13. $0 - (-\frac{3}{4})$

14. $0 - \frac{2}{3}$

15. $-[-300 + 450] - [230 - 1066]$

16. $[1492 - 1678] - [-44 + 12]$

17. $x + y - (x - y) - y$

18. $u - v - (u + v) + v$

19. $(h + 9) - (-7 + h)$

20. $-(10 - k) - (12 + k)$

Write the following phrases in algebraic symbols, and simplify.

B

21. -2 decreased by 7

22. -7 decreased by 18

23. b less $(b + 4)$

24. c less $(8 + c)$

25. Take $23\pi + 1$ from $23\pi - 1$.

26. Take $8\pi - 5$ from $-8\pi + 5$.

27. From the sum of 37 and -12 subtract 49.

28. From the sum of 42 and -51 subtract 27.

29. Subtract $-97 + a$ from $-32 + a$.

30. Subtract $18 - r$ from $19 - r$.

31. Prove: $-(a - b) = -a + b$.

32. Prove: $-(a + b + c) = (-a) + (-b) + (-c)$.

Using the convention on order of operations: $(-a) + (-b) + (-c) = -a - b - c$.

Which of these sets are closed under subtraction? Explain.

33. {odd integers}

34. {even integers}

35. {odd integers, 0}

36. $\{0, \frac{1}{2}, -\frac{1}{2}, \frac{2}{2}, -\frac{2}{2}, \frac{3}{2}, -\frac{3}{2}\}$

Remind students that zero is an even integer.

PROBLEMS

Solve these problems by using directed numbers.

Ⓐ

1. Find the difference in altitude between Salton Sea, California, 244 feet below sea level, and a spot in Death Valley, 276 feet below sea level.

2. Find the change in temperature on a winter day when the temperature dropped from 3° below zero to a low of 11° below zero.

3. At 6 P.M. the thermometer read 8° above zero. At midnight it read 5° below zero. Find the change in temperature.

4. In the New York City Rapid Transit System a high point is 161 feet above sea level, and a low point is 113 feet below sea level. Find the change in altitude in going from the high point to the low.

5. The Peloponnesian War began in 431 B.C. Peace was finally made in 404 B.C. How long did the war last?

6. The Greek mathematician Archimedes was born in 287 B.C. and died in 212 B.C. How long did he live?

7. John owes his father \$4.38. How much must John pay to acquire a credit of \$1.25 with his father?

8. In playing a game Ellen was 175 points "in the hole." How many points must she make to have a score of 250 points?

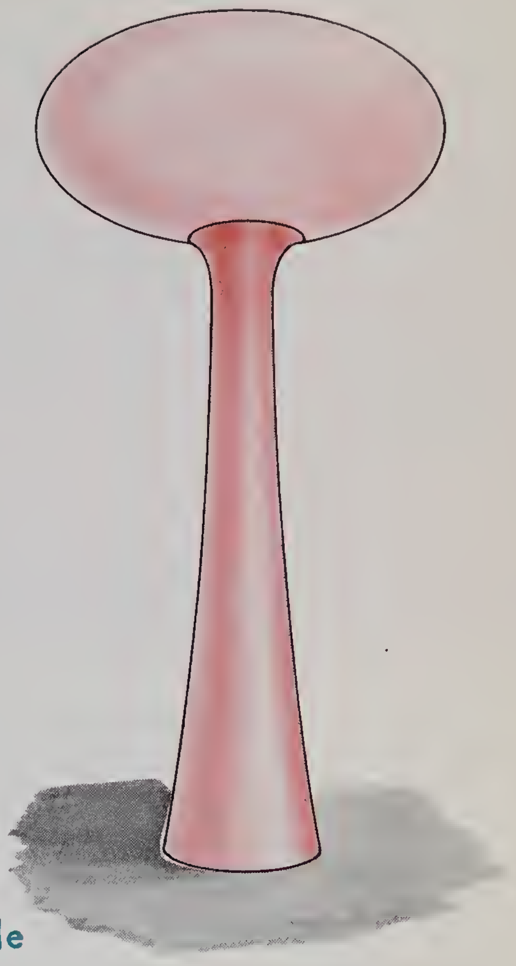
9. Carthage was destroyed in 146 B.C. How many years ago was that event? (Assume no year 0.)

10. If the sea is 37,800 feet deep and the highest mountain is 29,012 feet high, find the difference in elevation between them.

11. Mr. Lescaire had a bank balance of \$317.25 on Monday. On Friday the bank said he was overdrawn by \$9.47. How much had Mr. Lescaire spent during that period?

12. If New York's latitude is 41°N and Rio de Janeiro's is 23°S, find the difference in latitude between the two cities.

4–8 Multiplying Directed Numbers



Probably you were first introduced to multiplication in arithmetic by some explanation such as this: "When we write 3 × 2, we mean take 2 three times. If you have 3 boxes each with 2 apples, you have 3 × 2 apples. You also have 2 + 2 + 2 apples."

If you try to give meaning to $3 \times (-2)$, you cannot talk about apples in boxes, but you can talk about $(-2) + (-2) + (-2)$, so you can say that that is what $3 \times (-2)$ means. But, when you try to talk about $(-3) \times 2$, you have trouble. You cannot take 2 "minus three" times conveniently. To help solve this dilemma, consider a different kind of example.

Suppose water is flowing *into* a tank at the rate of 3 gallons per minute (*3*). You can make the following statements.

Pupils should not feel that this example shows the rules for multiplication. It just suggests what rules would fit this particular application.

1. Two minutes hence (2), there will be 6 gallons more (6) in the tank.

$$(2)(3) = 6$$

positive number × positive number gives positive number

2. Two minutes ago (−2), there were 6 gallons less (−6) in the tank.

$$(-2)(3) = -6$$

negative number × positive number gives negative number

Suppose that water is flowing *out* of the tank at the rate of 3 gallons per minute (−*3*). You can make these statements.

3. Two minutes hence (2), there will be 6 gallons less (−6) in the tank.

$$(2)(-3) = -6$$

positive number × negative number gives negative number

4. Two minutes ago (−2), there were 6 gallons more (6) in the tank.

$$(-2)(-3) = 6$$

negative number × negative number gives positive number

The rules suggested by these examples can be developed from the following assumptions for multiplication. These are the familiar properties stated for all directed numbers, not just arithmetic numbers.

For a, b, and c, members of the set of directed numbers:

1. The closure property: for every a and b, the product ab is a unique directed number.
2. The commutative property: $ab = ba$.
3. The associative property: $a(bc) = (ab)c$.
4. The distributive property: $a(b + c) = ab + ac$.
5. The multiplicative property of 0: $a \cdot 0 = 0 \cdot a = 0$.
6. The multiplicative property of 1: $a \cdot 1 = 1 \cdot a = a$.

This last property might make you curious about the product $a \cdot (-1)$. Would you guess that it would be $-a$? To verify this guess, show that the sum of $a(-1)$ and a is zero:

Then, as the additive inverse of a is unique, $a(-1)$ must be $-a$.

$$\begin{array}{rll}
a(-1) + a \stackrel{?}{=} 0 & & \\
a(-1) + a(1) \stackrel{?}{=} 0 & & \textbf{Multiplicative property of 1} \\
a[(-1) + 1] \stackrel{?}{=} 0 & & \textbf{Distributive property} \\
a(0) \stackrel{?}{=} 0 & & \textbf{Property of opposites} \\
0 = 0 & \checkmark & \textbf{Multiplicative property of zero}
\end{array}$$

Therefore, the ***multiplicative property of −1*** is: Emphasize this result and the example proving it. Illustrate graphically; see T.M. pg. 16 (14).

Multiplying any number by −1 gives its opposite.
For any a, $a(-1) = (-1)a = -a$.

A special case of this property occurs when $a = -1$; this gives $\mathbf{(-1)(-1) = 1}$. You now can justify the products you obtained in the four cases of multiplication illustrated by the water tank problem by writing the following chains of equalities: Point out the use of the commutative and associative properties.

1. $(2)(3) = 6$
2. $(-2)(3) = [(-1)(2)](3) = (-1)[2(3)] = (-1)(6) = -6$
3. $(2)(-3) = (2)[(-1)(3)] = [(-1)(2)](3) = (-1)[2(3)]$
   $= (-1)(6) = -6$
4. $(-2)(-3) = [(-1)(2)][(-1)(3)] = [(-1)(-1)][(2)(3)] = 1(6) = 6$

Similarly, for all numbers a and b: Have the students read these:

$$b(-a) = (-a)(b) = [-1(a)]b = (-1)(ab) = -ab$$

$$(-a)(-b) = [-1(a)][-1(b)] = [(-1)(-1)](ab) = 1(ab) = ab$$

b times the opposite of a = the opposite of ab.
The opposite of a times the opposite of b = ab.

Do you see that the following statements are true?

1. The absolute value of the product of two directed numbers is the product of the absolute values of the numbers.
2. The product of a positive and negative number is a negative number.
3. The product of two positive numbers or of two negative numbers is a positive number.

By pairing $(-1)(-1) = 1$, you can extend these rules to any number of factors.

1. The absolute value of an indicated product of numbers is the product of the absolute values of the numbers.
2. An indicated product containing an odd number of negative factors is a negative number.
3. An indicated product containing an even number of negative factors is a positive number.

Since the distributive property is assumed to hold for directed numbers, variables in the terms of an expression are treated as they have been previously.

EXAMPLE. Simplify: $6x - 4y - 5x + 8y + 7x - 9y$

Solution:

$$\begin{aligned} 6x - 4y - 5x + 8y + 7x - 9y &= (6 - 5 + 7)x \\ &\quad + (-4 + 8 - 9)y \\ &= 8x - 5y \end{aligned}$$

Encourage pupils to read this expression as "the sum of 6x, -4y, -5x, 8y, 7x, and -9y." See T.M. pg. 16 (15) for simplifying steps.

ORAL EXERCISES

Find each of the indicated products.

1. $(4)(5)$ 20
2. $(-6)(-2)$ 12
3. $(7)(-3)$ -21
4. $(2)(9)$ 18
5. $(-5)(-4)$ 20
6. $(-15)(-2)$ 30
7. $(-3)(5)$ -15
8. $-4(7)(-1)$ 28
9. $-1(1)(-1)$ 1
10. $\frac{1}{2}(-2)$ -1
11. $(-3)(-\frac{1}{3})$ 1
12. $4a(-5a)(10)$ $-200a^2$
13. $(-3)(2a)(a)$ $-6a^2$
14. $(7a)(0)(-6b)$ 0
15. $4(-\frac{1}{2})(-2b)b^2$ $4b^3$
16. $7(-\frac{1}{3})(-3d)d^3$ $7d^4$
17. $(-4)(-2)(0)(-1)$ 0
18. $(-x)(-xy^2)(-y)(0)$ 0
19. $(-2)^3$ -8
20. $(-3)^3$ -27
21. $2(-3)^4$ 162

$-(7x)$ or $(-7)x$ or $7(-x)$ read "the opposite of $7x$" or "negative 7 times x" o
"7 times the opposite of x."

Name the opposite of each of the following.

22. $7x$ $-7x$
23. $-3r$ $3r$
24. $-4a + 3b$ $4a - 3b$
25. $3t - 2s$ $-3t + 2s$
26. $-2x - 7$ $2x + 7$
27. $5k - 6$ $-5k + 6$

Subtract the lower expression from the one above it.

28. $$\begin{array}{r} -7x + 7y \\ 5x + (-5y) \\ \hline -12x + 12y \end{array}$$

29. $$\begin{array}{r} -3a + \quad 8b \\ 9a + (-9b) \\ \hline -12a + 17b \end{array}$$

30. $$\begin{array}{r} 8m + \quad 12n \\ -m + (-3n) \\ \hline 9m + 15n \end{array}$$

31. $$\begin{array}{r} -20b^2 \\ -33b^2 \\ \hline 13b^2 \end{array}$$

32. $$\begin{array}{r} 49x + 7y \\ 76x - 8y \\ \hline -27x + 15y \end{array}$$

33. $$\begin{array}{r} -29m^2 + 15m \\ -31m^2 + \quad 7m \\ \hline 2m^2 + 8m \end{array}$$

The following proofs are valid for all a, b, and c. Justify each step.

SAMPLE. To prove, $a(b - c) = ab - ac$

Steps:	*What you say:*
$a(b - c) = a[b + (-c)]$	Meaning of subtraction
$= a(b) + a(-c)$	Distributive property
$= ab + a[c(-1)]$	Multiplicative property of -1
$= ab + (ac)(-1)$	Associative property
$= ab + [-(ac)]$	Multiplicative property of -1
$= ab - ac$	Meaning of subtraction

This exercise shows that multiplication of directed numbers is distributive with respect to subtraction.

34. To prove, $-(-b) = b$

Mult. prop. of -1	$-(-b) = (-1)(-b)$
Mult. prop. of -1	$= (-1)[(-1)(b)]$
Assoc. prop. of mult.	$= [-1(-1)]b$
Mult. prop. of -1	$= 1 \cdot b$
Mult. prop. of 1	$= b$

35. To prove, $-(a + b) = -a - b$

Mult. prop. of -1	$-(a + b) = (-1)(a + b)$
Distrib. prop.	$= (-1)a + (-1)b$
Mult. prop. of -1	$= -a + (-b)$
Meaning of subt.	$= -a - b$

36. To prove, $-(a - b) = b - a$

Mult. prop. of -1	$-(a - b) = (-1)[a + (-b)]$
Distrib. prop.	$= (-1)a + (-1)(-b)$
Mult. prop. of -1	$= -a + b$
Comm. prop. of add.	$= b + (-a)$
Meaning of subt.	$= b - a$

WRITTEN EXERCISES

Evaluate each of the following numerical expressions (a) by combining terms, and (b) by using the distributive property.

SAMPLE. $(-3)[7-(-2)]$

Solution: **a.** $(-3)[7-(-2)]$
$= (-3)[7+2]$
$= (-3)(9)$
$= -27$

b. $(-3)[7-(-2)]$
$= (-3)(7)-(-3)(-2)$
$= -21-6$
$= -27$

A

1. $(-4)(10+11)$
2. $(-3)[(-2)+(-5)]$
3. $(20-21)(4)$
4. $(5)(-6-3)$
5. $0(-8+5)$
6. $[-3-(-7)]0$
7. $4(3)-4(13)$
8. $(-2)9+(-2)$
9. $(-99)(12)+(-12)$
10. $40(99)+(40)(-99)$
11. $(-13)(-7)+(-7)(13)$
12. $(-12)(4+\frac{1}{3})$
13. $14(-5-\frac{1}{7})$
14. $(-48)(-3\frac{1}{12})$
15. $-(-7+12)$
16. $-(-34-20)$
17. $.8-(2.5-.32)$
18. $6-(-1.9+.27)$
19. $-.3-.2(-.7-.3)$
20. $-.7-.2(-.1-.4)$

Combine the similar terms in each of the following expressions.

21. $2a-3+5a-10a+8$
22. $4r-7+3r+9-8r$
23. $-6x+5-2x+x-12$
24. $-8y+8-6y-y+14$
25. $-4nt+8-6-nt+3nt$
26. $-9hk+5hk-8+hk+5$
27. $4r+5s-r+s-6r$
28. $16a-9b-a+b-7b$
29. $1.5n-8-3.5n+5$
30. $\frac{1}{3}y-\frac{3}{2}-\frac{2}{3}y+1+y$
31. $-3a+b+4a+a-b$
32. $9u-u+5-6u-6+2u$
33. $33k^3-k^3+4k^3-40k^3$
34. $xyz-8xyz+5xyz-2xyz$
35. $-d^2-.1d^2+.5d^2+1.8d^2$
36. $5(r+s)-6(r+s)-8(r+s)$
37. $4x^2-5x-6x^2+7x$
38. $-13y+2y^2-4y^2+6y$
39. $m^2-2m-3-m^2+4m$
40. $u^3-2u^2+4u+3u^2-4u$

B

41. $-2(r+3s)+5(-r-s)$
42. $-7(a-3b)+9(-b+a)$
43. $3(p-2q)-(5q-2p)$
44. $-4(-7v+t)-(-t-3v)$

45. $2[3(a - 1) + 5] - 4$
46. $3[-6 + 2(a - 4)] + 26$
47. $-7n - 5[2(1 + 2n) - 3]$
48. $-10h - 4[-1 - 3(3h - 2)]$

Evaluate each of the following algebraic expressions, using $a = -1$, $b = .2$, $c = -2$, $x = .3$, and $y = -3$.

49. $.7xy - .5b$
50. $.3c - .2ab$
51. $-\frac{1}{2}b + .4y - \frac{1}{3}x$
52. $.2(a + 1)^2$
53. $a^2 + .2a + .1$
54. $1.3(y + 1)^3$
55. $y^3 - 3y^2 + 3y - 1$
56. $.04x^3 + .05y^3$
57. $.01c^4 - .01a^4$

C
58. $a(b + 3c)^3$
59. $c(2b - y)^2$
60. $x^2(y - a^3)$
61. $y^3(b^2 + a^5)$
62. $1.1(2a^2 - b)(2a^2 + b)$
63. $1.01(3y + x^2)(3y - x^2)$

4–9 Dividing Directed Numbers

Note analogy between division defined in terms of multiplication, and subtraction defined in terms of addition.

See T.M. pg. 16 (16).

You know the following two facts for arithmetic numbers: (1) division by zero is meaningless; (2) division is the inverse operation of multiplication, that is, $6 \div 2 = 3$ because $3 \cdot 2 = 6$; and division may be written in the form of a fraction, that is, $6 \div 2 = \frac{6}{2}$. To give the division of directed numbers the same properties, this definition is made: *for all directed numbers a and b, the directed number x satisfying the equation $xb = a$ is called the* **quotient** *of a and b* $\left(a \div b \text{ or } \frac{a}{b}\right)$.

You have been studying opposites, or additive inverses. Now you shall need to know about *multiplicative inverses* or *reciprocals*. Two numbers whose product is 1, *the identity element for multiplication*, are called **reciprocals**. For example,

$$6 \cdot \tfrac{1}{6} = 1, \quad .2 \cdot 5 = 1, \quad \frac{m}{n} \cdot \frac{n}{m} = 1, \quad 1 \cdot 1 = 1$$

$$-\tfrac{1}{5}(-5) = 1, \quad -.25(-4) = 1, \quad -1 \cdot (-1) = 1.$$

That each number except zero has a reciprocal is a basic assumption.

For every directed number a other than 0, there is a unique number, denoted by $\frac{1}{a}$, such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.

Do you see that a number and its reciprocal are either both positive or both negative numbers? See T.M. pg. 17 (17).

A statement about reciprocals that corresponds to the property of opposites on page 121 is:

For proof see T.M. pg. 17 (18).

The reciprocal of a product of two numbers, each different from 0, is the product of the reciprocals of the numbers.

$$\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$$

Reciprocals enable you to express a quotient as a product. If $b \neq 0$, the root of $xb = a$ is $a \cdot \frac{1}{b}$, as you can show.

$$xb = a$$

$$xb \cdot \frac{1}{b} = a \cdot \frac{1}{b}$$

$$x \cdot 1 = a \cdot \frac{1}{b}$$

$$x = a \cdot \frac{1}{b}$$

This last equation has just one root, $a \cdot \frac{1}{b}$. Checking in the original equation:

$$xb = a$$

$$\left(a \cdot \frac{1}{b}\right) b \stackrel{?}{=} a$$

$$a \left(\frac{1}{b} \cdot b\right) \stackrel{?}{=} a$$

$$a \cdot 1 \stackrel{?}{=} a$$

$$a = a \quad \checkmark$$

Since the one and only root of $xb = a$ is $a \cdot \frac{1}{b}$, it follows that:

$$a \div b = a \cdot \frac{1}{b}, \qquad b \neq 0$$

To perform a division, replace the divisor by its reciprocal, and multiply.

Does this rule give a meaningful expression for $a \div b$? As every number except 0 has a reciprocal, and $b \neq 0$, there is a number $\frac{1}{b}\cdot$ As $a \cdot \frac{1}{b}$ is a product, it represents a definite number. Thus, the set of directed numbers is closed under division, not including division by 0.

Notice how the properties of multiplication ar used here. (-12) (-1/3) =(-1) (12) (-1) (1/3) (-1) (-1) (12•1/3) =1•4 =4.
(-2) (1/5) =(-1) (2•1/5)=(-1) (2/5) =-2/5
1/-5 = - 1/5 because (-1/5) (-5) = (-1) (-1) (1/5•5) = 1. Recall the meaning of 1 ÷ -

EXAMPLE 1. $\dfrac{-12}{-3} = -12\left(-\dfrac{1}{3}\right) = 4$

EXAMPLE 2. $\dfrac{-2}{5} = -2\left(\dfrac{1}{5}\right) = -\dfrac{2}{5}$

EXAMPLE 3. $\dfrac{2}{-5} = 2\left(-\dfrac{1}{5}\right) = -\dfrac{2}{5}$

EXAMPLE 4. $\dfrac{2}{3} \div \left(-\dfrac{4}{5}\right) = \dfrac{2}{3}\left(-\dfrac{5}{4}\right) = -\left(\dfrac{2}{3}\cdot\dfrac{5}{4}\right) = -\dfrac{5}{6}$

Pupils may use either approach, $\frac{-12}{-3} = 4$ because $4(-3) = -12$, or because $\frac{-12}{-3} = -12(1/-3) = (-1)(-1)(12\cdot1/3) = 4$.

ORAL EXERCISES

Give each of the following quotients.

1. $\dfrac{12}{3}$ 4
2. $\dfrac{12}{-3}$ -4
3. $\dfrac{-12}{3}$ -4
4. $\dfrac{-12}{-3}$ 4
5. $\dfrac{0}{3}$ 0
6. $\dfrac{0}{-8}$ 0
7. $\dfrac{3}{3}$ 1
8. $\dfrac{-5}{-5}$ 1
9. $\dfrac{5}{15}$ $\frac{1}{3}$
10. $\dfrac{4}{-4}$ -1
11. $\dfrac{-3}{15}$ $-\frac{1}{5}$
12. $\dfrac{-15}{15}$ -1
13. $a \div 1$ a
14. $a \div (-1)$ $-a$
15. $6a \div (-2)$ $-3a$
16. $(-12b) \div 4$ $-3b$
17. $(-50a) \div (-10)$ $5a$
18. $(-60y) \div (-12)$ $5y$
19. $\dfrac{a}{a}$ for $a \neq 0$ 1
20. $\dfrac{-a}{a}$ for $a \neq 0$ -1

Solve each of the following equations.

21. $5r = -10$ $r = -2$
22. $6s = -18$ $s = -3$
23. $-2t = 8$ $t = -4$
24. $-3t = 6$ $t = -2$
25. $-4t = -8$ $t = 2$
26. $-3t = -9$ $t = 3$
27. $-5z = 0$ $z = 0$
28. $0 = -3z$ $z = 0$
29. $-4 = 3w$ $w = -\frac{4}{3}$
30. $3 = -5w$ $w = -\frac{3}{5}$
31. $-7 = -7x$ $x = 1$
32. $-5x = -5$ $x = 1$

Give the multiplicative inverse of each number.

33. 1 1

34. -1 -1

35. $\frac{3}{4}$ $\frac{4}{3}$

36. 10 $\frac{1}{10}$

37. $-\frac{9}{8}$ $-\frac{8}{9}$

38. $-\frac{3}{10}$ $-\frac{10}{3}$

39. $\frac{a}{2}, a \neq 0$ $\frac{2}{a}$

40. $\frac{3}{b}, b \neq 0$ $\frac{b}{3}$

41. $-c, c \neq 0$ $-\frac{1}{c}$

42. $-\frac{1}{d}, d \neq 0$ $-d$

In Exercises 43–46, find the value of *a*.

43. $\frac{1}{a} = -2$ $-\frac{1}{2}$

44. $\frac{1}{a} = -\frac{1}{3}$ -3

45. $\frac{1}{a} = -\frac{4}{3}$ $-\frac{3}{4}$

46. $\frac{1}{a} = -1$ -1

Determine which of the following sets of numbers are closed under (a) division, and (b) multiplication.
C = closed ; NC = not closed

47. $\{1\}$ (a) C (b) C

48. $\{-\frac{1}{2}, -1, \frac{1}{2}, 1\}$ NC(b)NC

49. $\{-1, 0, 1\}$ (a)NC(b)C

50. $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\}$ (a)NC (b)NC

51. $\{3, 1, \frac{1}{3}\}$ (a)NC (b)NC

52. $\{0\}$ (a)NC (b) C

53. {positive numbers} (a)C (b)C

54. {negative numbers} (a)NC (b)NC

55. $\{-1, -2, -3, -4, \ldots\}$ (a)NC (b)NC

56. $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\}$ (a)NC (b) C

WRITTEN EXERCISES

Find each of the indicated quotients.

A

1. $(-\frac{3}{7}) \div 3$

2. $\frac{-.25}{-5}$

3. $(-12) \div (-\frac{1}{3})$

4. $(-\frac{4}{9}) \div 4$

5. $\frac{-.36}{6}$

6. $(-16) \div (-\frac{1}{4})$

7. $(36) \div (-\frac{1}{2})$

8. $\frac{24.5}{-5}$

9. $(3b) \div (-\frac{1}{3})$

10. $(8) \div (-\frac{1}{4})$

11. $\frac{2.52}{-7}$

12. $(7c) \div (-\frac{1}{7})$

13. $(-15.6) \div 12$

14. $1.82 \div (-13)$

15. $(-.195) \div (-13)$

16. $(-39.1) \div 17$

17. $4.56 \div (-24)$

18. $(-.528) \div (-24)$

Evaluate each of the following expressions. Use $r = -2$, $s = -1$, $t = 3$, $u = -6$, $v = 10$.

B

19. $\frac{vu}{rst}$

20. $\frac{rtv}{su}$

21. $\frac{r^2}{s^3}$

22. $\frac{rt^2}{u}$

23. $\frac{uv^2}{r^3}$

24. $\frac{su^2}{t^2}$

25. $\frac{s^5}{rt}$

26. $\frac{s^6}{uv}$

27. $\frac{t^3}{r^2u}$

28. $\frac{r^3}{t^2v}$

29. $\frac{(r+2)^4}{u}$

30. $\frac{(u+6)^3}{r}$

C 31. $\dfrac{(rt)^2 - u^2}{v^2}$

32. $\left(\dfrac{u}{r}\right)^3 - t^3$

33. $\dfrac{(r + 1)^3 + s^7}{(u + 4)^2}$

34. $\dfrac{(v - 1)^2 - t^3}{(ru)^2}$

35. $u^2 + \left(\dfrac{v}{r}\right)^3$

36. $\dfrac{s^8 + (u - 1)^2}{(t + 2)^2}$

4–10 Averages and Directed Numbers (Optional)

The arithmetic average, or mean, of a set of n numbers is their sum divided by n. Sometimes you can reduce the labor by using directed numbers as follows:

1. Assume an average; that is, make a guess.
2. Express as a directed number the difference (or *deviation*) between each number and your guess.
3. Find the average of these differences (the average deviation).
4. Add the average deviation to the assumed average (your guess), and you have the correct average.

EXAMPLE **Excluding the ends, the linemen on the State University football team weighed as follows: 227 lb., 194 lb., 200 lb., 189 lb., 230 lb. Find the average weight.**

Solution:

1. Assume an average of 200 pounds.	**Assumed Average: 200 lb.**
2. Subtract the assumed average from each weight to get the deviations.	**Weight — Deviation** **227 — 27** **194 — −6** **200 — 0** **189 — −11** **230 — 30**
3. Divide the sum of the deviations by the number of cases to find the average deviation.	**Sum of deviations: 40** **Average deviation: $\frac{40}{5} = 8$**
4. Add the average deviation to the assumed average to find the true average.	**True average $= 200 + 8$** **$= 208$ lb., Answer.**

This practical application of directed numbers shows the usefulness of estimating an answer.

The better your guess, the smaller the average deviation, but what you guess really doesn't matter. Suppose you assumed 210 pounds as the average.

Weight	Deviation
227	17
194	-16
200	-10
189	-21
230	20

Sum of deviations: -10

Average deviation: $\frac{-10}{5} = -2$

$$\begin{aligned}\textbf{True average} &= 210 - 2\\ &= 208 \text{ lb., Answer.}\end{aligned}$$

To see why this method works, suppose M represents the true average of numbers represented by a, b, c, d, e, and f and assume an average m. The deviations are $a - m$, $b - m$, $c - m$, $d - m$, $e - m$, and $f - m$, where m is the assumed average.

$$\begin{aligned}\textbf{Sum of deviations} &= a + b + c + d + e + f - 6m\\ \textbf{Average deviation} &= \frac{a + b + c + d + e + f}{6} + \frac{-6m}{6}\\ \therefore\ \textbf{Average deviation} &= M - m\\ \text{or, } M &= m + \text{average deviation.}\end{aligned}$$

Of course, the same method would have worked for 7, 8, or any number of members, as well as for 6.

PROBLEMS

Do each problem by assuming an average and finding the deviations from it.

A

1. Find the average of the following tuition charges: \$2200, \$1980, \$2050, \$1880, \$2100, and \$1930.
2. Weather balloons released on successive days reached these altitudes (in meters): 7650, 8630, 5600, 9550, 8550, 7550, 8520. Find the average altitude reached by these balloons.
3. In test dives, a diving bell reached these depths: -5200 ft., -5600 ft., -5900 ft., -6100 ft., -6500 ft. Find the average depth.
4. Find the average of the following weights: 17.4 g., 16.6 g., 16.7 g., 17.9 g., 15.9 g., 17.3 g., and 17.5 g.

5. Fahrenheit temperature readings taken at 8 A.M. each morning during one week were: $13°$, $5°$, $-6°$, $-9°$, $-14°$, $-4°$, $2°$. Find the average 8 A.M. temperature for that week.

6. The following daily net changes in the price of one stock were observed in the course of two weeks: $-1\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{1}{2}$, $2\frac{1}{2}$, $1\frac{1}{2}$, $-\frac{1}{2}$, $3\frac{1}{2}$, 0, $-\frac{1}{4}$, $4\frac{1}{2}$. Find the average daily net change in price.

7. At different times, eight scientists determined experimentally the value of a constant, as follows: 4.177, 4.188, 4.196, 4.196, 4.186, 4.188, 4.184, 4.181. Find the average value.

8. A high school physics class was trying to verify a rule. They obtained the following values for a constant: 1654.8, 1655.6, 1654.8, 1643.5, 1645.1, 1646.0, 1649.6, 1645.8. Find the average value.

Chapter Summary

Inventory of Structure and Method

1. The number scale can be extended to the left as well as to the right of the zero-point. Points to the left correspond to **negative numbers,** whose numerals include **minus** signs. Points to the right correspond to positive numbers, whose numerals may include plus signs. For example, $^{-}3$ marks the point 3 units to the left of 0; $^{+}3$, or 3, the point 3 units to the right of 0.

   A number a is greater than every number to its left and less than every number to its right on the number line.

2. The sum $a + b$, for any numbers a and b on the number line is found by moving from a a distance $|b|$ units in the direction associated with b. Every number a on the number line has an **opposite** or **additive inverse** such that $a + (-a) = 0$. Thus, $-(^{+}3) = {}^{-}3$ and $-(^{-}3) = {}^{+}3$. $-0 = 0$. The addition of directed numbers can be developed without reference to the number line if you assume closure, commutative, and associative properties, and properties of zero and of opposites.

3. The difference $a - b$ is defined as that number which added to b gives a. Subtracting b is the same as adding the opposite of b. The set of directed numbers is closed under subtraction.

4. You can deduce other properties of multiplication of directed numbers from the closure, commutative, associative, and distributive properties and properties of 1 and 0. You have $(-1) \cdot a = -a$. For a and b, any

directed numbers, $|ab| = |a| \cdot |b|$, $ab > 0$ if a and b are both positive or both negative, and $ab < 0$ if one factor is positive and the other, negative. If $a = 0$ or $b = 0$, $ab = 0$.

5. If $b \neq 0$, the quotient $a \div b$, or $\frac{a}{b}$, is defined as that number which multiplied by b gives a; $\frac{a}{0}$ is not defined. Every number a, $a \neq 0$, has a **reciprocal** or **multiplicative inverse,** $\frac{1}{a}$; a and $\frac{1}{a}$ are both positive or both negative. Zero has no reciprocal. $\frac{a}{b} = a\left(\frac{1}{b}\right)$, $b \neq 0$.

6. (Optional) A relatively easy way of finding an average A of a set of n numbers is to use a guessed average G. Subtract G from each number to find its ***deviation.*** $A = G +$ the average deviation.

Vocabulary and Spelling

two-way scale (*p. 111*)
signs of direction (*p. 112*)
positive number (*p. 112*)
negative number (*p. 112*)
magnitude (*p. 112*)
directed number (*p. 112*)
signed number (*p. 112*)
displacement (*p. 116*)
identity element for addition (*p. 117*)
opposite of a number (*p. 120*)
additive inverse (*p. 120*)
absolute value (*p. 123*)
reciprocal (*p. 138*)
identity element for multiplication (*p. 138*)
multiplicative inverse (*p. 138*)
average (*p. 142*)
deviation (*p. 142*)

Chapter Test

4–1 Use a number to express the following.

1. **a.** 3 seconds before blast-off
b. the longitude of the Greenwich meridian

4–2 **2.** The greater of the two numbers, 3 and -5, is $\underline{?}$.

3. Graph the solution set for the variable on the number line. Consider the set of directed numbers as the replacement set.

a. $-2 \leq r \leq 0$ **b.** $p < -2$

4. Translate into an open sentence and give the replacement set for the variable: In making a tool, a company specifies that the diameter should be 3 inches, but may be as much as 1% off.

4-3 5. Add: $(47) + [(-71) + (-14)]$

6. A submarine submerged to a depth of 125 feet rises 80 feet. What is its new position with respect to the surface?

4-4 7. Using $\{-3, -1, 0, 3, 4\}$ as the replacement set for s, solve

a. $-s = 3$ b. $-s > 0$

4-5 8. If $|x| = 2$, then x may equal ___?___ or ___?___.

4-6 9. Add: $(-92) + (-36) + (153) + (-87)$

4-7 10. Perform the indicated operations: $15 - 44 - 1 + 7 - 12$

11. Subtract the sum of 19 and -5 from their difference.

12. A family moves from a city whose average temperature is 61.3° to a city whose average temperature is 47.6°. Find the change in average temperature.

4-8 13. Perform the indicated operations: $59(-18) - 32(-59)$

14. Simplify: $3(a - b) - 7a + 3b$

15. Evaluate: $-5m^4 \cdot \frac{1}{9}n^{17}$, when $m = -3$ and $n = -1$

16. Multiply and simplify: $5(ab^4 - 8b^5) - 3(a^3b - 3a^2b^2)$

4-9 17. Solve: a. $-1.56 = 13t$ b. $-16h = -12$

18. Write the multiplicative inverse of: a. $-\frac{1}{8}$ b. $1\frac{3}{4}$

19. Find the quotient: $36x^9y^8 \div (-24)$

4-10 20. (Optional) Using $-3°$ as the assumed average, find the true average of these hourly temperatures:

$-5°, -5°, -3°, -1°, 0, 3°, 10°, 13°$.

Chapter Review

4-1 Directed Numbers *Pages 111–113*

1. Numerals with plus signs designate points to the ___?___ of the zero-point.

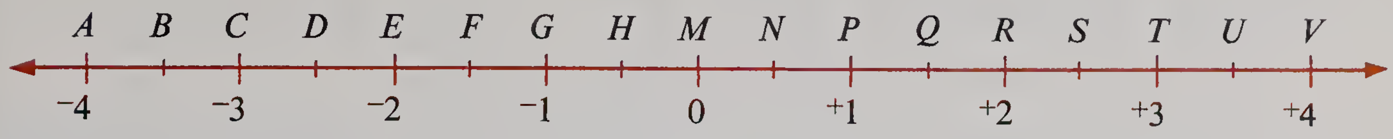
2. Numerals with ___?___ signs designate points to the left of zero.

3. Zero is neither ___?___ nor ___?___.

4. The number corresponding to the point 4.5 units to the left of the origin is written ___?___.

5. Positive and negative numbers are known as ___?___ or ___?___ numbers because their numerals contain signs of direction.

Questions 6–9 refer to the number line below.



Identify each of the following quantities by a lettered point.

6. A 3-point penalty

7. A discharge of 2.5 amperes

8. A rise of $1\frac{1}{2}$ points

9. The ground floor of a house

4-2 Comparing Numbers *Pages 114–116*

In the following, consider to the right and up as positive directions.

10. A number on the horizontal number line is ___?___ than any number to the left of it.

11. Zero is less than any number to the ___?___ of it on the horizontal scale.

12. Saying that $n < 0$ is equivalent to saying that n is a ___?___ number.

13. A number on the vertical scale is ___?___ than any number above it.

State which number in each pair is the greater.

14. $-3, -2$ **15.** $-3, 2$ **16.** $-3, 0$ **17.** $-3, -5$

In Exercises 18–20, consider the replacement set of the given variable to be $\{-5, -4, -2, 0, 1, 4\}$. *Find the solution set in each case.*

18. $n > 0$ **19.** $s \leq -2$ **20.** $-3 < w < 1$

4-3 Addition on the Number Line *Pages 116–120*

21. Addition of 5 means a displacement of 5 units to the ___?___.

22. Addition of -3 means a displacement of 3 units to the ___?___.

23. A displacement of -2 from -5 brings you to ___?___.

Perform the indicated additions.

24. $[(-24) + (-37)] + 45$ **25.** $48 + (-71) + (-19 + 42)$

Using the set of directed numbers as the replacement set for the variable, solve each of the following.

26. $m + (-21) = -12$ **27.** $r + (-17) = -17$

4-4 The Opposite of a Directed Number *Pages 120–123*

28. The opposite of a positive number is the __?__ number having the __?__ absolute value.

29. The opposite of zero is __?__.

30. The identity element for addition is __?__.

31. When the sum of two numbers is __?__, each is the opposite or the __?__ __?__ of the other.

32. The opposite of the sum of two numbers is the sum of the __?__ of the numbers.

If $\{-4, -2, -1, 0, 1, 4\}$ *is the replacement set for* x*, find the solution set.*

33. $-x = 1$ **34.** $-x < 1$ **35.** $-x > -2$

4-5 Absolute Value *Pages 123–124*

36. The absolute value of a number corresponds to its __?__ from zero, regardless of the direction.

37. The absolute value of a number is also called its __?__.

38. The absolute value of -3 is written __?__ and equals __?__.

For each number, give another having the same absolute value.

39. 3.2 **40.** -12 **41.** $3\frac{1}{5}$ **42.** $-.085$

For Exercises 43–46, find the solution set. The replacement set is $\{-5, -4, -2, 0, 1, 4\}$.

43. $4 = |u|$ **44.** $-3 = |t|$ **45.** $-4 < |v|$ **46.** $2 \leq |-s|$

4-6 Adding Directed Numbers *Pages 124–128*

47. The sum of positive numbers is a __?__ number.

48. The sum of negative numbers is a __?__ number.

49. When you add a positive and a negative number,

- **a.** The numerical value of the sum is the __?__ between the greater absolute value and the smaller.
- **b.** The sum is positive if the smaller absolute value belongs to the __?__ number.
- **c.** The sum is negative if the greater absolute value belongs to the __?__ number.

In Exercises 50–51, find the sum.

50. $(17) + (-32) + (-28) + (13)$

51. $(-48) + (73) + (21) + (-46)$

4-7 Subtracting Directed Numbers *Pages 128–132*

In Exercises 52–53, (a) rewrite the expression as simply as possible, and (b) find the sum.

52. $(382) + (-425) + (-36) + (291)$

53. $(-78) + (112) + (259) + (-407)$

54. $n - 58 - (-91)$

55. $-39 - (-77 + t)$

56. Subtraction is the $\underline{\ ?\ }$ of addition.

57. Subtracting -5 is the same as adding $\underline{\ ?\ }$.

In each of Exercises 58–59, (a) rewrite the expression in terms of addition only, (b) find the result.

58. $26 - 59 - (-12)$ **59.** $(-17) - (83) - (-62)$

60. From the sum of 38 and -83 subtract -29.

61. Subtract the sum of -47 and 74 from -12.

62. $r + 28 - (-81)$

63. $-32 - s + 47$

64. Find the change in temperature when it dropped from 1° below zero to 8° below zero.

65. An ancient Greek lived from 421 B.C. to 373 B.C. How old was he when he died?

4-8 Multiplying Directed Numbers *Pages 133–138*

66. The product of two positive numbers or of two negative numbers is a $\underline{\ ?\ }$ number.

67. The product of a positive number and a negative number is a $\underline{\ ?\ }$ number.

68. The multiplicative property of -1 is that multiplying a number by -1 gives the $\underline{\ ?\ }$ of the number.

In Exercises 69–73, simplify the expression.

69. $-38(7)$ **70.** $-14(31 - 12)$ **71.** $(-6\frac{1}{2})(17) - 6\frac{1}{2}(-15)$

72. $\frac{h}{4} - (-8)$ **73.** $-3 - (-\frac{1}{6}k)$

In each of the following expressions, combine similar terms.

74. $13a - 20b - 3a - 11b$ **75.** $32m - 48mn + mn - m$

76. $-83(27)(-7)(0) = \underline{\ ?\ }$

77. $(-43)(-52)(10)(-1)^{32} = \underline{\ ?\ }$

Evaluate each expression using $m = -1, r = 2, s = -2, t = 3$.

78. $-5st^2$ **79.** $-5m(r^3 - s^3)$

4-9 Dividing Directed Numbers *Pages 138–142*

80. The quotient of two positive or two negative numbers is $\underline{\ ?\ }$.

81. The quotient of a positive and a negative number is $\underline{\ ?\ }$.

82. Zero may not be used as a $\underline{\ ?\ }$.

83. When the dividend is 0 and the divisor is not 0, the quotient is $\underline{\ ?\ }$.

In Exercises 84–85, find the quotient.

84. $(-30) \div (-45)$ **85.** $(-18) \div [3 \div (-\frac{1}{6})]$

Solve for the indicated variable.

86. $-14v = 126$ **87.** $-36w = -\frac{9}{4}$ **88.** $1 = -\frac{3}{2}x$

89. The product of a number and its reciprocal is $\underline{\ ?\ }$.

90. Every number, except $\underline{\ ?\ }$, has one, and only one, reciprocal or $\underline{\ ?\ }$ $\underline{\ ?\ }$.

91. The quotient of two directed numbers may be expressed as the product of the dividend and the $\underline{\ ?\ }$ of the divisor.

92. The identity element for multiplication is $\underline{\ ?\ }$.

Give the reciprocal of each of the following.

93. -2 **94.** $2\frac{1}{5}$ **95.** $\frac{2}{3}$ **96.** $\dfrac{2}{a-b}, a \neq b$

State whether each of the following sets is closed under division.

97. $\{\frac{1}{2}, 1, 2\}$ **98.** {the reciprocals of the directed numbers}

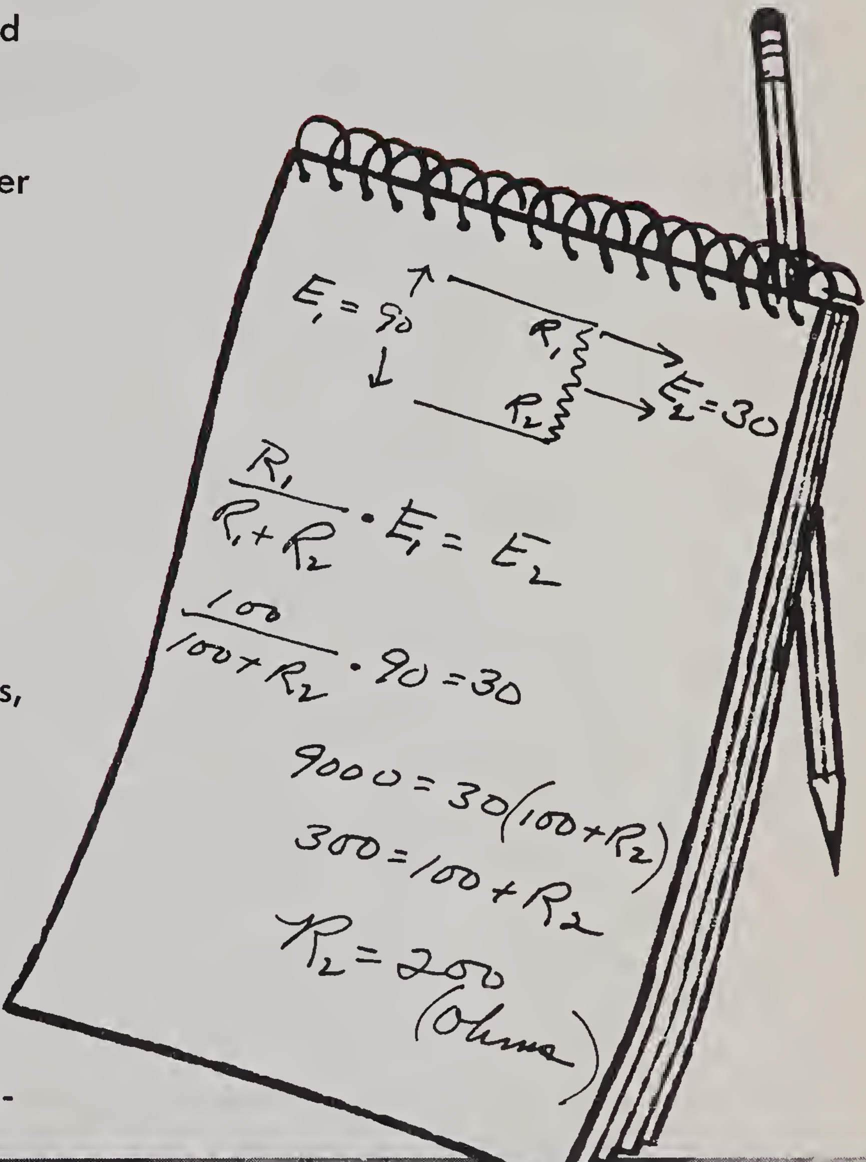
4-10 Averages and Directed Numbers (Optional) *Pages 142–144*

99. The difference between the value being considered and the assumed average is called the $\underline{\ ?\ }$ and is expressed as a $\underline{\ ?\ }$ number.

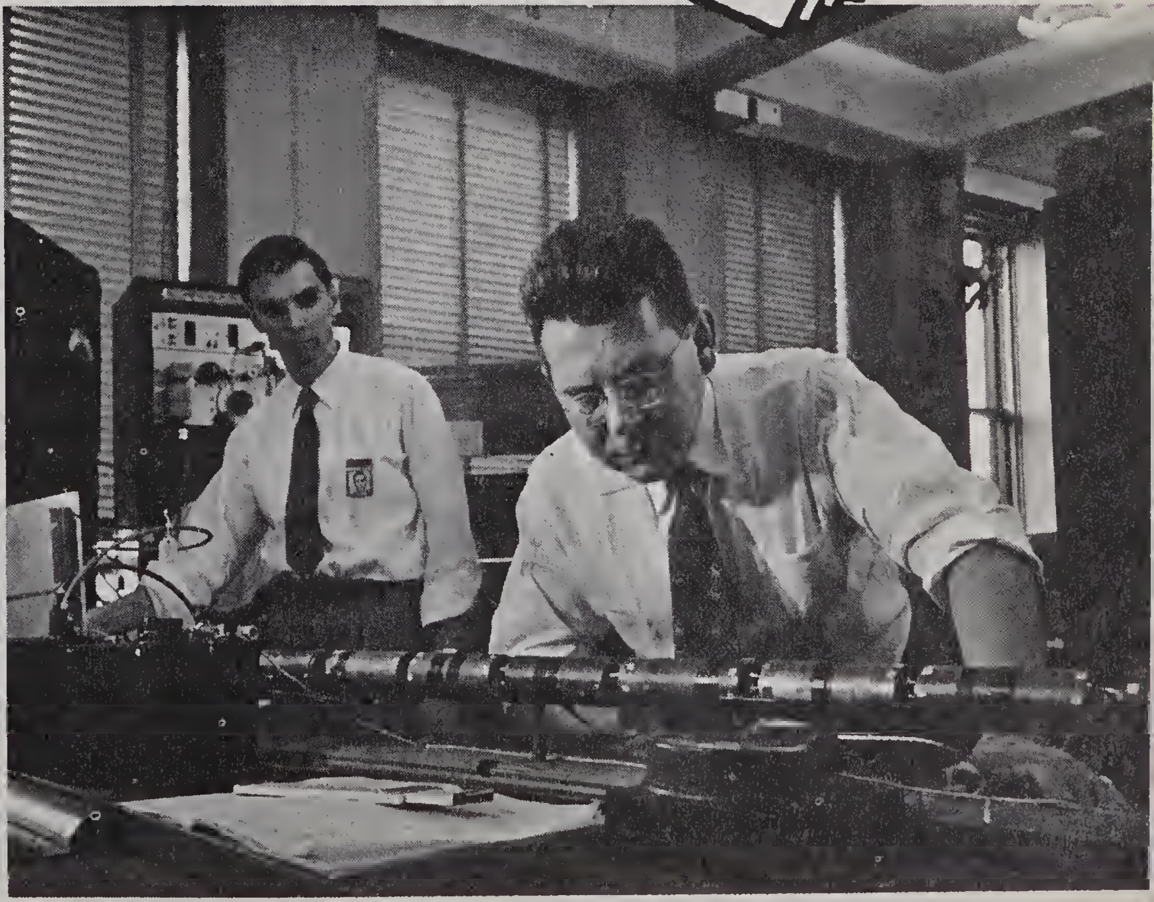
100. Using 5 as an assumed average, find the average of -12, -3, 0, 2, 7, 13, 14.

Electrical Engineers and Mathematics

In planning the vast and intricate networks of modern communications, the engineer must have a sound mathematical background. One particularly significant achievement of electronic engineering, the successful operation of underseas cables, for example, would not be possible if compensations were not made for the loss of power (due to the cable's resistance to the electric current) over the long distance to be covered. The engineer must calculate the power loss and design the devices, called repeaters, which amplify the current at regular intervals and prevent the signal from dying out. The photograph shows a laboratory model of a cable repeater which is being tested under a variety of extreme conditions. More than 100 such repeaters are used in an Atlantic cable; they must function perfectly for years, without needing any maintenance.



Illustrated on the work pad is a more general type of problem in electrical engineering. In order to adapt a motor for the available source of current, an engineer wishes to reduce the voltage from 90 volts to 30 volts. He installs a resistor network consisting of a 100-ohm resistor, R_1, and another resistor, R_2, whose strength he must determine. The measurement of the resistance in ohms (R), is the ratio of the voltage across the resistor to the current through the resistor, or $R = \frac{E}{I}$. By applying the formula $\frac{R_1}{R_1 + R_2} \cdot E_1 = E_2$, the engineer calculates that the second resistor, R_2, must be 200 ohms in strength.

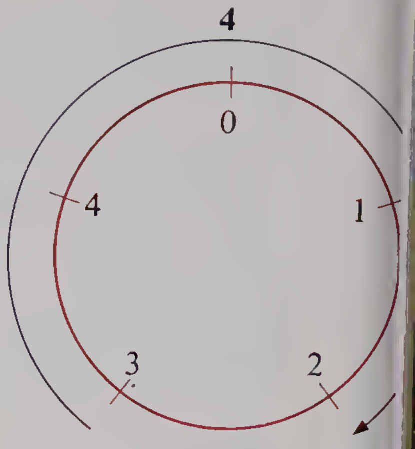


Extra for Experts

Clock Arithmetic

A number system is specified by giving a set of numbers and telling how to add and multiply the numbers. However, the names *addition* and *multiplication* may not mean the familiar operations of adding and multiplying directed numbers.

Consider a system consisting of just five numbers $\{0, 1, 2, 3, 4\}$. To compute a sum in this system, imagine a number line which forms a closed circle like a five-hour clock; then start at the first addend, and count clockwise the number of spaces named by the second addend. For example, $3 + 4 = 2$ would be computed as shown in the figure. By similar means, verify each sum in the addition table.



$3 + 4 = 2$

To multiply in this set of clock numbers, take $a \times b$ to mean $b + b + b + \ldots b$, to a terms. Thus, $2 \times 3 = 3 + 3 = 1$, and $3 \times 4 = 4 + 4 + 4 = 2$. By counting, check each product in the multiplication table.

Addition Table

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Multiplication Table

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Questions

Compute in the system of clock numbers $\{0, 1, 2, 3, 4\}$ for Questions

1. **a.** $(3 + 3) + 4$ and $3 + (3 + 4)$ **b.** $(2 + 3) + 1$ and $2 + (3$

c. What property of numbers do these results illustrate?

2. Does the system of clock numbers have the commutative propert
addition? multiplication? Give two examples to illustrate each an

3. Is the system closed under addition? subtraction? multiplication?

4. What is the identity element for addition? for multiplication?

5. Solve the following equations in the system.

   a. $x + 3 = 2$ b. $2 + y = 0$ c. $s + 4 = 3$

6. Solve in the system.

   a. $3x = 2$ b. $3 = 2w$ c. $2t + 1 = 4$

7. Why are negative numbers unnecessary in clock arithmetic?

8. In the clock arithmetic of a twelve-hour clock dial:

   a. What number is the identity element for addition? Give four examples.

   b. Does this number also have the multiplicative property of zero? Give four examples.

Just for Fun

Magic Squares

A magic square is divided into a number of smaller squares, called *cells*, with a different number written in each cell, so that the sum of each row, of each column, and of each diagonal is the same. A magic square of the third order has three rows and three columns. Illustrated is a pure magic square of the third order.

8	1	6
3	5	7
4	9	2

A pure magic square is a magic square containing consecutive integers. There are eight ways to arrange the numbers in a pure magic square of the third order. The number 5 is always in the center, and even numbers are in the four corners. You can easily make a magic square of the third order which is not pure by multiplying the number in each cell by -1 or 2 or any other number. How does the distributive property assure you that such a square will still be magic?

You may construct a magic square of 9, 25, 49, or any other number of cells (so long as the number is the square of an odd number) by following these rules.

1. Write the digits in order, putting each in a separate cell. Begin by writing 1 in the middle cell of the top row.

2. Move from cell to cell by going up and to the right one step at a time. If you find yourself going out of the square, or getting into a cell that is already filled, make the move and then proceed as follows:

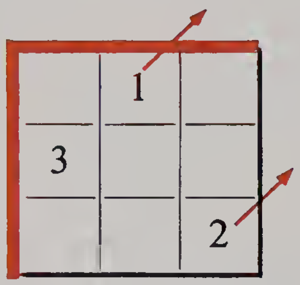
   a. When a move takes you out at the top of the square, drop down to the lowest cell in that column. When a move takes you out at the right of the square, shift to the cell farthest left in that row.

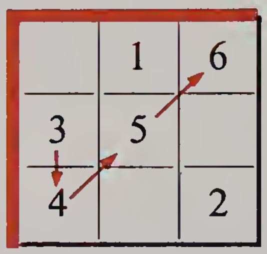
b. When a move takes you to a cell already filled, drop down one row, instead. When a move takes you out the upper right-hand corner, drop down one row, instead.

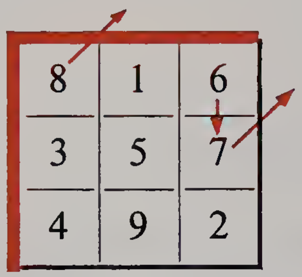
Using these rules to construct a pure magic square of the third order, as the figure shows, your very first move takes you out at the top of the square, so you drop down to the bottom of that column, and put 2 in the lowest cell. Your second move takes you out at the right of the square, so you shift to the left of that row, and put 3 in the left-hand cell.

Your next move also leads to difficulty; it takes you to a cell already filled. So you drop down one row, and put 4 directly beneath 3. Your next two moves are unobstructed.

You are now at the upper right-hand corner of the square. Therefore, you drop down one row, and put 7 beneath 6. The next move takes you out at the right again, so you shift left. And the final move takes you out at the top, so you drop to the bottom!







Of course, the reason your moves take you out so often is that the square is small. Try building a magic square of the fifth order, and one of the seventh order.

In a pure magic square of the third order, as you surely have discovered, the sum of each row, column, and diagonal is 15. In a pure magic square of the fifth order the sum is 65, and in one of the seventh order it is 175. You can find this sum for any pure magic square if you know how many cells are on each side. If n represents this number, the sum is $\frac{1}{2}(n^3 + n)$.

Third order: $\frac{1}{2}(3^3 + 3) = \frac{1}{2} \cdot 30 = 15$

Fifth order: $\frac{1}{2}(5^3 + 5) = \frac{1}{2} \cdot 130 = 65$

Seventh order: $\frac{1}{2}(7^3 + 7) = \frac{1}{2} \cdot 350 = 175$

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Of course, if the square is not pure you multiply this result by the smallest number in the square.

So far, nothing has been said about magic squares containing an even number of cells. But there are such squares, and you can find the sum of their columns, rows, and diagonals by the same formula. You cannot construct them by the same rules, however. They are based on an entirely different principle.

THE HUMAN EQUATION

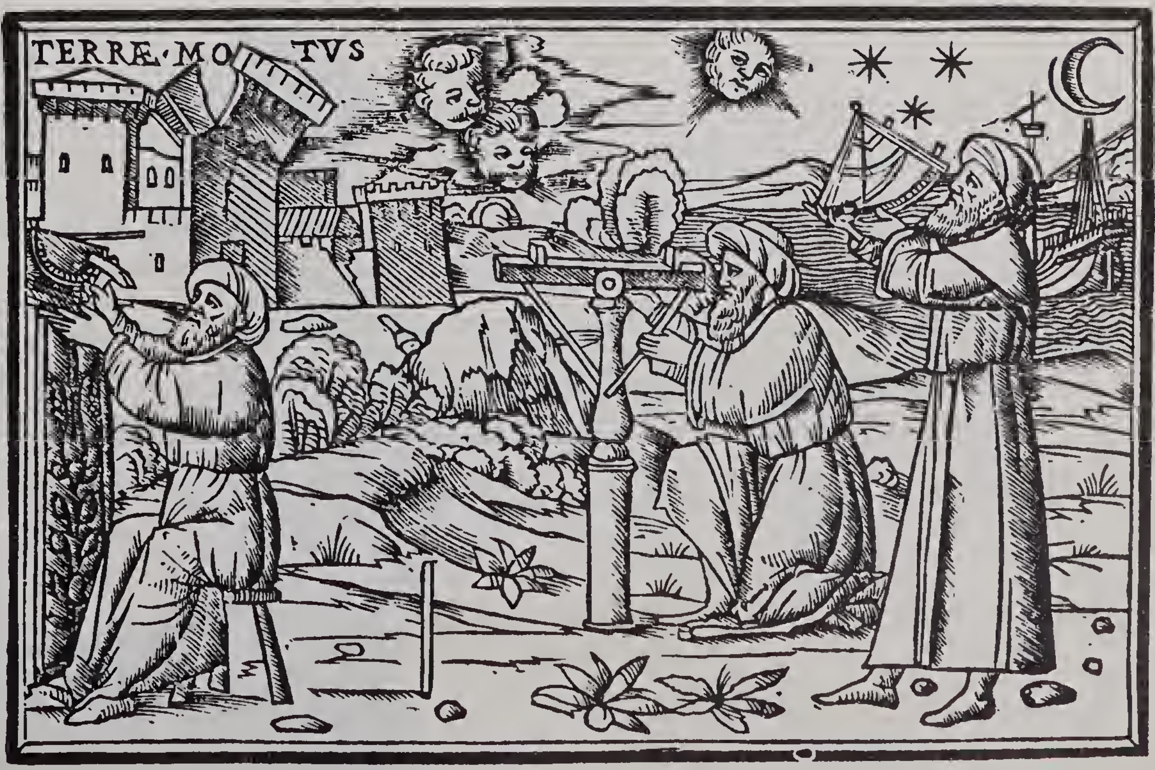
At the
Court of the Caliphs

Aladdin, Sindbad, and Harun al-Rashid — these names are familiar to you from *The Arabian Nights.* Have you supposed them to be the names of mythical characters? Harun al-Rashid, at least, was real. He reigned in Bagdad from 796 to 808. He not only went about among his subjects, as recorded in *The Book of a Thousand Nights and a Night;* he also encouraged his nobles to study science and mathematics.

During his reign the Mohammedans were invading and conquering the non-Moslem lands to the west of them, including northern Africa. Thus the learning that centered at the University of Alexandria became a prize of war: books in Greek were brought to Bagdad from Alexandria. It was Harun al-Rashid who caused them to be translated into Arabic.

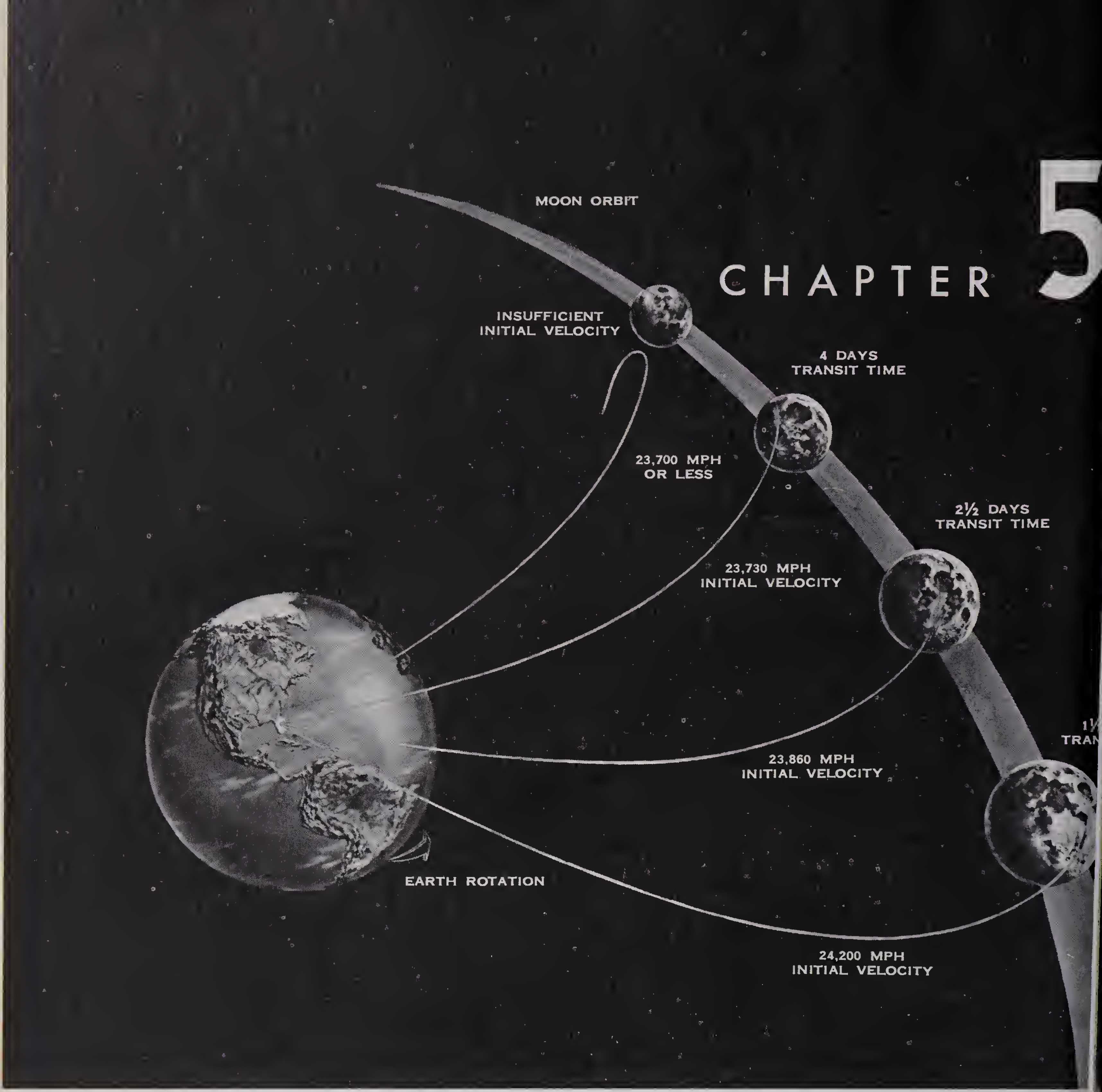
Soon Mohammedan scholars were engaged in the serious study of mathematics. Among these scholars was the caliph's son, Al-Mamun. By the time he succeeded his father as caliph, there was, in the words of *The Arabian Nights* (for Al-Mamun also figures in those tales), "none more accomplished in all branches of knowledge than he." It was only natural that his court should include many learned men.

One of Al-Mamun's scholarly courtiers — the greatest mathematician in all Arabia — was Al-Khowarazmi. He wrote a book with this title: *ilm al-jabr wa'l muqabalah.* The book was called *al-jabr* for short. This shortened title has been used for many books from Al-Khowarazmi's day to the present. Of course, the spelling has changed. Now it is usually spelled *algebra.*



Arabian scholars at work. When this picture was made, Bagdad was the center of learning. Interest in astronomy made knowledge of algebra necessary, and the development of algebra made possible increased knowledge of astronomy.

CHAPTER 5


MOON ORBIT
INSUFFICIENT INITIAL VELOCITY
4 DAYS TRANSIT TIME
23,700 MPH OR LESS
2½ DAYS TRANSIT TIME
23,730 MPH INITIAL VELOCITY
1½
TRAN
23,860 MPH INITIAL VELOCITY
EARTH ROTATION
24,200 MPH INITIAL VELOCITY

Equations, Inequalities, and Problem Solving

As the picture indicates, missiles differ. So do numbers. Although most of your concern in this course has been with equal expressions, it is necessary to consider conditions which may exist among numbers that are unequal. A grasp of such situations is necessary for a full understanding of numbers. Also, complex problems which are solvable only in terms of inequalities arise in applied mathematics. Here is an example:

> The time of flight of a ballistic missile fired at the moon is extremely sensitive to the initial velocity of the final stage. An initial velocity of 23,700 m.p.h. will be insufficient to reach the moon. However, increasing this value by only 30 m.p.h. would make a four-day earth-moon mission possible. Increasing this velocity an additional 130 m.p.h. would reduce the transit time to 2.5 days.

OPEN SENTENCES IN THE SET OF DIRECTED NUMBERS

5–1 Transforming Equations

Transformations used in the set of arithmetic numbers to replace one equation by an equivalent equation are valid also in the set of directed numbers. For example, the same directed number may be added to or subtracted from the value of each member of an equation without changing the solution set of the equation. Also, multiplying or dividing each member by a nonzero directed number produces an equivalent equation. Use this section to review properties of equality by asking pupils to tell why each step in the transformation produces an equivalent equation. See 5-1 T.M. pg. 18.

EXAMPLE 1. Solve: $x + 36 = 1 - 4(x - 5)$

Solution:

1. Copy the equation; use the distributive property; combine similar terms.

$$x + 36 = 1 - 4(x - 5)$$
$$x + 36 = 1 - 4x + 20$$
$$x + 36 = 21 - 4x$$

2. Subtract 36 from or add -36 to each member.

$$x + 36 - 36 = 21 - 4x - 36$$
$$x = -15 - 4x$$

3. Add $4x$ to each member.

$$x + 4x = -15 - 4x + 4x$$
$$5x = -15$$

4. Divide each member by 5.

$$\frac{5x}{5} = \frac{-15}{5}$$
$$x = -3$$

Check is left to you.

Check to guard against numerical mistakes.

EXAMPLE 2. Solve for t:

$$I = prt$$

Solution:

$$I = prt$$
$$\frac{I}{pr} = \frac{prt}{pr}$$
$$\frac{I}{pr} = t$$

EXAMPLE 3. Solve for g:

$$d = \tfrac{1}{2}gt^2$$

Solution:

$$d = \tfrac{1}{2}gt^2$$
$$2(d) = 2(\tfrac{1}{2}gt^2)$$
$$2d = gt^2$$
$$\frac{2d}{t^2} = \frac{gt^2}{t^2}$$
$$\frac{2d}{t^2} = g$$

ORAL EXERCISES

Solve for the variable in red, or for the indicated variable.

1. $x + 5 = -7$ $\{-12\}$
2. $x + 3 = -8$ $\{-11\}$
3. $y - 8 = -12$ $\{-4\}$
4. $y - 1 = -29$ $\{-28\}$
5. $-2z = +8$ $\{-4\}$
6. $-3z = +15$ $\{-5\}$
7. $\frac{t}{3} = -11$ $\{-33\}$
8. $\frac{t}{-5} = -9$ $\{45\}$
9. $5k = 4k - 1$ $\{-1\}$
10. $-2 = -2 + 5k$ $\{0\}$
11. $-12m + 8 = -13m$ $\{-8\}$
12. $-6m + 2 = -7m$ $\{-2\}$
13. $5r + 5 = 0$ $\{-1\}$
14. $-7r + 7 = 0$ $\{$
15. $3x = a$ $\{\frac{a}{3}\}$
16. $\frac{x}{2} = b$ $\{2b\}$
17. $\frac{x}{a} = bc$ $\{\frac{x}{ab}\}$
18. $\frac{1}{2}y = a - \frac{1}{2}y$ $\{y\}$

WRITTEN EXERCISES

Solve for the variable in red, or for the indicated variable.

A

1. $18y = 203 - 11y$
2. $15x = 144 - 9x$
3. $7 = \frac{n}{2} - 1$
4. $9 = \frac{x}{5} - 1$
5. $19 = 11 - \frac{n}{5}$
6. $20 = 1 - \frac{7s}{2}$
7. $360 + 36z = 30z$
8. $714 + 38r = 21r$
9. $39b = 171 + b$
10. $106x = 540 + 34x$
11. $-20y = 221 + 6y$
12. $-47w = 40 + 13w$
13. $rsx = 4rs$
14. $mnx = mn$
15. $x + 6a = 7a$
16. $x - 4a = a$
17. $\frac{2y}{3} + 7 = 5$
18. $\frac{3w}{4} + 13 = 7$
19. $\frac{5y}{.2} = .3$
20. $\frac{3x}{.5} = .6$
21. $w + l = 3w - 2l$
22. $t - 2h = 5t + 3h$
23. $y - a = 2b$
24. $3a = p + 2c$
25. $p = a - prt$
26. $p = a - prt$
27. $3x - 7b = x$
28. $3m + 4y = 2y$
29. $p = a + b + c$
30. $s = c + o + p$
31. $P = \frac{W}{t}$
32. $p = \frac{w}{A}$
33. $A = \frac{bh}{2}$
34. $V = \frac{Bh}{3}$
35. $A = 4\pi r^2$
36. $C = 2\pi r$
37. $V = \frac{1}{3}\pi r^2 h$
38. $V = \frac{4}{3}\pi r^3$
39. $V = \frac{1}{3}Bh$

Equations and formulas in Exercises 13-16 and 21-39 contain more than one variable.

B

40. $9 - \frac{1}{2}(3n - 6) = n + 2$
41. $2n - \frac{2}{3}(4n + 12) = n + 7$
42. $5(x + 1) = 4(x + 2)$
43. $3(8x - 2) = 3(4 + 2x)$
44. $4(3n + 2) = 6(3 - n) + 8$
45. $5(x + 1) = 3(x + 2)$
46. $5r + (r + 15) - 3(r + 2) = 0$
47. $69 = 4(h + 5) - (h - 1)$
48. $5(y + 2) = 13 + 4(2y - 1)$
49. $5(y + 2) = 6 + 3(2y - 1)$
50. $3(x - 9) = 74 - 2(61 + 2x)$
51. $2w - 4(w + 2) = 5(w + 4)$

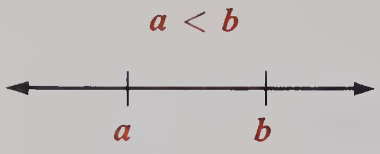
5–2 The Properties of Inequality

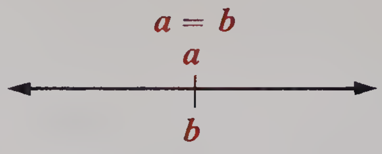
When you consider two different numbers, you know that one is larger than the other. The *order property of numbers* is usually stated:

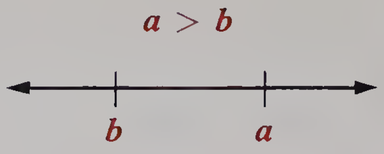
> For each pair of directed numbers a and b, one and only one of the following statements is true: $a < b$; $a = b$; $a > b$.

The properties of order are all accepted here as postulates. See 5-2 T.M. pg. 18.

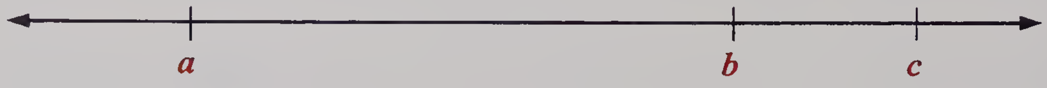
The meaning of this fact on the number line is shown in these figures.







You can see from the following number line that if point a lies to the left of point b and if point b lies to the left of point c, then point a is to the left of point c.



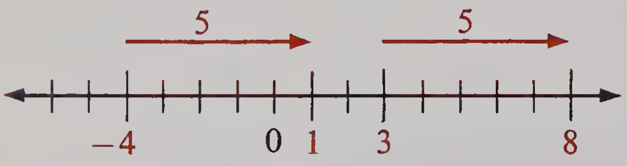
This illustrates the ***transitive property of inequality*** in the set of directed numbers:

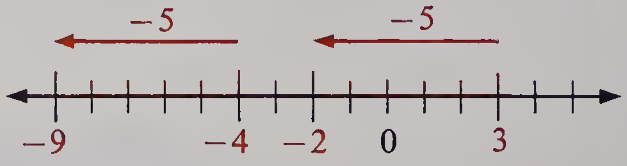
Point out that the substitution principle also gives this property: if $a < b$ and $c = a$ and $d = b$, then $c < d$.

For any directed numbers a, b, and c:

1. if $a < b$ and $b < c$, then $a < c$; similarly,
2. if $a > b$ and $b > c$, then $a > c$.

On the number line, -4 lies to the left of 3. If you move 5 units from -4 and also 5 units *in the same direction* from 3, you arrive at points in the same order as -4 and 3.





From $$-4 < 3$$

it follows that $$-4 + (5) < 3 + (5) \quad \text{or} \quad 1 < 8$$

and $$-4 + (-5) < 3 + (-5) \quad \text{or} \quad -9 < -2.$$

This illustrates the ***additive property of inequality:***

For any directed numbers a, b, and c:

1. if $a < b$, then $a + c < b + c$, and $a - c < b - c$; similarly,
2. if $a > b$, then $a + c > b + c$, and $a - c > b - c$.

Since $a - c = a + (-c)$ and $b - c = b + (-c)$, the second part of each of these statements follows from the first.

Notice what happens if you multiply each member of the inequality $-4 < 3$ by 2. Since $(-4)(2) = -8$ and $(3)(2) = 6$, and also $-8 < 6$, it follows that

$$(-4)(2) < (3)(2)$$

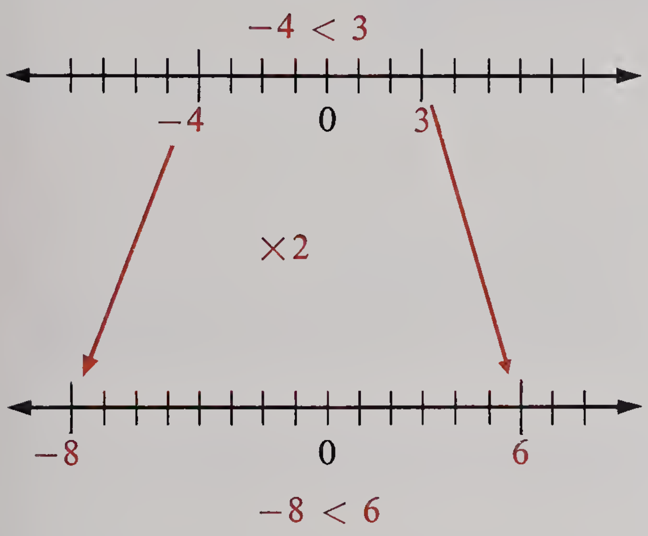
In an able class you may wish to develop some of the theorems suggested in 5-2 T.M. pgs. 18-21.

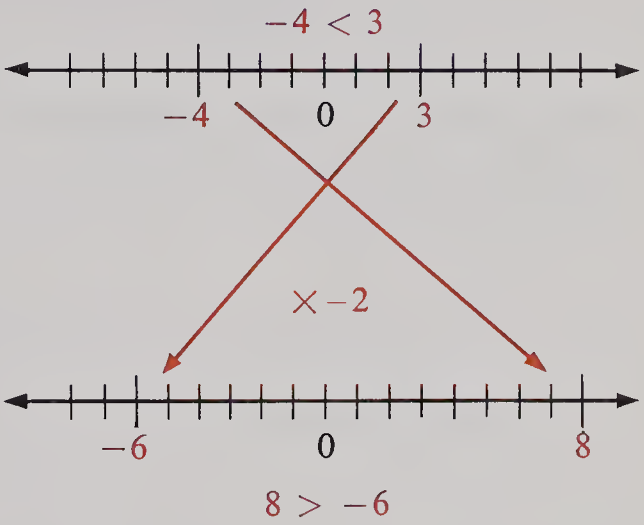
Thus, multiplying each member by 2 preserves the *order*, *direction*, or *sense* of the inequality.

On the other hand, multiply each member of $-4 < 3$ by -2. Since $8 > -6$, it follows that

$$(-4)(-2) > (3)(-2).$$

Thus, multiplying each member of the inequality by -2 reverses the sense of the inequality.





When you multiply an inequality by a directed number, you must take into account the direction associated with the multiplier. The ***multiplicative property of inequality*** states:

Replace a by 0 and b by c: if $0 < c$, then $0 < c^2$ or $c^2 > 0$.

For any directed numbers a, b, and c, if $a < b$, then

1. $ac < bc$, when $c > 0$,
2. $ac = bc$, when $c = 0$,
3. $ac > bc$, when $c < 0$; similarly,
4. $\frac{a}{c} < \frac{b}{c}$, when $c > 0$, and
5. $\frac{a}{c} > \frac{b}{c}$, when $c < 0$.

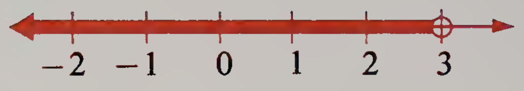
Ask students why Steps 4 and 5 follow from 1 and 3. See T.M. pg. 20 (1).

You can find the solution set of an inequality by transforming it into an **equivalent inequality**, that is, one with the same solution set, by use of the properties of inequality.

Replace a by c and b by 0: if $c < 0$, then $c^2 > 0$. Thus, if $c \neq 0$, $c^2 > 0$.

EXAMPLE Solve the inequality $7x - 13 < 3x - 1$, and graph its solution set.

Solution:

1. Copy the inequality.	$7x - 13 < 3x - 1$
2. Add 13 to each member.	$7x - 13 + 13 < 3x - 1 + 13$ $7x < 3x + 12$
3. Subtract $3x$ from each member.	$7x - 3x < 3x + 12 - 3x$ $4x < 12$
4. Divide each member by 4. Ask students to suggest processes that produce equivalent inequalities. See T.M. pg. 21 (2).	$\frac{4x}{4} < \frac{12}{4}$ $x < 3$ ∴ **The solution set is {all directed numbers less than 3}**
5. Graph the solution set.	

If you transform an inequality such as

$$12 \geq -6r$$

by dividing by negative six, remember to reverse the order of the inequality:

$$-2 \leq r.$$

ORAL EXERCISES

In addition to answers given, description of soln. set should be required as in sa

State the transformation you use to solve each of these inequalities.

SAMPLE. $-\frac{h}{5} > 12$

What you say: Multiply each member by −5 and reverse the order of the inequality; $h < -60$.
The solution set is {all directed numbers less than −60}.

A = add to each member. S = subtract from each member.
M = multiply each member by; D = divide each member by;

rev. ord. = reverse order of inequality.

1. $2n < -8$ D2; $n < -4$
2. $3k > -21$ D3; $k > -7$
3. $t + 5 \geq -1$ S5; $t \geq -6$
4. $s + 2 \leq -1$ S2; $s \leq -3$
5. $5 > \frac{x}{4}$ M4; $20 > x$
6. $2 < \frac{z}{7}$ M7; $14 < z$
7. $-3 < m - 4$ A4; $1 < m$
8. $-9 > n - 8$ A8; $-1 > n$
9. $-12z > -60$ D−12, -12 rev. ord.; $z < 5$
10. $-13w \geq -39$ D−13, rev. ord.; $w \leq 3$
11. $-\frac{p}{3} \leq 2$ M−3, rev. ord.; $p \geq -6$
12. $-\frac{q}{6} < 5$ M−6, rev. ord.; $q > -30$
13. $5m \geq 0$ D5; $m \geq 0$
14. $7n \leq 0$ D7; $n \leq 0$
15. $0 < -3s$ D−3, rev. ord.; $0 > s$
16. $0 > -4u$ D−4, rev. ord.; $0 < u$
17. $.1d \geq -7$ D.1; $d \geq -70$
18. $.3e \leq -8$ D.3; $e \leq -\frac{80}{3}$

$-z > -5$
$z < 5$

WRITTEN EXERCISES

Solve each inequality. In Exercises 1–22, show also the graph of the solution set.

A

1. $5a - 1 \geq 9$
2. $6b - 11 \leq 13$
3. $2 - 3s < 11$
4. $5 - 2t > 17$
5. $7 + \frac{z}{4} \leq 0$
6. $8 + \frac{w}{3} \geq 0$
7. $-14 \leq 3t - 2$
8. $26 \geq -10 + 4n$
9. $\frac{2}{3}p - 1 > -3$
10. $\frac{3}{4}q - 2 < -5$
11. $3x + 5 < x - 5$
12. $y - 1 > 9 - 4y$

13. $-6w + 5 + w > -(13 + w)$
14. $-6v - 2 + v < 13 + v$
15. $6x - 3(4 - 2x) \geq 0$
16. $-y + 2(9 - y) \leq 0$
17. $-12\left(\frac{t}{6} - \frac{1}{3}\right) < 2t$
18. $15\left(\frac{1}{5} - \frac{w}{3}\right) > -2w$
19. $0 \geq -17(6r - 2)$
20. $0 \leq -51(4v + 3)$
21. $-2(3m - 6) < 6(2 + m)$
22. $15(-4 - z) > -5(12 - 3z)$

B

23. $\frac{3}{10}(2t - 20) + 5 \leq \frac{t - 60}{10}$
24. $\frac{5}{6}(18 - g) - 1 \geq \frac{g + 90}{6}$
25. $2(h - 1) < 2h$
26. $3(1 + y) > 3y$

27. $3(|x| + 1) \leq 3 - |x|$
28. $-2(2|n| - 1) \geq \frac{1}{2}(4 - 6|n|)$
29. $-2(1 - 7b) < 7(2b - 5) - 17$
30. $-3(2 + 5c) > 1 - 5(3c - 1)$

31. $7(d + 3) - 5(d - 3) \geq 2(d + 20) - 4$
32. $3(k - 4) - (k + 8) \leq -2 + 2(k - 9)$

5–3 Pairs of Inequalities (Optional)

Often you are interested in variables whose values must satisfy *two* inequalities *at the same time.* For example, the solution set of the sentence $-1 < x < 4$ consists of those numbers for which *both* of the inequalities $-1 < x$ *and* $x < 4$ hold.

Similarly, to solve the inequalities

1. $$-2 < x + 4 \leq 5$$

you must find the values of x for which

$$-2 < x + 4 \quad \text{and} \quad x + 4 \leq 5.$$

Ask students to discuss the solution sets in terms of union and intersection. See Extra for Experts in Chapters 1 and 2.

Simplifying each of these inequalities by subtracting 4 from each member, you find

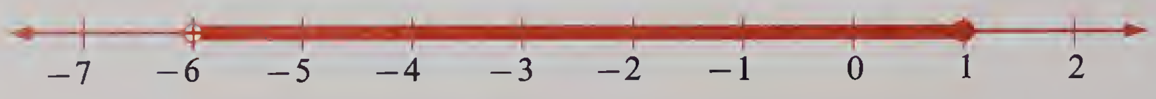
$$-2 - 4 < x + 4 - 4 \quad \text{and} \quad x + 4 - 4 \leq 5 - 4$$

$$-6 < x \quad \text{and} \quad x \leq 1.$$

Thus, the pair of inequalities **1** is equivalent to the pair of inequalities

2. $$-6 < x \leq 1$$

The graph appears as:



Sometimes you want to find the values of a variable for which *at least one* of a pair of inequalities is true. The inequality

$$|2y - 1| > 5$$

will be satisfied provided either

$$(2y - 1) > 5 \quad \text{or} \quad -(2y - 1) > 5$$

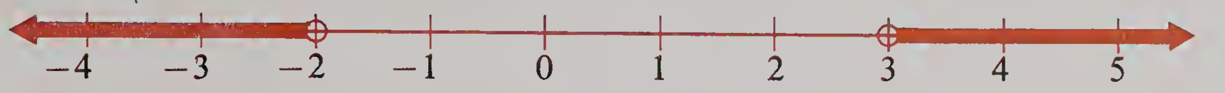
To simplify these inequalities, first multiply each member of the second inequality by -1 and obtain this pair:

$$2y - 1 > 5 \quad \text{or} \quad 2y - 1 < -5$$

Add 1 to each member: $2y > 6$ or $2y < -4$

Divide each member by 2: $y > 3$ or $y < -2$

Thus the graph of the solution set of $|2y - 1| > 5$ consists of the *two* sets of points indicated below.



In dealing with two inequalities it is essential to decide whether you want the set of numbers satisfying *both* of them or the set satisfying *at least one* of them. For example, the set satisfying *both* of the inequalities $x \leq 3$ *and* $x > 3$ is the *empty set*, whereas the set of numbers satisfying *at least one* of them is the set of *directed numbers!*

WRITTEN EXERCISES

Solve each pair of inequalities, and graph the solution set.

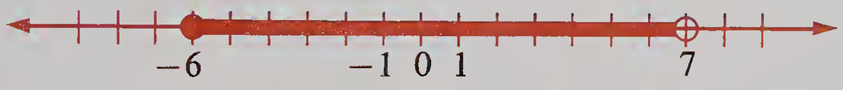
SAMPLE 1. $t + 2 \geq -4$ and $t - 3 < 4$

Solution:

$$t + 2 \geq -4 \qquad\qquad t - 3 < 4$$
$$t + 2 - 2 \geq -4 - 2 \qquad\qquad t - 3 + 3 < 4 + 3$$
$$t \geq -6 \quad \text{and} \quad t < 7$$

$$\therefore \quad -6 \leq t < 7$$

Answer.



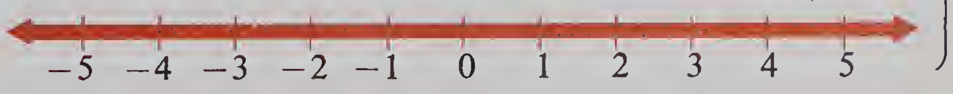
SAMPLE 2. $t + 2 \geq -4$ or $t - 3 < 4$

Solution:

$$t + 2 \geq -4 \quad \text{or} \quad t - 3 < 4$$
$$\therefore \quad t \geq -6 \quad \text{or} \quad t < 7$$

Every number satisfies at least one of these inequalities.

The solution set is the set of directed numbers, the graph being the entire number line: **Answer.**



A

1. $-3 \leq x + 4 \leq 0$

2. $1 < 5 + y \leq 7$

3. $-8 \leq -1 + 3a < 11$

4. $-7 \leq 4b - 5 \leq 19$

5. $|x| \geq 0$

6. $|b - 1| > 0$

7. $m - 1 > -1$ and $m - 2 \leq 0$

8. $4 + n < -3$ and $-2 + n \geq -9$

9. $5 > 1 - 2r$ and $3r > 2r + 1$

10. $6v < 7v + 4$ and $2 - 3v \geq -1$

11. $|3 - w| > 2$

12. $1 < |4 - k|$

13. $|r| - 3 \geq -1$

14. $|u| - 4 \geq 3$

B

15. $5 - 2v > 7$ or $4v < v + 9$

16. $11p \leq 5p - 12$ or $1 - 4p > 13$

17. $(x - 1)(x + 2) > 0$

18. $(y + 4)(y + 3) > 0$

19. $(r - 2)(r - 3) < 0$

20. $(s + 1)(s - 1) < 0$

THE ANALYSIS OF PROBLEMS

5–4 A Plan for Solving Problems

See 5-4 T.M. pg. 22.

You have already solved some problems using methods outlined in Chapter 2 (page 57). The first step helps you to determine what you are to find. To take the second step, you must know what facts the problem gives you; therefore, ask yourself these questions: What does the problem ask? What facts does the problem give? It is helpful, when the problem permits, to make a sketch that illustrates it.

EXAMPLE 1 **A certain roll has 10 fewer calories than twice the number of calories in a slice of white bread. Together they contain at least 185 calories. Find the smallest possible number of calories in the slice of white bread.**

Emphasize the four steps to encourage orderly procedure, not to stifle ingenuity. Challenge students to invent alternative solutions.

Solution:

1 **What does the problem ask? The smallest possible number of calories in a slice of white bread.**

Let x = this number of calories.

The roll has 10 fewer than twice the number of calories in the slice of bread.

Therefore, $2x - 10$ will represent the number of calories in the roll.

2 **What other facts does the problem give?**

$$\underbrace{\text{calories in roll}}_{2x - 10} \quad \underbrace{\text{together with}}_{+} \quad \underbrace{\text{calories in bread}}_{x} \quad \underbrace{\text{are at least}}_{\geq} \quad \underbrace{185}_{185}$$

3 Solve the inequality:

$$\begin{aligned} 2x - 10 + x &\geq 185 \\ 3x &\geq 195 \\ x &\geq 65 \end{aligned}$$

---

4 Check in the words of the original problem.

If the slice of bread contains at least 65 calories, then the roll has at least $2x - 10 = 2(65) - 10 = 120$ calories.

Is the sum of $65 + 120$ at least 185? $185 \geq 185$ ✓

The slice of white bread contains at least 65 calories, Answer.

EXAMPLE 2 Ben's age is four years less than three times that of his younger sister Amy. Half of Ben's age increased by Amy's age is 2 years more than twice Amy's age. Find their ages.

Solution:

1 Let n = Amy's age in years.

Then $3n - 4$ = Ben's age.

2

$$\underbrace{\text{Half of Ben's age}}_{\frac{1}{2}(3n - 4)} \quad \underbrace{\text{increased by}}_{+} \quad \underbrace{\text{Amy's age}}_{n} \quad \underset{=}{\overset{\text{is}}{\downarrow}} \quad \underbrace{\text{2 years}}_{2} \quad \underbrace{\text{more than}}_{+} \quad \underbrace{\text{twice Amy's age}}_{2n}$$

3

$$\begin{aligned} \tfrac{3}{2}n - 2 + n &= 2 + 2n \\ \tfrac{1}{2}n &= 4 \\ n &= 8 \\ 3n - 4 = 3(8) - 4 &= 20 \end{aligned}$$

4 Is Amy's age 8 years? Is Ben's age 20 years?

a. Half of Ben's age added to Amy's age $= \frac{20}{2} + 8 = 18$

b. Two years more than twice Amy's age $= 2 + 2(8) \overset{?}{=} 18$

$18 = 18$ ✓

$$\left.\begin{aligned} \text{Amy's age} &= 8 \text{ years} \\ \text{Ben's age} &= 20 \text{ years} \end{aligned}\right\} \text{Answer.}$$

PROBLEMS

Take the four steps given on page 57 in solving each problem, making a sketch when possible.

A

1. Michael and Robert are going fishing. Michael owns the boat; therefore the boys have agreed that he is to get 5 more fish than Robert. If the total catch is 19 fish, how many will each receive?
2. The Red Cross knitted 50 sweaters within ten days. The Junior Red Cross assisted, contributing 2 dozen fewer than the senior organization. How many did the Junior Red Cross knit?
3. The Jowett family budgets part of its weekly income of $150 for food. Half the remainder of the income exceeds the amount spent on food by from $15 to $30. How much do they spend on food per week?
4. Tom and Otto picked 36 quarts of berries. Tom picked 3 more than half the number Otto picked. How many did each pick?
5. Mrs. Abbott decided on Christmas Day to save $150 for next Christmas. She started saving $5 a week, but at the end of 12 weeks saw she could reduce that amount. By how much could she reduce her weekly saving and still have $150 or more at the end of the year?
6. The length of a playground exceeds twice its width by 25 feet, and 650 feet of fencing are needed to enclose it. Find its dimensions.
7. The length of a rectangle exceeds three times the width by 6 feet, and the perimeter is 188 feet. Find the dimensions of the rectangle.
8. In a certain puzzle, the larger of two numbers must exceed three times the smaller by 5, and their difference must be at least 31. Find the least possible value of the smaller number.
9. To be called "Limited" a train's average speed must be 5 miles an hour more than twice the average speed of a "Local." If the Limited travels at 63 miles an hour, what is the speed of the Local?
10. A child's bank contained twice as many nickels as pennies and two-thirds as many dimes as nickels, the total value being at least $3.65. Find the smallest possible number of coins in the bank.
11. Bill Jones wanted Sally Smith's telephone number. Sally said that ninety added to her age equaled six times her telephone number, minus 6060. Bill knew that Sally was eighteen years old, but he didn't know enough algebra to call her. Find Sally's telephone number.
12. The area of a twelve-foot square equals the area of a rectangle 9 feet wide. Find the length of the rectangle.

13. A rectangle is 9 feet by 8 feet. Its area is three times the area of a rectangle 12 feet long. Find the width of the second rectangle.

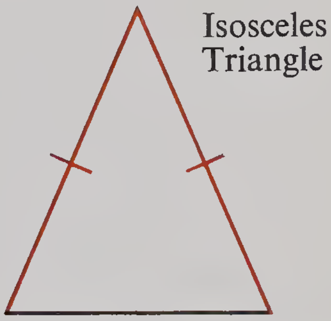
14. In a new school building, 270 cubic feet of air are to be allowed for each pupil. To meet this requirement, what should be the height of the ceiling of a classroom, 30 feet by 24 feet, seating 36 pupils?

15. A coal bin is 15 feet long, 10 feet wide, and 9 feet high. If $10\frac{1}{2}$ tons of egg coal which runs 28 pounds to the cubic foot are put in, to what height will the coal reach?

16. A bicycle wheel has a diameter of 2 feet. How many revolutions will it make in going 5500 feet? (*Hint:* The rule for finding the circumference of a circle is given by the equation $c = \pi d$; use $\pi = \frac{22}{7}$.)

B **17.** David is making a model of a rectangular solid from a piece of wire 52 inches long. The length is to be twice the width, and the height is to be 1 inch more than the width. What are the dimensions of the solid?

18. An isosceles triangle is a triangle having two sides equal in length. The base of an isosceles triangle is a whole number and is 4 feet less than the sum of the two equal sides. The perimeter is a whole number between 0 and 75 feet. Find the possible lengths of each side.



19. Mrs. Fry weighs 50 pounds less than her husband. Their combined weight is at least 220 pounds more than that of their daughter, who weighs half as much as Mr. Fry. What is Mrs. Fry's minimum weight?

20. Mr. Martin earns three times as much in his regular job as he does as a writer. His total income is at least $14,000 more than that of his sister, who earns only half as much as Mr. Martin does in his regular job. What is the least amount he earns in his regular job?

21. Farmer Brown needs .03 acre of land to grow 1 bushel of corn and .06 acre to grow 1 bushel of wheat. He has at most 480 acres of land for planting and wants to use at least half of that acreage. If he decides to grow twice as much corn as wheat, find (a) the maximum and (b) the minimum number of bushels of corn he can grow.

22. In a factory the time required to assemble a table is 20 minutes and a chair, 30 minutes. The factory has at least 126 hours of labor available each day, but can provide as much as 140 hours daily. If four times as many chairs as tables are produced, find the minimum and the maximum number of chairs the factory can produce.

This concept depends on every integer's having an immediate successor in the set of integers. Does ½ have an immediate successor in the set of arithmetic numbers? You meet this idea again in Chapter 11.

5–5 Problems about Consecutive Integers

Integer is another name for any whole number, positive, negative, or zero. The integers have many interesting properties, and to talk about them you need a few descriptive terms. An integer which is twice some integer is called *even;* all others are called *odd.* For example, 2, 126, 0, −10 are even integers; 3, −15, 77 are odd.

The word **consecutive** is used here to mean *following in order*, just exactly as in ordinary language when you say, "I got *A* in algebra for three consecutive weeks." Counting by ones gives *consecutive integers:* 1, 2, 3, 4, ... The three largest consecutive two-digit integers are 97, 98, 99. Likewise, −6, −5, −4, −3, is a set of consecutive integers. Counting by twos from an even integer gives *consecutive even integers:* 2, 4, 6, 8, or −4, −2, 0. Counting by twos from an odd integer gives *consecutive odd integers:* 15, 17, 19, or −5, −3, −1, 1. Some consecutive multiples of five are 5, 10, 15, 20.

Two consecutive integers differ by 1; two consecutive even integers differ by 2. Two consecutive odd integers also differ by 2. If x represents any integer, then $x + 1$ is the next larger integer and $x - 1$ is the next smaller integer. If x represents an even integer, then $x + 2$ is the next larger even integer. What is the next smaller even integer? If x is an odd integer, then $x + 2$ is the next larger odd integer. What is the next smaller odd integer?

EXAMPLE **Find three consecutive integers whose sum is 48.**

Solution:

1 **Let n = the first integer.**

Then $(n + 1)$ = the second integer and $(n + 2)$ = the third integer.

2 **The sum of the integers is 48:** $n + (n + 1) + (n + 2) = 48$

3 **Solve the equation:**

$$\begin{aligned} 3n + 3 &= 48 \\ 3n &= 45 \\ n &= 15 \\ \therefore\ n + 1 &= 16 \\ \text{and}\quad n + 2 &= 17 \end{aligned}$$

Some pupils may believe that $n + 2$ must be an even integer because of the 2. This example should correct that impression.

4 **Is the sum of these integers 48?** $15 + 16 + 17 \overset{?}{=} 48$

$48 = 48$ ✓

The three consecutive integers are 15, 16, 17, Answer.

ORAL EXERCISES

1. Represent 4, 6, 7, and 8 in terms of k, if k equals 5. $k-1, k+1, k+2, k+3$
2. Let m represent 14. How can you represent the numbers 13, 15, 16, and 17 in terms of m? $m-1, m+1, m+2, m+3$
3. Let n be the first number in the series 10, 12, 14, and 16. Express the other numbers in terms of n. $n+2, n+4, n+6$
4. Represent 5, 7, 9, and 11 in terms of h, letting $h = 5$. $h, h+2, h+4, h+6$
5. Let n represent any even integer. What is the next even integer? the preceding even integer? $n+2$; $n-2$
6. Let x represent any odd integer. What is the next odd integer? the preceding odd integer? $x+2$; $x-2$
7. Let x represent any even integer. Is $(x + 1)$ an even integer or an odd integer? $(x - 1)$? Odd; odd
8. Let n be any integer. Is $2n$ an even integer or an odd integer? Is $(2n + 1)$ an even integer or an odd integer? $(2n - 1)$? Even; odd; odd
9. Represent 9, 8, and 7, in that order, letting $n = 9$. $n, n-1, n-2$
10. Represent the numbers 46, 47, and 48, letting $k = 48$. $k-2, k-1, k$
11. If $(x + 7)$ is an integer, what is the next smaller integer? $x+6$
12. If $(z - 1)$ is an integer, what is the next larger integer? z

This idea is important.

PROBLEMS

A

1. The sum of two consecutive integers is 57. Find the numbers.
2. The sum of two consecutive integers is 75. Find the numbers.
3. Find three consecutive odd integers whose sum is 57.
4. Find three consecutive odd integers whose sum is 111.
5. Find four consecutive even numbers whose sum is 100.
6. Find four consecutive even numbers whose sum is 164.
7. Find three consecutive integers, if the sum of the first and third is 128.
8. Find four consecutive integers, if the sum of the third and fourth is 63.

9. George Dean plans to use 60 inches of lumber for four shelves whose lengths are to be a series of consecutive even numbers. How long shall he make each shelf?

10. The sides of a triangle are consecutive numbers. If the perimeter of this triangle is 240 feet, find the length of each side.

11. The smaller of two consecutive even integers is 2 more than twice the larger. Find the numbers.

12. The larger of two consecutive odd integers is 4 less than $\frac{1}{3}$ the smaller. Find the numbers.

B 13. Find four consecutive integers such that five times the fourth diminished by twice the second is 7.

14. Find four consecutive even integers such that four times the fourth decreased by one-half the second is 9.

15. Three times the smaller of two consecutive odd integers is less than twice the larger. What are the largest possible values for the integers?

16. Three consecutive even integers are such that their sum is more than 24 decreased by twice the third integer. What are the smallest possible values for the integers?

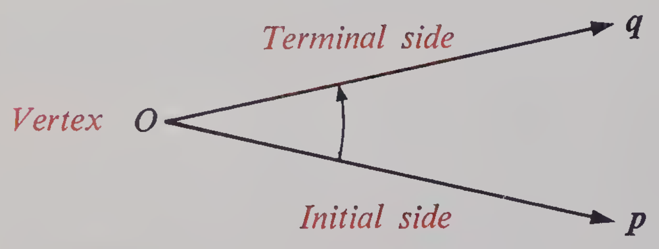
17. The larger of two consecutive integers is greater than 4 more than half the smaller. What are the smallest possible values for the integers?

18. Three consecutive integers are such that the sum of the first and third is less than 18 increased by half the second. What are the largest possible values for the integers?

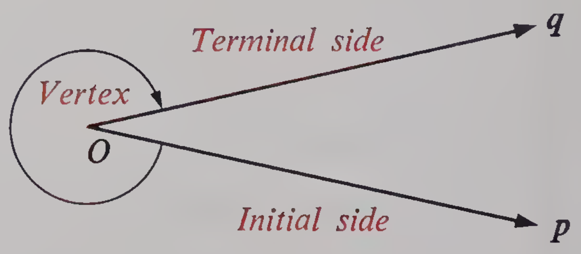
5–6 Problems about Angles

Although fairly simple, this section gives a useful tie-in with geometry.

Think of the figure composed of two rays p and q drawn from a point O. Then think of the ray q as having turned or rotated about O, starting at p and going to its indicated position. As shown, the rotation may be clockwise or counterclockwise.



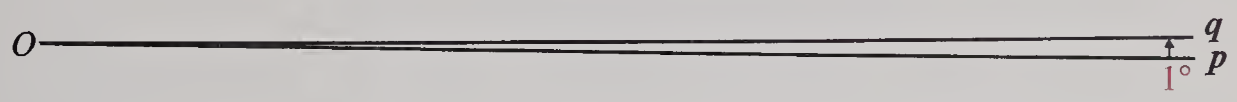
Counterclockwise Rotation

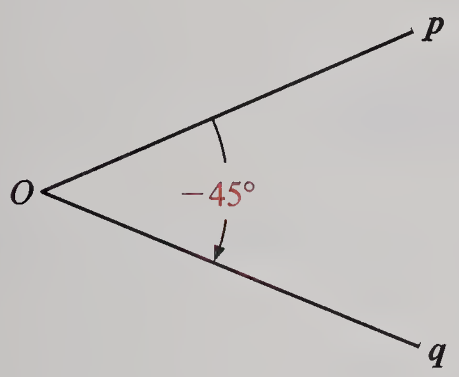


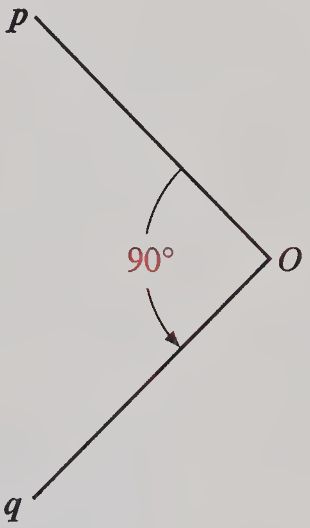
Clockwise Rotation

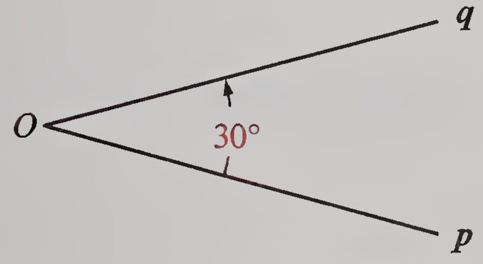
The figure composed of two rays drawn from a point, together with the rotation that sends one ray into the other is called a **directed angle**. Counterclockwise rotation yields a **positive directed angle**; clockwise rotation yields a **negative directed angle**. Ray p is the *initial side* of the angle and ray q is the *terminal side*. The point O is the *vertex* of the angle.

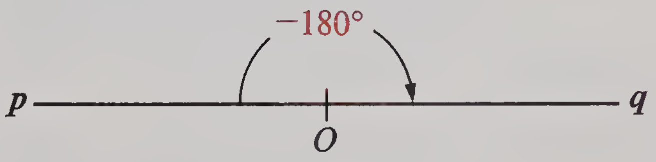
A common unit of measure of an angle is a *degree*, written as 1°. A **degree** is $\frac{1}{360}$ of a complete rotation of a ray about a point. The directed angles whose measures are 1°, 30° (read "30 degrees"), 90°, 180°, −45°, −180°, and −360° are shown:

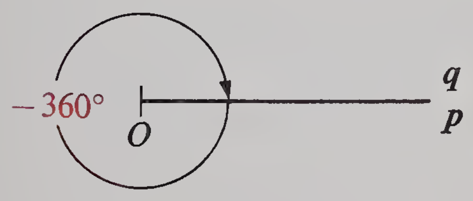


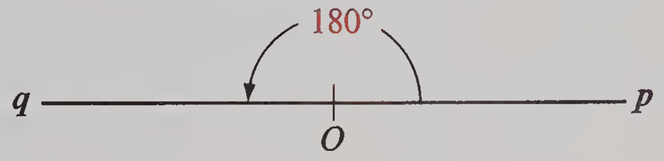






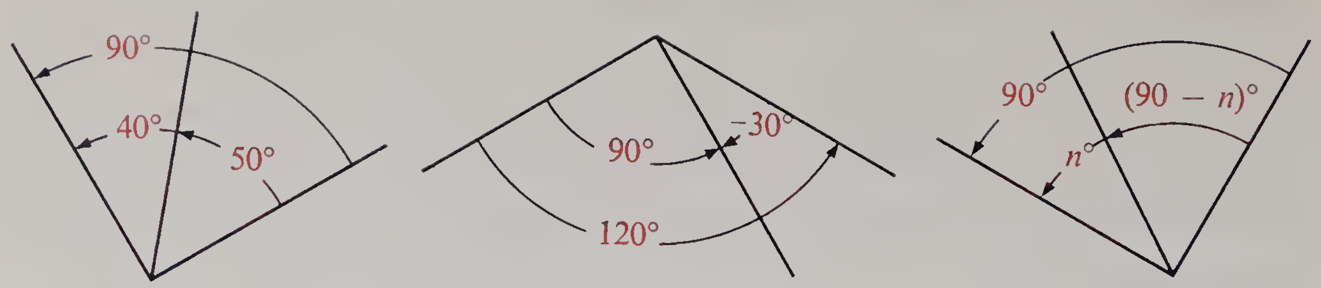




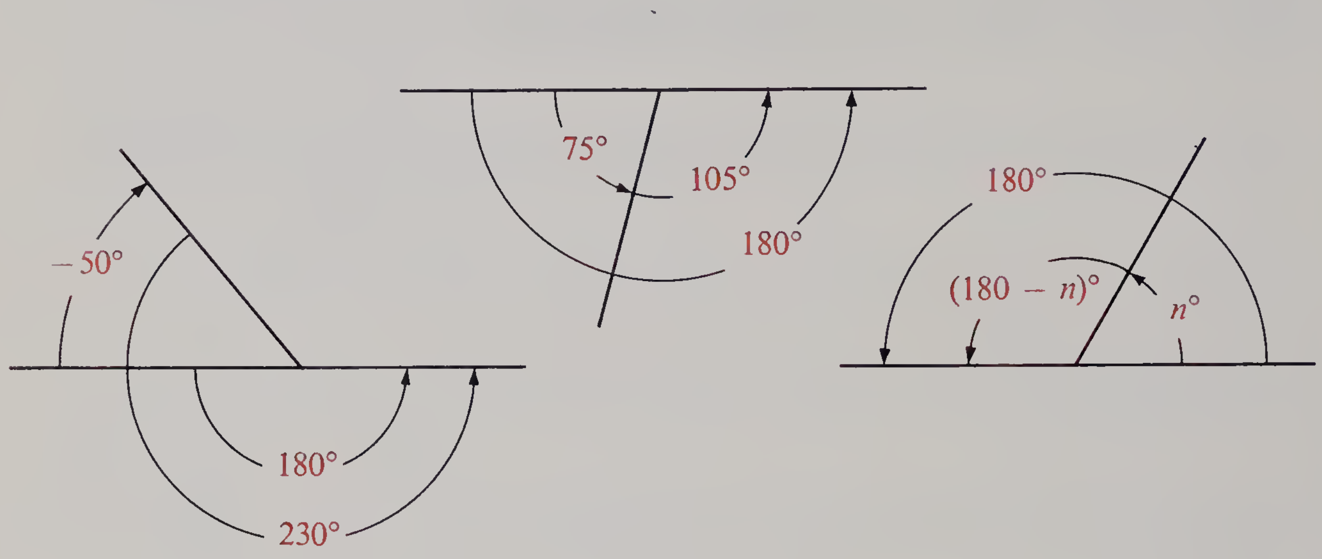


Two angles are **complementary angles** if the sum of their measures is 90°. Each is the *complement* of the other. If an angle contains n degrees, its complement contains $(90 - n)$ degrees.

Two angles are **supplementary angles** if the sum of their measures is 180°. Each is the *supplement* of the other. If an angle contains n degrees, its supplement contains $(180 - n)$ degrees. The diagrams on the next page show complementary and supplementary angles.



Complementary Angles



Supplementary Angles

EXAMPLE. **How large is an angle whose supplement contains 21° less than four times its complement?**

Solution:

1

Let n = the number of degrees in the angle.

Then $(90 - n)$ = the number of degrees in its complement,

and $(180 - n)$ = the number of degrees in its supplement.

2

$$\underbrace{\text{The supplement}}_{(180 - n)} \quad \underset{=}{\overset{\text{is}}{\downarrow}} \quad \underbrace{\text{four times the complement}}_{4(90 - n)} \quad \underset{-}{\overset{\text{less}}{\downarrow}} \quad \underset{21}{\overset{21}{\downarrow}}$$

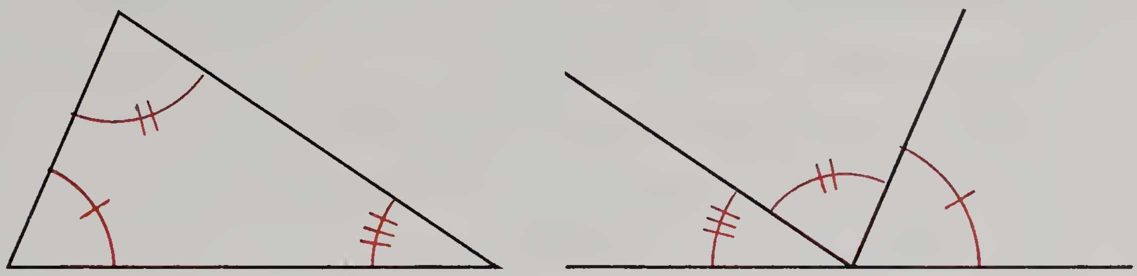
Steps 3 and 4 are left to you.

The three line segments that compose a triangle intersect by pairs and so form three angles. If you tear off the corners from any paper triangle and fit them together as shown in Figure 5–1, you will notice

Have students do this experiment. It provides experience in drawing tentative conclusions inductively from experimentation.

that the three angles fit together to form a straight angle. This suggests a property of all triangles which is proved in geometry.

The sum of the measures of the angles of any triangle is 180°.



· *Figure 5–1* ·

ORAL EXERCISES

Give the complement of each angle.

1. 20° 70°
2. 70° 20°
3. 25° 65°
4. 15° 75°
5. 60° 30°
6. 80° 10°
7. −12° 102°
8. −22° 112°
9. 33° 57°
10. 73° 17°
11. $3x$ degrees $(90-3x)$ degrees
12. $2n$ degrees $(90-2n)$ degrees
13. −110° 200°
14. −120° 210°
15. −90° 180°
16. −0° 90°

State the number of degrees in the supplement of each angle.

17. 50° 130°
18. 70° 110°
19. 105° 75°
20. 115° 65°
21. −94° 274°
22. −84° 264°
23. a degrees $(180-a)$ degrees
24. $2a$ degrees $(180-2a)$ degrees
25. −150° 330°
26. −135° 315°
27. 0.75° 179.25°
28. $\frac{1}{3}$° $179\frac{2}{3}$°
29. $(n-3)$ degrees $(183-n)$ degrees
30. $(x-10)$ degrees $(190-x)$ degrees
31. $(2n+5)$° $(175-2n)$°
32. $(3x-10)$° $(190-3x)$°

WRITTEN EXERCISES

In each exercise, two angles of a triangle are given. Find the number of degrees in the third angle.

A

1. 20°, 70°
2. 30°, 80°
3. 60°, 60°
4. 110°, 40°
5. 100°, 60°
6. 13°, 139°
7. $n°$, $2n°$
8. $\frac{3}{4}n°$, $\frac{1}{4}n°$
9. $m°$, $n°$
10. $x°$, 90°
11. $n°$, $(n+30)°$
12. $2n°$, $(n-20)°$

In Exercises 13–18, find the number of degrees in $a + b$, if the measures of a and b are as indicated.

13. $a = 30°$, $b = \frac{1}{2}$ of a complete rotation clockwise

14. $b = 15°$, $a = \frac{1}{4}$ of a complete rotation clockwise

15. $a =$ a positive straight angle, $b =$ a negative straight angle

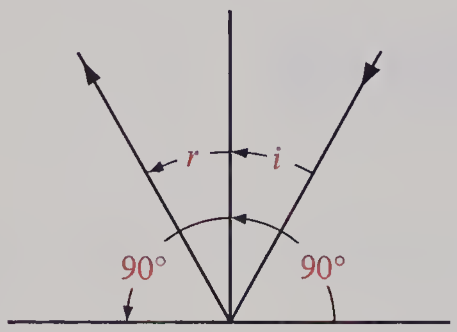
16. $a = \frac{3}{4}$ of a complete rotation clockwise
$b = \frac{1}{2}$ of a complete rotation counterclockwise

17. $a = \frac{1}{6}$ of a complete rotation counterclockwise
$b = \frac{1}{8}$ of a complete rotation clockwise

18. $a = \frac{1}{4}$ of a complete rotation counterclockwise
$b = \frac{1}{4}$ of a complete rotation clockwise

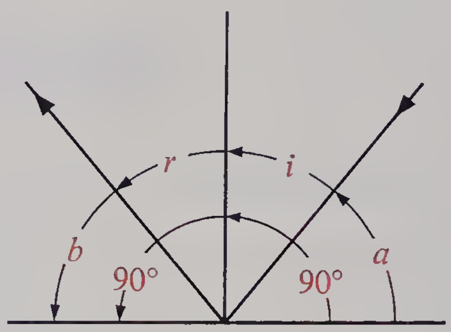
Exercises 19 and 20 refer to the Law of Reflection: $i = r$.

19.



$i = (2n + 30)°$
$r = (4n - 10)°$
Find n.

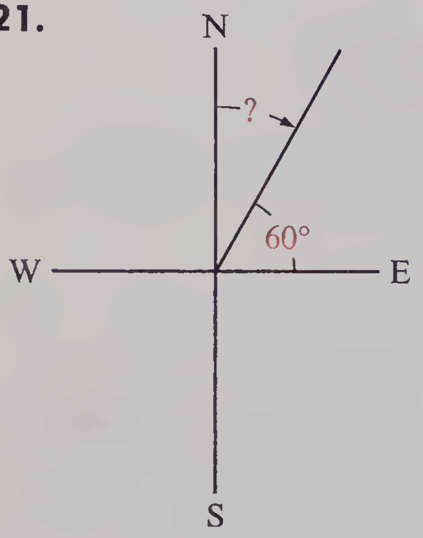
20.



$a = 2m°$
$b = (m + 10)°$
Find m.

Exercises 21–26 refer to the science of navigation in which a compass direction is expressed as a bearing. The bearing of a line of motion is the angle it makes with the north line, measured clockwise from north, through a point at which the observations are made. Find each bearing.

21.



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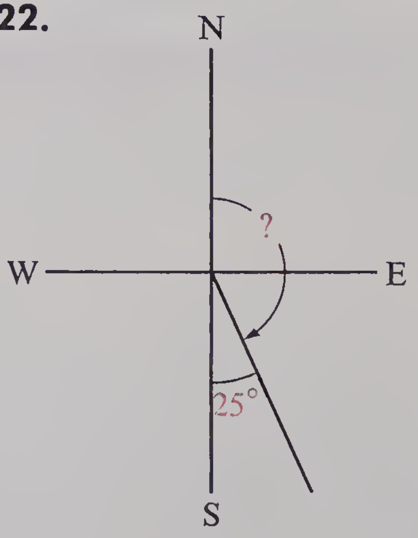

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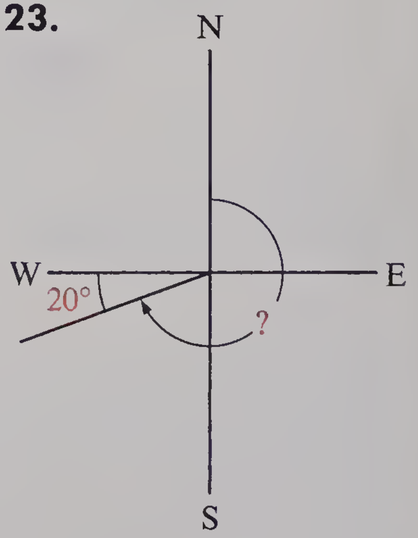

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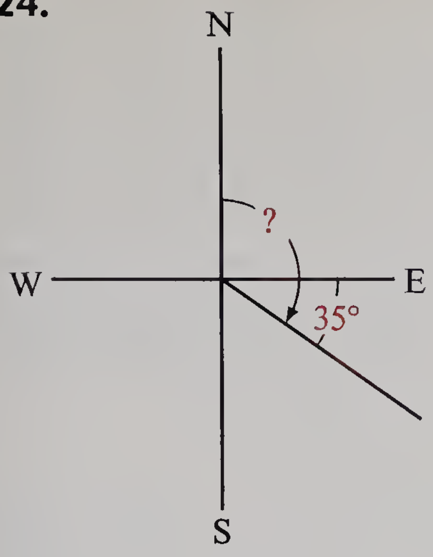

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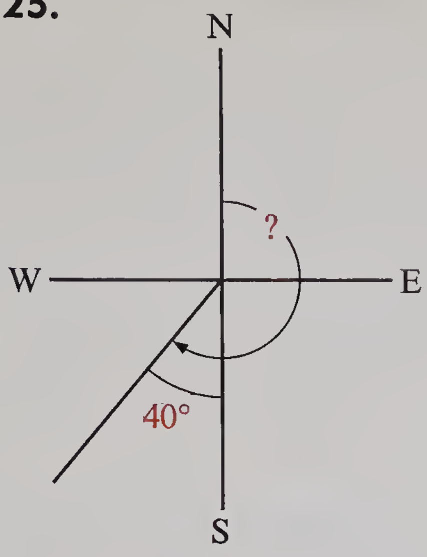
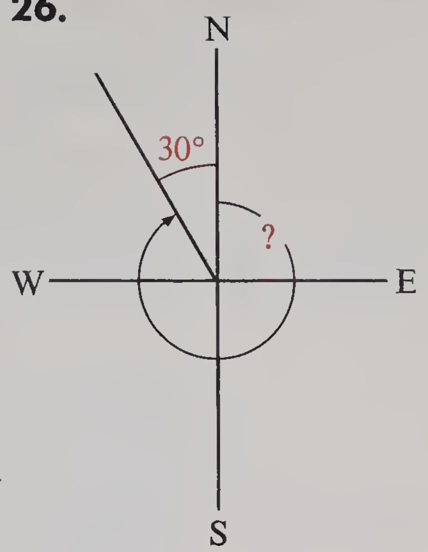


26.



PROBLEMS

A

1. An angle is 12° more than its complement. Find the number of degrees in the complement.
2. Find two complementary angles if one is 28° less than the other.
3. An angle is 15° less than twice its complement. Find the angle.
4. Find two complementary angles if one is 18° less than 3 times the other.
5. Find two supplementary angles if one is four times the other.
6. Find two supplementary angles if one is five times the other.
7. One angle of a triangle is twice as large as another. The third angle contains 5° more than the larger of these. Find each angle.
8. One angle of a triangle is three times as large as another. The third angle is 20° less than the sum of the first two angles. Find the number of degrees in each angle.
9. In any isosceles triangle two angles are equal to each other. The third angle of one isosceles triangle is 36° less than the sum of the other two. Find each angle of the triangle.
10. Two angles of a triangle are equal, but the third angle is 23° less than $2\frac{1}{2}$ times the sum of the first two. How many degrees are in each angle?
11. One angle of a triangle exceeds another by 23°. The third angle is 6° less than the sum of the other two. Find the angles.
12. A triangle is to be drawn in which one angle is 18° larger than another, and the third, 12° less than the sum of the others. Find the angles.
13. How large is an angle whose complement contains 5° more than half its supplement?
14. How large is an angle whose supplement contains 12° less than twice its complement?

5–7 Uniform Motion Problems

An object which moves without changing its speed is said to be in **uniform motion**. Often, charts can help you in organizing the given facts in problems involving uniform motion. The basic principle you will need in such cases is:

$$\text{distance} = \text{rate} \times \text{time}$$
$$d = r \cdot t$$

See 5-7 T.M. pg. 22.

EXAMPLE 1. **(Motion in Opposite Directions) Mr. Rush and Mr. Slow arrange to meet at an airport that is between, and in a straight line with, their home airports. Mr. Rush's jet travels at 600 miles per hour; Mr. Slow's plane travels at 320 miles per hour. They leave their home airports, which are 1380 miles apart, at the same time. If each plane is scheduled for a nonstop flight, in how many hours will they meet?**

Solution:

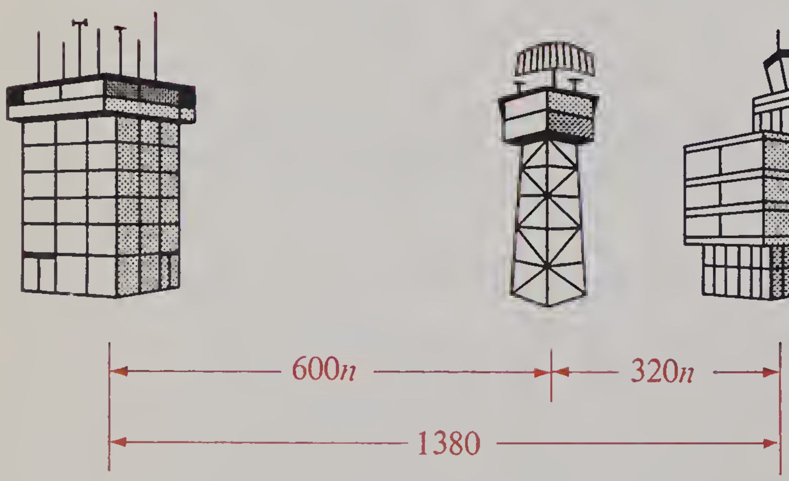
1 **Let n = the number of hours before the men meet.**

2 **Make a sketch illustrating the given facts.**

Mr. Rush's jet rate is 600 m.p.h.
Mr. Slow's plane rate is 320 m.p.h.
Total distance is 1380 miles.
Each travels the same number of hours.

Arrange the facts in chart form.

A suggestion, not a demand! See T.M. pg. 22 (3).



Rule	r	$\cdot\ t$	$= d$
Mr. Rush	600	n	$600n$
Mr. Slow	320	n	$320n$

$$\underbrace{\text{Mr. Rush's jet distance}}_{600n} + \underbrace{\text{Mr. Slow's plane distance}}_{320n} = \underbrace{\text{Total distance}}_{1380}$$

3 Solve the equation:

$$\begin{aligned} 600n + 320n &= 1380 \\ 920n &= 1380 \\ n &= 1\tfrac{1}{2} \end{aligned}$$

4 To check whether the men met in $1\frac{1}{2}$ hours, you have to answer this question: How far did each man fly?

$$\begin{aligned} \text{Mr. Rush flew } 600n \text{ miles:}\quad 600 \cdot 1\tfrac{1}{2} &= 900 \text{ miles} \\ \text{Mr. Slow flew } 320n \text{ miles:}\quad 320 \cdot 1\tfrac{1}{2} &= \underline{480 \text{ miles}} \\ \text{The sum of these distances } &\stackrel{?}{=} 1380 \text{ miles} \\ 1380 &= 1380 \quad \checkmark \end{aligned}$$

The men will meet in $1\frac{1}{2}$ hours, Answer.

EXAMPLE 2. (Motion in the Same Direction) An airplane which maintains an average speed of 350 miles per hour passed an airport at 8 A.M. A jet following that course, at a different altitude, passed the same airport at 10 A.M. and overtook the airplane at noon. At what rate was the jet flying?

Solution:

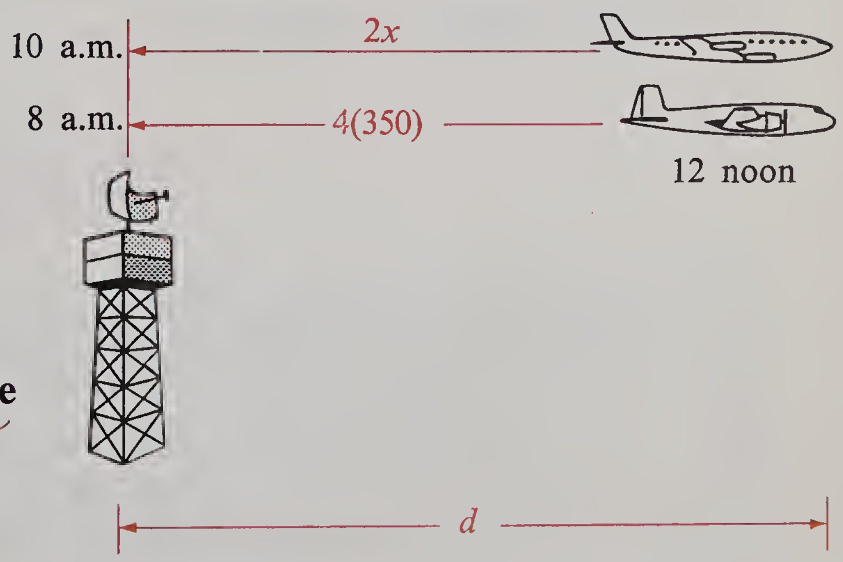
1 Let x = the rate of the jet in m.p.h.

2 Make a chart of the facts given in the problem.

Rate of airplane is 350 m.p.h.
Periods of time under consideration
Airplane: 8 A.M. to noon, or 4 hours
Jet: 10 A.M. to noon, or 2 hours
Each plane covered the same distance.

Make a sketch illustrating the facts given in the problem.

Rule	r ·	t =	d
Airplane	350	4	1400
Jet	x	2	$2x$



$$\underbrace{\text{Distance of jet}}_{2x} = \underbrace{\text{Distance of airplane}}_{1400}$$

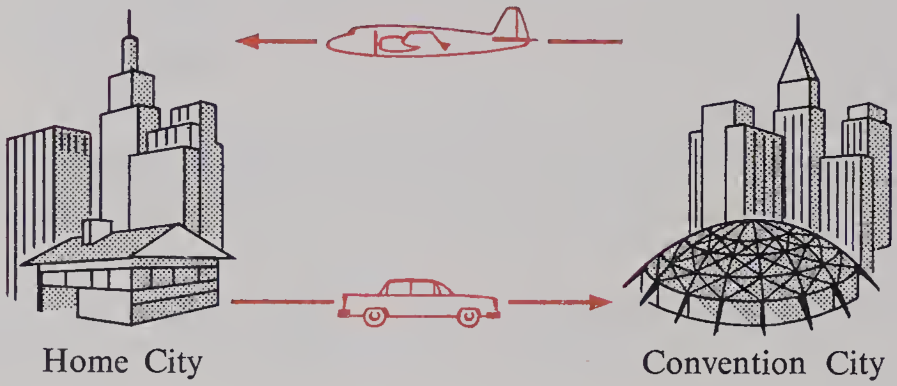
Steps 3 and 4 are left to you.

EXAMPLE 3. (Round Trip) A man leaves his home and drives to a convention at an average rate of 50 miles per hour. Upon arrival, he finds a telegram advising him to return at once. He catches a plane that takes him back at an average rate of 300 miles per hour. If the total traveling time was $1\frac{3}{4}$ hours, how long did it take him to fly back? How far from his home was the convention?

Solution:

1 Let h = number of hours flown.

Then $\frac{7}{4} - h$ = number of hours driven.



2 The given facts are these: a. The total time is $1\frac{3}{4}$ hours. b. The driving rate is 50 miles per hour. c. The flying rate is 300 miles per hour. d. The number of miles driven is the same as the number of miles flown.

Rule	r ·	t =	d
Driving	50	$\frac{7}{4} - h$	$50(\frac{7}{4} - h)$
Flying	300	h	$300h$

$$\underbrace{\text{Distance driven}}_{50(\frac{7}{4} - h)} \overset{=}{\downarrow} \underbrace{\text{Distance flown}}_{300h}$$

$$50(\tfrac{7}{4} - h) = 300h$$

Steps 3 and 4 are left to you.

PROBLEMS

Make a drawing and a chart for each problem. Then form the equation, solve it, and check your answer in the words of the problem.

A 1. A westbound jet leaves Central Airport traveling 625 miles an hour. At the same time, an eastbound plane departs at 325 miles an hour. In how many hours will the planes be 1900 miles apart?

(continued on page 181)

The following insert provides a review of number properties and a feeling for the development of the number system.

Number System Structure

Faced with such problems as counting the animals in his flock or comparing the size of his warrior band with that of his enemy, man eventually conceived the *natural* or *counting numbers*. To count you need a *first number*, and you have to know what number comes next after any given number. Hence, two important properties of the set of natural numbers are: (*i*) there is a first natural number, namely, 1; (*ii*) every natural number has an *immediate successor* in the set, such that between a natural number and its immediate successor no other natural numbers can be inserted. Consequently, the set of natural numbers is usually pictured as a succession of equally spaced points extending without end in one direction along a line.

The operations of addition and multiplication arise when you seek to count the members in the set formed by combining the elements in two or more sets having no members in common. For example, if you have 3 pennies in one row and 2 in another row, you have, in all, 5 pennies, a fact expressed by the symbols: $3 + 2 = 5$. On the other hand, if you have 3 pennies in each of 2 rows, you have 6 pennies in all: $3 \times 2 = 6$. This suggests the **property of closure for addition and multiplication**:

> **For any natural numbers a and b the sum $a + b$ and the product ab are both definite natural numbers.**

This assumption implies not only that sums and products exist in the set of natural numbers, but also that they are unique. If you replace a and b by natural numbers c and d such that $a = c$ and $b = d$, then $a + b = c + d$ and $ab = cd$.

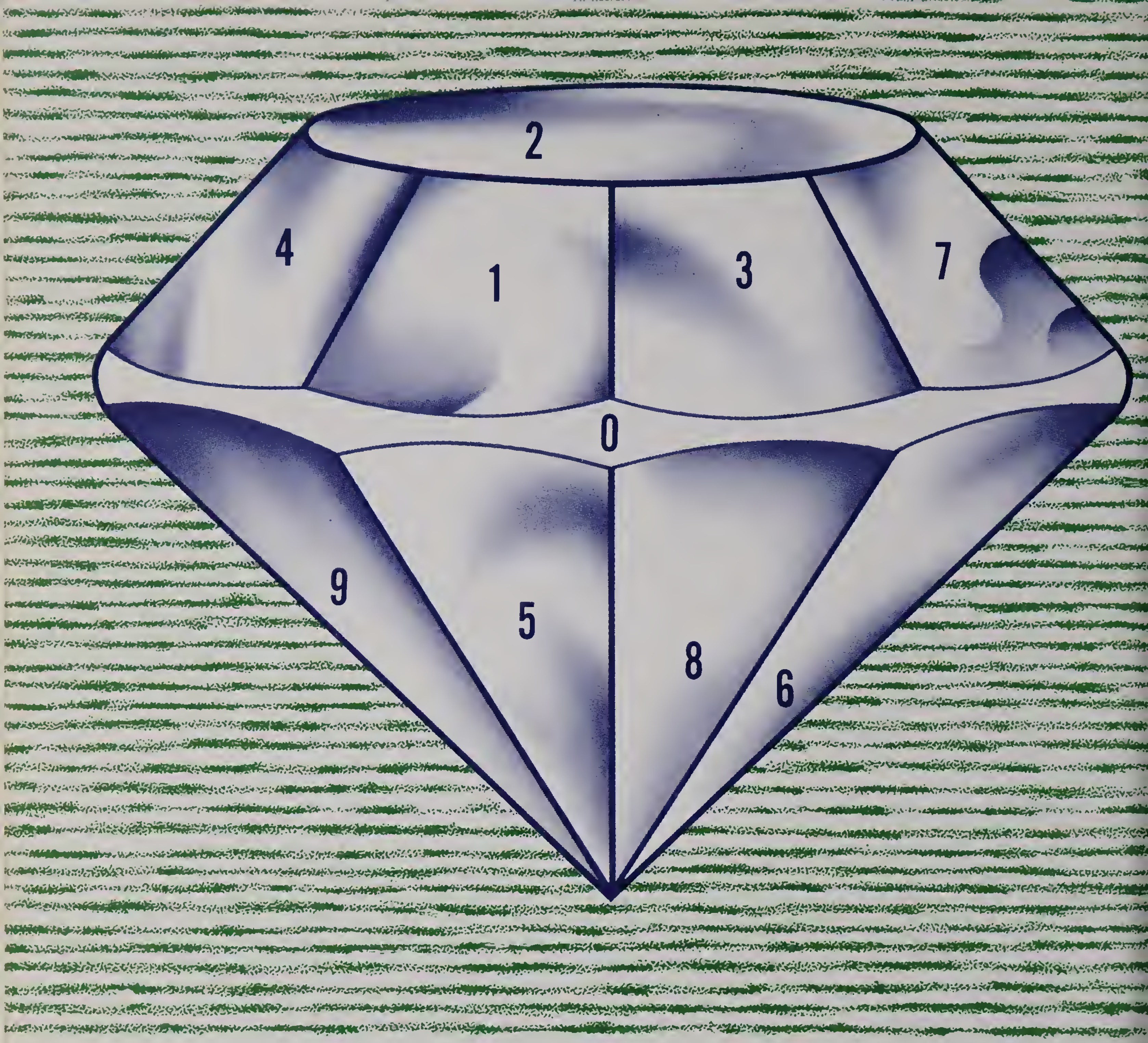
The invention of a number for the empty set was a fairly late but significant advance in algebra. When we adjoin 0 to the set of natural numbers, we call the enlarged set, the system of *whole numbers*. Zero, the first whole number, is the unique number such that for every number a, $0 + a = a + 0 = a$.

Order in the set of whole numbers can be defined in terms of addition: $a < b$ or $b > a$ provided that the equation $a + x = b$ has a solution in the set of positive whole numbers. For example, $2 < 3$ because $2 + 1 = 3$. Whole numbers are also called integers. The *positive integers* are the whole numbers greater than 0. If $a \leq b$, $b - a$ is the whole number whose sum with a is b. For example, $3 - 2 = 1$, but $2 - 3$ does not exist in the set. The quotient $\frac{a}{b}$ exists in this set only if $b \neq 0$ and a is a multiple of b; that is, $a = qb$ for a whole number q. In this case, $\frac{a}{b} = q$.

Operations with the whole numbers are assumed to have the basic properties of arithmetic stated on page B. Representing the structure of the system of whole numbers by the incomplete diamond pictured on that page suggests that a number system containing only whole numbers is also incomplete.

Once man advanced from the counting process to the problem of measuring such quantities as length and weight, he found that whatever standard measuring units he used, he met lengths and weights which did not contain the standard unit a whole number of times. Therefore, he divided his standard units into halves, thirds, quarters, and so on, thus introducing such common fractions as $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

(Continued on page E)



WHOLE NUMBERS

COMMUTATIVE PROPERTIES

$a + b = b + a$; $ab = ba$

ASSOCIATIVE PROPERTIES

$(a + b) + c = a + (b + c)$; $(ab)c = a(bc)$

DISTRIBUTIVE PROPERTIES

$a(b + c) = ab + ac$; $a(b - c) = ab - ac$

PROPERTIES OF ONE AND ZERO

$1 \cdot a = a \cdot 1 = a$; $0 + a = a + 0 = a$; $0 \cdot a = a \cdot 0 = 0$

PROPERTIES OF EQUALITY

$a = a$.

If $a = b$, then $b = a$.

If $a = b$ and $b = c$, then $a = c$.

If $a = b$, then:

1) $a + c = b + c$ and $ac = bc$;

2) $a - c = b - c$, if $a \geq c$;

3) $\frac{a}{c} = \frac{b}{c}$, if $c \neq 0$ and a is a multiple of c.

PROPERTIES OF INEQUALITY

One, and only one, of these statements is true for each a and each b: $a = b$, $a < b$, $b < a$.

If $a < b$ and $b < c$, then $a < c$.

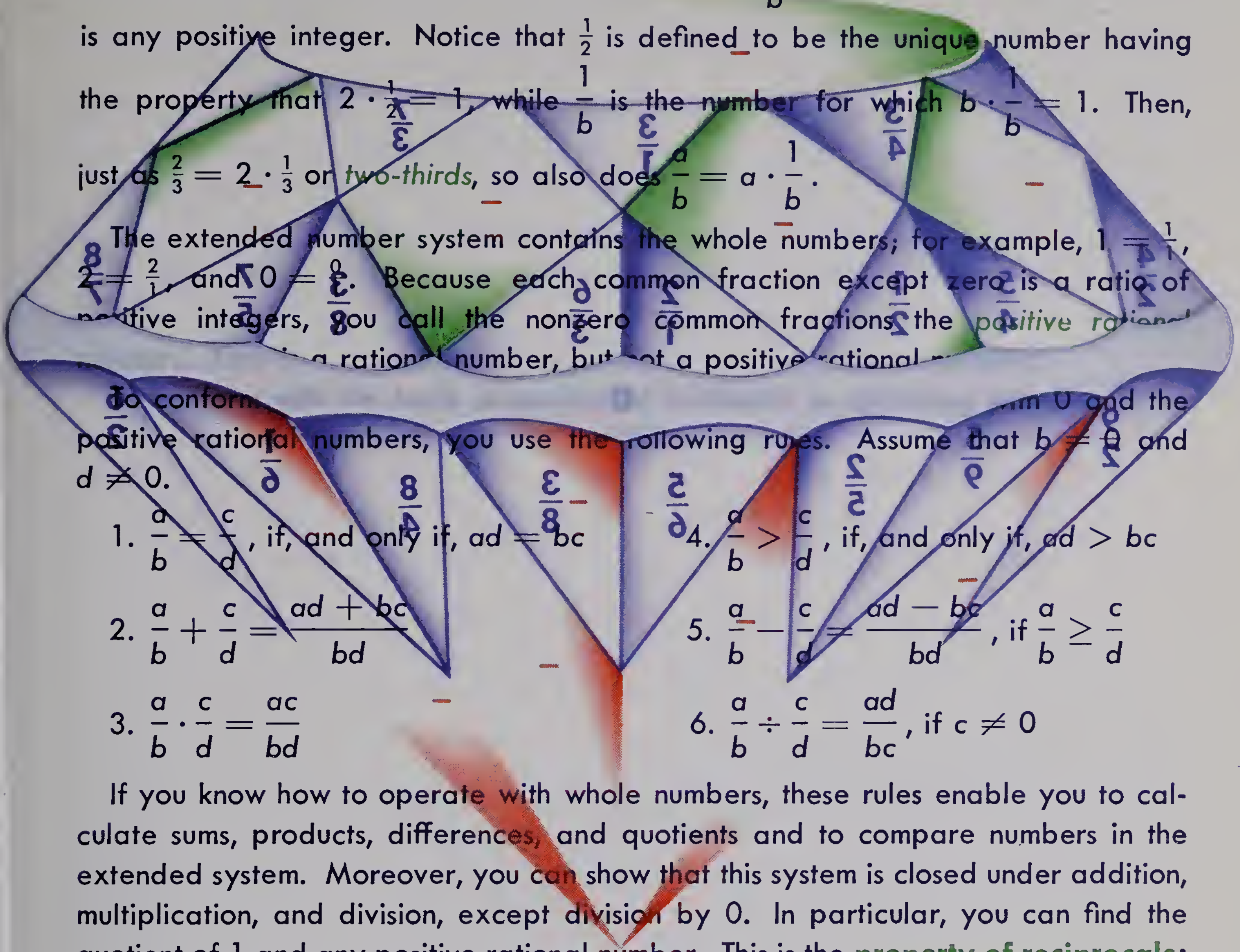
If $a < b$, then:

1) $a + c < b + c$, and, if $0 < c$, $ac < bc$;

2) $a - c < b - c$, if $a \geq c$;

3) $\frac{a}{c} < \frac{b}{c}$, if $0 < c$ and a and b are multiples of c.

Later appeared fractions like $\frac{2}{3}$, $\frac{3}{4}$, and, in general, $\frac{a}{b}$, where a is any integer and b is any positive integer. Notice that $\frac{1}{2}$ is defined to be the unique number having the property that $2 \cdot \frac{1}{2} = 1$, while $\frac{1}{b}$ is the number for which $b \cdot \frac{1}{b} = 1$. Then, just as $\frac{2}{3} = 2 \cdot \frac{1}{3}$ or *two-thirds*, so also does $\frac{a}{b} = a \cdot \frac{1}{b}$.



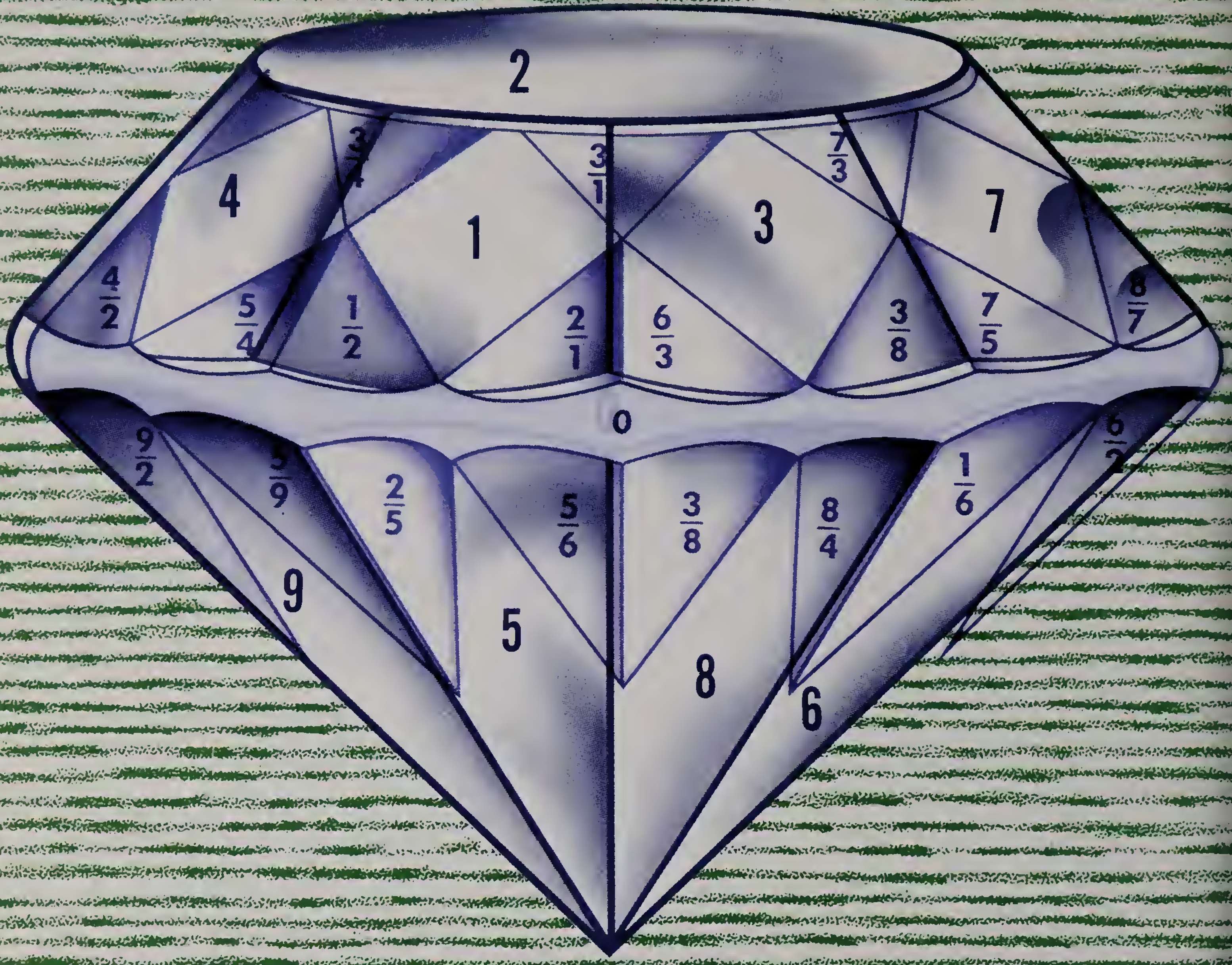
The extended number system contains the whole numbers; for example, $1 = \frac{1}{1}$, $2 = \frac{2}{1}$, and $0 = \frac{0}{1}$. Because each common fraction except zero is a ratio of positive integers, you call the nonzero common fractions the *positive rational* [illegible] is a rational number, but not a positive rational [illegible]

To conform [illegible] 0 and the positive rational numbers, you use the following rules. Assume that $b \neq 0$ and $d \neq 0$.

1. $\frac{a}{b} = \frac{c}{d}$, if, and only if, $ad = bc$
2. $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$
3. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
4. $\frac{a}{b} > \frac{c}{d}$, if, and only if, $ad > bc$
5. $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$, if $\frac{a}{b} \geq \frac{c}{d}$
6. $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$, if $c \neq 0$

If you know how to operate with whole numbers, these rules enable you to calculate sums, products, differences, and quotients and to compare numbers in the extended system. Moreover, you can show that this system is closed under addition, multiplication, and division, except division by 0. In particular, you can find the quotient of 1 and any positive rational number. This is the **property of reciprocals:**

> **For every nonzero number a there is a unique number, denoted by $\frac{1}{a}$ and called the *reciprocal* or *multiplicative inverse* of a, such**



POSITIVE RATIONAL NUMBERS AND ZERO

The set of positive rational numbers and zero satisfies all the basic laws of arithmetic stated on page B. In addition, it has the following properties:

PROPERTIES OF CLOSURE: For all numbers a and b and nonzero number c in the set $a + b$, ab and $\frac{a}{c}$ are definite numbers in the set.

PROPERTIES OF RECIPROCALS: 1) For every nonzero number a in the set there is in the set a unique number, denoted by $\frac{1}{a}$, such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.

2) $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$

PROPERTY OF DENSITY: Between any two different numbers in the set is another number in the set.

Since the set is closed under division, excluding division by zero, the last of the properties of equality and of inequality on page B can be stated more simply.

If $a = b$ and $0 \neq c$, then $\frac{a}{c} = \frac{b}{c}$. If $a < b$ and $0 < c$, then $\frac{a}{c} < \frac{b}{c}$.

Later appeared fractions like $\frac{2}{3}$, $\frac{3}{4}$, and, in general, $\frac{a}{b}$, where a is any integer and b is any positive integer. Notice that $\frac{1}{2}$ is defined to be the unique number having the property that $2 \cdot \frac{1}{2} = 1$, while $\frac{1}{b}$ is the number for which $b \cdot \frac{1}{b} = 1$. Then, just as $\frac{2}{3} = 2 \cdot \frac{1}{3}$ or *two-thirds*, so also does $\frac{a}{b} = a \cdot \frac{1}{b}$.

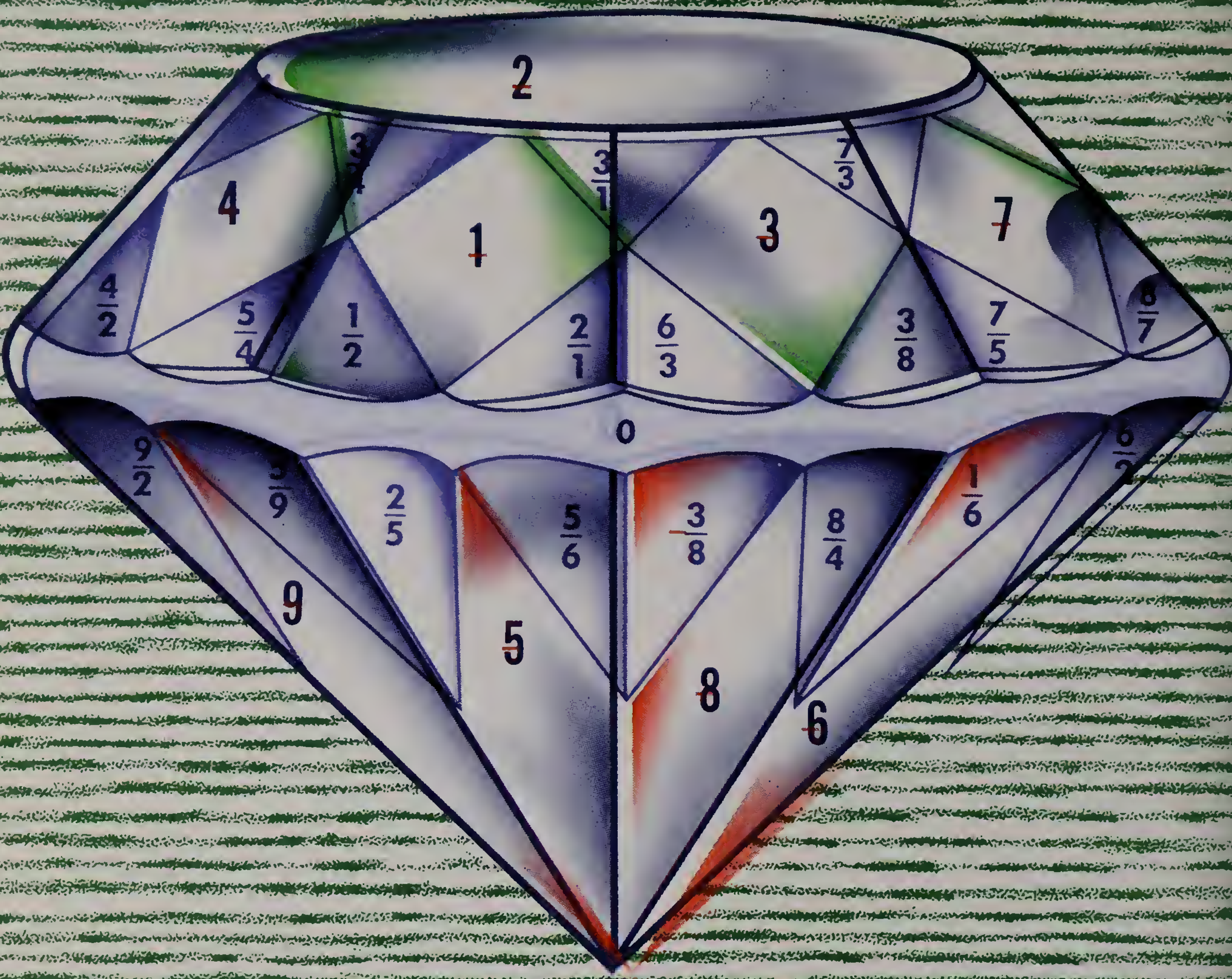
The extended number system contains the whole numbers; for example, $1 = \frac{1}{1}$, $2 = \frac{2}{1}$, and $0 = \frac{0}{1}$. Because each common fraction except zero is a ratio of positive integers, you call the nonzero common fractions the *positive rational numbers*. Zero is a rational number, but not a positive rational number.

To conform with the basic properties of arithmetic in operating with 0 and the positive rational numbers, you use the following rules. Assume that $b \neq 0$ and $d \neq 0$.

1. $\frac{a}{b} = \frac{c}{d}$, if, and only if, $ad = bc$
2. $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$
3. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
4. $\frac{a}{b} > \frac{c}{d}$, if, and only if, $ad > bc$
5. $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$, if $\frac{a}{b} \geq \frac{c}{d}$
6. $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$, if $c \neq 0$

If you know how to operate with whole numbers, these rules enable you to calculate sums, products, differences, and quotients and to compare numbers in the extended system. Moreover, you can show that this system is closed under addition, multiplication, and division, except division by 0. In particular, you can find the quotient of 1 and any positive rational number. This is the **property of reciprocals:**

> **For every nonzero number a there is a unique number, denoted by $\frac{1}{a}$ and called the *reciprocal* or *multiplicative inverse* of a, such**



RATIONAL NUMBERS

Besides having all the properties stated on pages B and C, the set of rational numbers has the following properties.

PROPERTIES OF OPPOSITES:

1) For every rational number a there is a unique rational number, denoted by $-a$, such that $a + (-a) = (-a) + a = 0$.
2) $-(a + b) = (-a) + (-b)$
3) $-a = (-1)a$.

PROPERTY OF CLOSURE: For all rational numbers a and b, $a - b$ is a definite rational number.

Since the set is closed under subtraction, you may state an equality and an inequality property more simply than on page B. Furthermore, the set has an additional property of inequality.

If $a = b$, then $a - c = b - c$, If $a < b$, then $a - c < b - c$.

If $a < b$ and $0 > c$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

Later appeared fractions like $\frac{2}{3}$, $\frac{3}{4}$, and, in general, $\frac{a}{b}$, where a is any integer and b is any positive integer. Notice that $\frac{1}{2}$ is defined to be the unique number having the property that $2 \cdot \frac{1}{2} = 1$, while $\frac{1}{b}$ is the number for which $b \cdot \frac{1}{b} = 1$. Then, just as $\frac{2}{3} = 2 \cdot \frac{1}{3}$ or *two-thirds*, so also does $\frac{a}{b} = a \cdot \frac{1}{b}$.

The extended number system contains the whole numbers; for example, $1 = \frac{1}{1}$, $2 = \frac{2}{1}$, and $0 = \frac{0}{1}$. Because each common fraction except zero is a ratio of positive integers, you call the nonzero common fractions the *positive rational numbers*. Zero is a rational number, but not a positive rational number.

To conform with the basic properties of arithmetic in operating with 0 and the positive rational numbers, you use the following rules. Assume that $b \neq 0$ and $d \neq 0$.

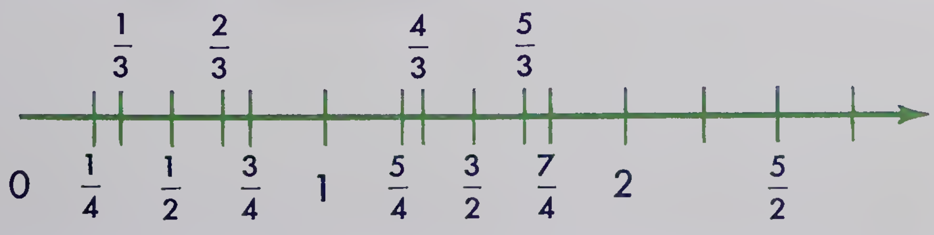
1. $\frac{a}{b} = \frac{c}{d}$, if, and only if, $ad = bc$
2. $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$
3. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
4. $\frac{a}{b} > \frac{c}{d}$, if, and only if, $ad > bc$
5. $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$, if $\frac{a}{b} \geq \frac{c}{d}$
6. $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$, if $c \neq 0$

If you know how to operate with whole numbers, these rules enable you to calculate sums, products, differences, and quotients and to compare numbers in the extended system. Moreover, you can show that this system is closed under addition, multiplication, and division, except division by 0. In particular, you can find the quotient of 1 and any positive rational number. This is the **property of reciprocals**:

> **For every nonzero number a there is a unique number, denoted by $\frac{1}{a}$ and called the *reciprocal* or *multiplicative inverse* of a, such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.**

For example, the reciprocals of 3, $\frac{1}{5}$, and $\frac{7}{4}$ are $\frac{1}{3}$, 5, $\frac{4}{7}$.

To picture the system of positive rational numbers and 0 on a line, start with the representation of the set of whole numbers. Then divide the unit intervals into halves, thirds, and so on, thus locating points to be labeled $\frac{1}{2}, \frac{3}{2}, \ldots; \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \ldots;$ and so on. By continuing this procedure, you pair every positive rational number r with a point whose distance from the point 0 is given by r. This geometric picture

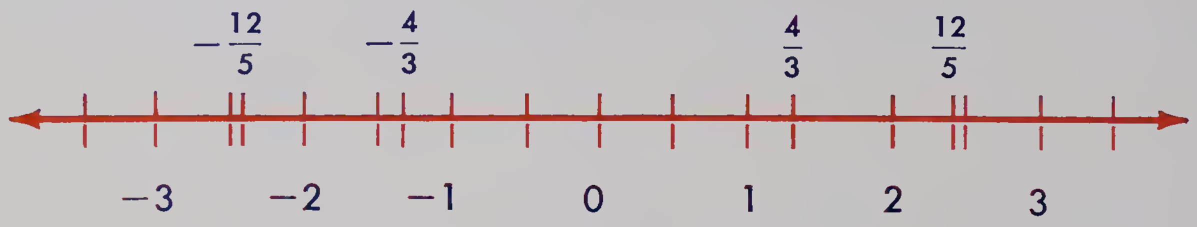


of the set of positive rational numbers and 0 suggests that this set has an order pattern completely different from that of the set of whole numbers. It is not true in the extended system that every number has an immediate successor; for example, you can find a rational number as close to $\frac{1}{2}$ as you like. The system of positive rational numbers and 0 has the **property of density**:

> **Between every pair of rational numbers is another rational number.**

In extending the number system, you have gained not only more numbers, but also more number properties. The growing structure of the number system is suggested by the development of the diamond and by the additional properties stated on page C.

Although the system of positive rational numbers and zero is adequate for measuring many quantities, still other quantities can be specified only by giving a direction as well as a magnitude. Thus, a bank balance includes debits as well as credits, and a displacement involves direction as well as distance. Moreover, subtraction is not always possible in the system of common fractions. To remedy this situation, as well as to answer the need for *directed numbers*, we extend the number system again by inventing *negative rational numbers:* -1, $-\frac{3}{4}$, $-\frac{1}{5}$, and so on. Order in the new system is shown on the number line with negative numbers paired with points to the left of 0.



Numbers like 1 and -1 or $\frac{4}{3}$ and $-\frac{4}{3}$ are called *opposites* or *negatives* of one another. In general, $-a$ stands for *the opposite of a*, so that -3 is negative 3, while $-(-3)$ is 3. The positive and negative rational numbers and 0 form the system of *rational numbers*. Fundamental in this system is the **property of opposites:**

> **For every number a there is a unique number $-a$, called the *opposite* or *additive inverse* of a, such that $a + (-a) = (-a) + a = 0$.**

This property together with the basic laws of arithmetic and the rules for calculating with 0 and the positive numbers enables you to discover the rules for operating with directed numbers. First define $|a|$, the *absolute value of a*, to be the greater of a nonzero number a and its opposite $-a$. Define $|0|$ to be 0. The following are the rules of operation for directed numbers.

$$\begin{aligned} a + b &= |a| + |b|, \text{ if } a \geq 0 \text{ and } b \geq 0 \\ &= -(|a| + |b|), \text{ if } a \leq 0 \text{ and } b \leq 0 \\ &= |a| - |b|, \text{ if } a \geq |b| \text{ and } b < 0 \\ &= -(|b| - |a|), \text{ if } |b| \geq a \geq 0 \text{ and } b < 0 \end{aligned}$$

$$\begin{aligned} a \cdot b &= |a| \cdot |b|, \text{ if } a \geq 0 \text{ and } b \geq 0 \\ &= |a| \cdot |b|, \text{ if } a \leq 0 \text{ and } b \leq 0 \\ &= -(|a| \cdot |b|), \text{ if } a \geq 0 \text{ and } b \leq 0 \\ &= -(|a| \cdot |b|), \text{ if } a \leq 0 \text{ and } b \geq 0 \end{aligned}$$

$$a - b = a + (-b)$$

$$\frac{a}{b} = a \cdot \frac{1}{b}, \text{ if } b \neq 0$$

Since you can add, subtract, multiply, and divide (except by 0) in the system of rational numbers and since these operations satisfy the basic laws of arithmetic, you might believe that we now have the complete number system. Certainly, as the many faceted diamond shown on page D suggests, the set of rational numbers has a highly developed structure. However, even the ancient Greeks realized that the rational numbers were inadequate for making certain measurements. The third transparency will consider the question of whether every distance can be measured by a rational number, and will extend the number system still further.

2. At a certain time two airplanes start from the same airport and travel in opposite directions at 350 miles an hour and 325 miles an hour, respectively. In how many hours will they be 2025 miles apart?

3. A train starts from Grand City and travels toward Belleville 388 miles away. At the same time a train starts from Belleville and runs at the rate of 47 miles an hour toward Grand City. They pass each other 4 hours later. Find the rate of the train from Grand City.

4. At 12 noon, two river steamers are 120 miles apart. They pass St. Louis at 6 P.M. headed in opposite directions. If the northbound boat steams at 9 miles per hour, find the rate of the southbound boat.

5. A train left Omaha at 9 A.M. traveling at 50 miles per hour. At 1 P.M. a plane also left Omaha and traveled in the same direction at 300 miles per hour. At what time did the plane overtake the train?

6. A freight train left Beeville at 5 A.M. traveling 30 miles per hour. At 7 A.M. an express train traveling 50 miles per hour left the same station. When did the express overtake the freight?

7. An airplane traveling 280 miles per hour leaves San Francisco $13\frac{1}{2}$ hours after a steamship has sailed. If the plane overtakes the ship in $1\frac{1}{2}$ hours, find the rate of the steamship.

8. Mr. Gomez is 213 miles on his way when his secretary starts out to overtake him in a plane going 320 miles an hour. How fast is Mr. Gomez traveling if the plane overtakes him in three-quarters of an hour?

9. A troop of scouts hiked to the county scout cabin at the rate of 2 miles per hour. They rode back to headquarters at 18 miles per hour. If the round trip took 10 hours, how far is headquarters from the cabin?

10. While waiting for a connecting train, a traveler takes a bus ride at 10 miles per hour to a certain point and then walks back at 2 miles an hour. If he returns 3 hours after he left, how far did he walk?

11. Tom sets out on a hike. After walking for a while at 5 miles an hour, he discovers that he has forgotten his lunch. A passing truck takes him home at 20 miles an hour. When he gets home, he finds that he has lost exactly one hour. How far had he walked?

12. Dick's motorboat can make an average of 8 miles an hour. One day he sets out for a trip, only to have the motor break down. Dick rows back at 2 miles an hour. When he reaches his dock, he finds that he has been gone 5 hours. How far has he rowed?

13. Jim took a trip of 1020 miles. He traveled by train at 55 miles an hour and the same number of hours by plane at 285 miles an hour. How many hours did the trip take?

14. Mrs. Asbury traveled south, half the time by automobile and half the time by train. She averaged 45 miles per hour by automobile and 50 miles per hour by train. The total trip was 665 miles. How long was Mrs. Asbury traveling?

15. A bus and a train start for the same destination at the same time. The highway runs along the railroad track. The bus averages 31 miles an hour, and the train averages 39 miles an hour. In how many hours will they be 24 miles apart?

16. A jet plane traveling 600 miles per hour can make a certain trip in 33 hours less time than a train traveling at 50 miles an hour. How long does the train trip take?

17. In a run around a 150-yard track, Stan and Walter ran at 325 yards a minute and 300 yards a minute respectively. In how many minutes will Stan have run a track length farther than Walter?

18. In a race, Ned is 50 feet in front of Jed after 10 seconds. How fast can Ned run if Jed can run 20 feet per second?

B **19.** A private airplane had been flying for 1 hour when a change of wind direction doubled the effective rate. If the entire trip of 240 miles took $2\frac{1}{2}$ hours, how far did the plane go in the first hour?

20. A ship must average 22 miles per hour to make its ten-hour run on schedule. During the first four hours, bad weather caused it to reduce its speed to 16 miles per hour. What should its speed be for the rest of the trip to keep the ship to its schedule?

21. Mr. Hardy's car breaks down at a village, and he is told that it will take $1\frac{1}{2}$ hours to repair. He takes a bus into the country and intends to walk back over an alternate route which is no shorter than the bus route. If he walks at 4 miles per hour and the bus averages 20 miles per hour, what is the greatest distance from the village he could ride?

22. A salesman starts at 9 A.M. to deliver certain parts to Mr. Rector, install them, and return. Although he could go 12 miles per hour on his bicycle, the salesman does the 6 miles to the Rector place at 8 miles per hour. When he arrives, he finds a message saying that he must be back by 11 A.M. at the latest. Find the maximum time he can spend in installing the parts.

5–8 Mixture Problems

Often a chemist mixes solutions of different strengths of a chemical to obtain a solution of a desired strength. Similarly, merchants mix goods of two or more qualities in order to sell the blend at a given price. All these problems are solved in the same way. The sum of the values or units of weight of the original ingredients must equal the value or weight of the final mixture.

EXAMPLE. **At a Book Fair, 600 books were sold, some pocket editions at 35 cents each and the rest hard-covered books at 50 cents each. The total receipts were equivalent to the last year's intake when the same number of books were sold at an average price of 40 cents per book. How many of each kind of book were sold?**

Solution:

1 **Let n = number of pocket editions sold.**

Then $600 - n$ = number of hard-covered books sold.

2

Kind	Number	Unit price in cents	Total receipts in cents
Pocket editions	n	35	$35n$
Hard-covered books	$600 - n$	50	$50(600 - n)$
Total	600	40	$40 \cdot 600$

$$\underbrace{\text{Total receipts from pocket editions this year}}_{35n} + \underbrace{\text{Total receipts from hard-covered books this year}}_{50(600 - n)} = \underbrace{\text{Total receipts from all books last year}}_{40 \cdot 600}$$

Steps 3 and 4 are left to you.

PROBLEMS

1. At its opening, the Burnside Public Market decides to distribute 1000 souvenirs of two kinds. Some souvenirs cost 20 cents each, and the others cost 25 cents each. If a total of $220 is allowed for the souvenirs, how many of each kind are needed?
2. The school store sold 348 notebooks the first day of school, some at 25 cents each, the rest at 38 cents each. The total receipts for notebooks was $100.91. How many of each kind were sold?

3. A confectioner makes 100 pounds of candy to sell at $1.75 a pound. He mixes candy worth $1.65 a pound with some worth $1.90 a pound. How many pounds of each does he use?

4. A merchant mixes tea worth 90 cents a pound with some worth $1.50 a pound to make 20 pounds of a blend which he can sell at $1.20 a pound. How many pounds of each kind of tea does he use?

5. How much 84-cent coffee must be blended with 24 pounds of 60-cent coffee if the blend is to be sold at 68 cents a pound?

6. Tickets to the Saturday movie cost 18 cents for children and 42 cents for adults. A total of $69.12 was collected for 328 tickets. As a check on the ticket seller, find how many adults attended.

7. Billy Little's bank opened when it registered $1.00. Billy counted the coins and said to his brother, "Altogether there are 48 pennies and nickels. Bet you can't figure out how many of each kind there are!" Show how you would answer the challenge.

8. A man purchases some three-cent stamps and some one-cent stamps for $3.05. There are 19 more three-cent stamps than one-cent stamps. How many of each kind does he buy?

Ⓑ **9.** A nut shop sells almonds for $1.80 a pound, walnuts for $1.30 a pound, and peanuts for 65 cents a pound. The shopkeeper makes a mixture of 21 pounds of these nuts to sell for $1.00 a pound. He uses twice as many pounds of peanuts as walnuts. How much of each does he use?

10. Fred paid $3.00 for stamps. He bought three times as many 3-cent stamps as 6-cent stamps, twice as many $1\frac{1}{2}$-cent stamps as 3-cent stamps, and the same number of 1-cent stamps as of $1\frac{1}{2}$-cent stamps. How many stamps of each sort did Fred buy?

11. Your boss asks you to change a $20 bill so that there will be twice as many quarters as half dollars, and half as many single dollar bills as half dollars. Show him politely how smart you are by explaining how this is impossible.

12. Cathy purchased 100 items in a stationery store for one dollar. She bought pencils at 10 cents each, 9 times as many erasers at 5 cents each, and clips at two-for-a-penny. How many of each did Cathy buy?

Ⓒ **13.** How much of an alloy which contains 6 grams of gold in 10 grams must be melted with another alloy which contains 3 grams of gold in 10 grams, to give an alloy containing 5 grams of gold in 10?

14. A chemist has two solutions of sulfuric acid. The first is half sulfuric acid and half water; the second is three-fourths sulfuric acid and one-fourth water. He wishes to make 10 liters of a solution which is two-thirds sulfuric acid and one-third water. How many liters of each available solution should he use?

THE HUMAN EQUATION

A Legendary Hero and a Resourceful Archbishop

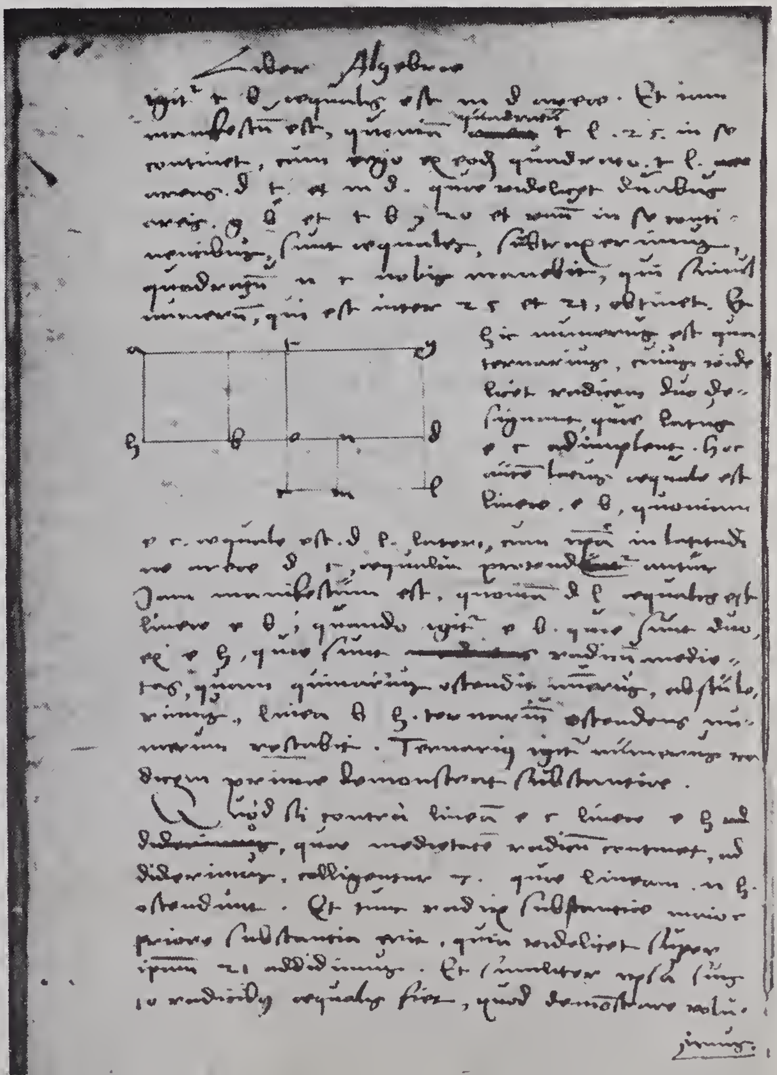
If you study Spanish or read Spanish folk tales in translation, you are almost sure to encounter the legendary hero El Cid. He lived about 1100, and was *not* a mathematician. He played a colorful part in the fighting whereby the Spanish Christians wrested Toledo from the Mohammedans, who had held this Spanish city for four centuries. This victory led directly to the spread of algebra throughout Europe. It did so because the newly appointed Archbishop of Toledo (who was *not* a mathematician) was a resourceful man.

When Toledo returned to Christian rule, it was a great center of learning. Here were books (in Arabic) which contained knowledge long since forgotten in the rest of Europe. Here, too, were Arabic books which recorded new knowledge.

The Archbishop was determined that this learning should be made available to the scholars of Europe. The books should be translated into Latin. But who was to do the translation? Few men knew both Arabic and Latin.

The answer was teamwork. The people of Toledo spoke Spanish as well as Arabic. And most of the scholars of Europe knew Spanish, or could learn it quickly. So the Archbishop set up teams, each consisting of a scholar and a Toledo citizen. The citizen would read an Arabic book aloud, translating into Spanish as he went along, and the scholar would write the Latin of what he heard, translating from Spanish as he went along.

One of the learned men who came to Spain in those days was an Englishman, called Robert of Chester. He it was who translated Al Khowarizmi's masterpiece, *ilm al-jabr wa'l muqabalah*, and thus made algebra available to Europeans.



Page from a copy of Robert of Chester's Latin translation of Al Khowarizmi's algebra. This manuscript was painstakingly copied from Robert's original by Johann Scheubel, a mathematician.

Chapter Summary

Inventory of Structure and Method

1. You can obtain (or derive) equivalent equations in the set of directed numbers by applying the same properties of equality as in the set of arithmetic numbers.
2. If a and b represent directed numbers, exactly one of the following holds: $a < b$, $a = b$, or $a > b$. Graphically, $a < b$, or $b > a$ means that a is to the left of b on the number line.
3. The **transitive property of inequality** is:

   If $a < b$ and $b < c$, then $a < c$ and similarly, if $a > b$ and $b > c$, then $a > c$.

   The **additive property** (this includes subtraction) **of inequality** is:

   For any number c, if $a < b$, then $a + c < b + c$; if $a > b$, then $a + c > b + c$.

   The **multiplicative property** (this includes division) **of inequality**:

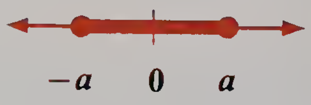
   For c *positive*

   if $a < b$, then $ac < bc$, and if $a > b$, then $ac > bc$;

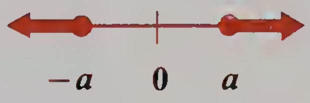
   For c *negative*

   if $a < b$, then $ac > bc$, and if $a > b$, then $ac < bc$.
4. You may obtain equivalent inequalities by applying the additive and multiplicative properties in much the same way as for an equation, except that you reverse the sense of the inequality if you multiply or divide its members by a negative number.
5. The sentence $b < x + a < c$ represents the two inequalities $b < x + a$ *and* $x + a < c$ whose separate solution sets are $b - a < x$ and $x < c - a$. Their common solution set $b - a < x < c - a$ is therefore the solution set of $b < x + a < c$. This solution set may be obtained from the given sentence by subtracting a from each member.

   The inequality $|x| \leq a$ will be satisfied only when $x \geq -a$ *and* $x \leq a$. This pair of inequalities will in turn be satisfied at the same time by the elements of the solution set: $-a \leq x \leq a$.



   On the other hand, the inequality $|x| \geq a$ will be satisfied when $x \geq a$ *or* $x \leq -a$.



6. In solving problems, start by asking yourself two questions: What does the problem require? What facts are given or available? A sketch may help identify the available facts.
7. The rule $d = r \cdot t$ can be applied to problems involving moving objects, whether they move in the same or in different directions on the same straight line.

Vocabulary and Spelling

order property of numbers (*p. 159*)
order, direction or sense, of an inequality (*p. 161*)
equivalent inequality (*p. 161*)
integer (*p. 170*)
consecutive integers (*p. 170*)
even integer (*p. 170*)
odd integer (*p. 170*)
directed angles (*p. 173*)
positive angle (*p. 173*)
negative angle (*p. 173*)
initial side (of an angle) (*p. 173*)
terminal side (of an angle) (*p. 173*)
vertex (of an angle) (*p. 173*)
degree (angle measure) (*p. 173*)
complementary angles (*p. 173*)
supplementary angles (*p. 173*)

Chapter Test

5–1 Solve and check each equation.

1. $\dfrac{3y}{5} + 63 = 27$

2. $5s - 3(4s + 7) = 9 - s$

3. Solve for x: $\dfrac{x}{a} - c = b$

4. Solve for l: $p = 2(l + w)$

5–2 Perform each indicated operation, and write the resulting sentence.

5. **a.** Add -4 to each member of $x + 4 < -1$.
b. Divide $21 > -7v$ by -7.

Solve each inequality, and graph its solution set.

6. $8x + 2 > 5x - 4$

7. $47 \leq 7 - 5(3t - 2)$

5–3 **8.** (Optional) **a.** $|1 - y| \geq 2$ **b.** $|3m - 6| > 9$

5–4 **9.** In a rectangle, the length and width, combined, total 36 yards. Twice the length is 8 yards less than three times the width. Find the dimensions.

10. Typist A can address 50 fewer than twice the number of envelopes addressed by typist B in an hour. Together they address at least 175 envelopes hourly. Find the least number of envelopes typist A addresses in an hour.

5-5 **11.** Find three consecutive odd integers such that the sum of the first and third exceeds the second by 27.

5-6 **12.** The complement of an angle is 18 degrees less than twice the angle. Find the angle and its complement.

13. A triangle is to be drawn with one angle 12° greater than another, and the third 12° less than the sum of the other two. Find the angles.

5-7 **14.** An airplane left Seattle at 1 P.M. and flew directly east at 340 miles per hour. At 2:30 P.M., a jet plane traveling at 595 miles per hour started east from the same airport. In how many hours did the jet overtake the first plane?

5-8 **15.** Adult tickets for a lecture at the school auditorium were 50 cents each, but students were admitted for 20 cents. The auditorium, seating 700, was filled; a total of $305 was realized. How many adults and how many students attended the lecture?

Chapter Review

5-1 Transforming Equations *Pages 157–159*

1. In solving an equation, transformations used in the set of arithmetic numbers are valid also in the set of __?__ numbers.

2. If the same number is added to or subtracted from each member of an equation, the resulting equation is __?__ to the original equation.

3. Multiplying or dividing each member of an equation by a nonzero number produces an equation having the same __?__ __?__.

Of what transformation operation is each of the following an example?

4. $a = b$; so $a + b = 2b$

5. $c = d$; so $c - 3 = d - 3$

6. $m = r$; so $1 = \frac{r}{m}\ (m \neq 0)$

7. $b = c$; so $bat = cat$

In each of the following, the variables represent directed numbers which make the first sentence true. Give a reason that justifies each lettered statement.

8. $5 - 5k = 0$
$-5k = -5$ (a)
$k = 1$ (b)
$5 - 5(1) = 0$ (c)

9. $-4 = \frac{3}{2}p - 4$
$0 = \frac{3}{2}p$ (a)
$0 = 3p$ (b)
$0 = p$ (c)

Solve each equation.

10. $15n = 2n - 1.3$

11. $10 = 8 - \frac{4}{3}w$

12. $7 + 4(t - 3) = t - 5$

13. $2 - (r - 3) = 3r + 3(r + 4)$

Solve each of the following equations for the variable in red.

14. $cz = d$

15. $2y - 3p = 5y$

16. Solve for r: $t = \frac{d}{r}$

17. Solve for a: $m = \frac{1}{2}(a + b)$

18. Solve for i: $p = a - i$

19. Solve for C: $F = \frac{9}{5}C + 32$

20. $\frac{n}{a} = c$

21. $x + 4a = -x$

5-2 The Properties of Inequality *Pages 159–163*

22. If $a \neq b$, then $a > b$ or a __?__ b.

In Exercises 23–33, a, b, *and* c *are any directed numbers.*

23. On a number line, if a is to the left of b and b is to the left of c a is to the __?__ of c.

24. If $a > b$ and $b > c$, then a __?__ c.

25. If $a < b$, then $a + c$ __?__ $b + c$.

26. If $a < b$, then $a - c$ __?__ $b - c$.

27. If $a < b$ and $c > 0$, then ac __?__ bc.

28. If $a < b$ and $c < 0$, then ac __?__ bc.

29. If $a < b$ and $c = 0$, ac __?__ bc.

30. If $a < b$ and $c > 0$, then $\frac{a}{c}$ __?__ $\frac{b}{c}$.

31. If $a < b$ and $\frac{a}{c} > \frac{b}{c}$, then c __?__ 0.

32. If $a \neq 0$, then a^2 __?__ 0, and $-a^2$ __?__ 0.

33. If each member of $-5a < 25$ is divided by -5, then a __?__ -5.

34. You can transform inequality $4x \geq 9x + 10$ into $-5x \geq 10$ by adding ___?___ to each member.

35. If you transform an inequality by multiplying or dividing by any negative number, you must ___?___ the sense of the inequality.

36. If $-5x \geq 10$, then x ___?___ -2.

Solve each inequality, and graph its solution set.

37. $4n - 1 < n + 2$ **38.** $4 - 3m < 8 + 5m$

5-3 Pairs of Inequalities (Optional) *Pages 164–166*

39. The solution set of the sentence $-3 < x < 2$ consists of those numbers for which ___?___ and ___?___ both are true.

40. The inequality $|3x - 2| \geq 4$ will be satisfied if $3x - 2 \geq$ ___?___ or $3x - 2 \leq$ ___?___.

41. Solve the inequality: $|3x - 2| \geq 4$.

42. The inequality $|3x - 2| < 4$ is equivalent to $3x - 2 >$ ___?___ and $3x - 2 <$ ___?___.

43. Solve the inequality: $|3x - 1| > 4$.

44. The set satisfying both $x < 1$ *and* $x > 1$ is the ___?___ set.

45. The set satisfying $x \leq 1$ *or* $x > 1$ is the set of ___?___ numbers.

5-4 A Plan for Solving Problems *Pages 166–169*

46. In an election, *A* received half as many votes as *B*, and *C*'s votes exceeded *B*'s by 110. If 1000 votes were cast, find the results of the election.

47. One side of a triangle must exceed twice a second side by 5 inches, while the third side must be twice the second side. If the perimeter is to be at least 5 inches, find the smallest possible value of the longest side.

5-5 Problems about Consecutive Integers *Pages 170–172*

48. Write the five consecutive even integers that follow -3.

49. Find three consecutive integers whose sum is 2274.

50. Find five consecutive odd integers whose sum is 85.

5-6 Problems about Angles *Pages 172–177*

51. If two angles of a triangle are complementary, the third angle must contain ___?___ degrees.

52. One of two complementary angles is 24 degrees less than the other. Find the angles.

53. Find two supplementary angles, one of which is five times as large as the other.

54. A triangle is to be drawn in which two angles are equal and the third is at most 12 degrees less than twice their sum. Find the smallest possible value of one of the equal angles.

5-7 Uniform Motion Problems *Pages 178–182*

55. Amy walks to the repair shop at 2 miles an hour, picks up her bicycle, and rides home at 10 miles an hour. If the round trip took $1\frac{1}{2}$ hours, how far is the shop from Amy's house?

56. Mr. Ford traveled from Axton to Trent via Grenby, which is midway between them. On the first leg of his trip, he was able to ride at 45 miles an hour, but from Grenby to Trent, poorer roads caused him to go at 36 miles an hour. Still, the whole trip took 1 hour. How far is Grenby from Axton?

57. A bus and a train leave the same station at the same time and travel in the same direction along parallel routes. The train runs twice as fast as the bus. They are from $12\frac{1}{2}$ to 15 miles apart at the end of half an hour. Find the smallest and largest possible value of each rate.

58. Jim took an 80-mile trip, part by train and the rest by automobile. If he had traveled an extra 10 miles of the trip by automobile, he would have gone just twice as far by train as by automobile. How many miles did he actually travel by train?

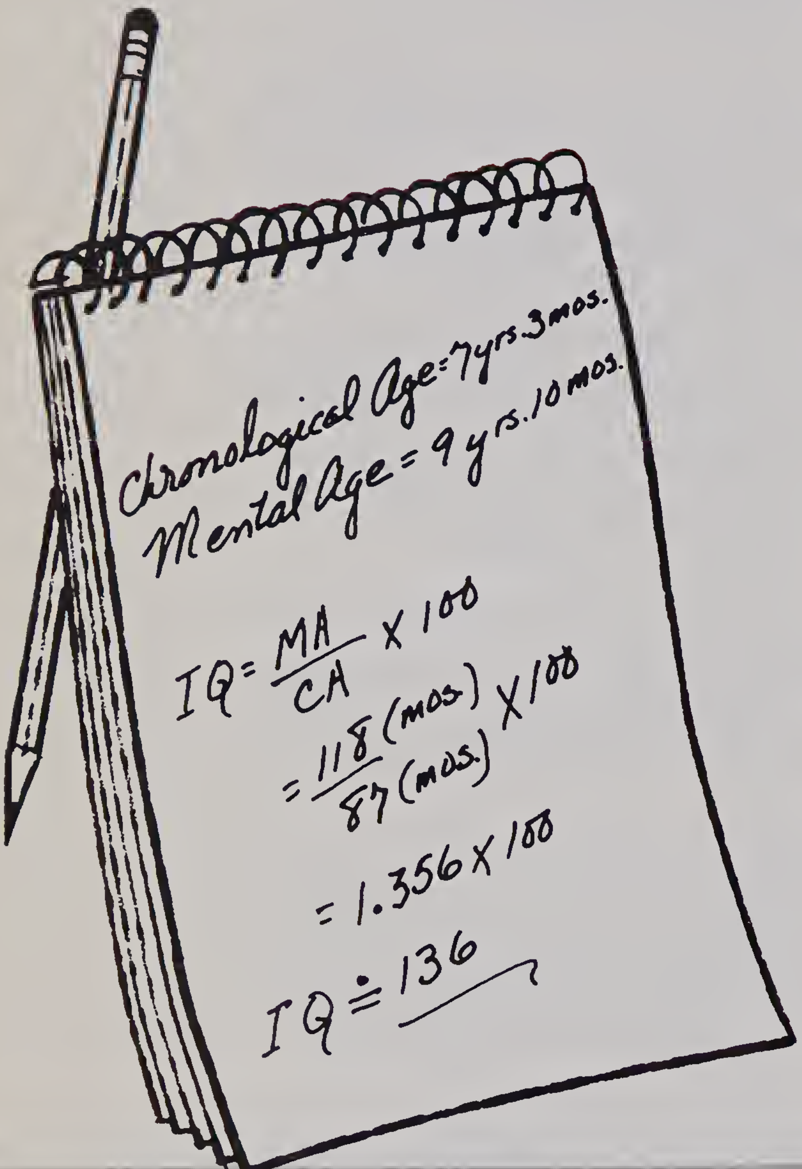
5-8 Mixture Problems *Pages 182–184*

59. How many pints of oil worth 12 cents a pint must be mixed with 100 pints worth 5 cents a pint to produce an oil which can be sold at 7 cents a pint?

60. A confectioner has 10 pounds of nut-and-fruit candy which isn't selling well at $1.25 a pound. He buys creams at 65 cents a pound and mixes them with his candy. He puts out an assortment which he can sell at 85 cents a pound. How many pounds of creams does he buy?

61. An after-school performance was attended by 115 persons. Students were charged 15 cents, but all others paid 35 cents. In all, $25.65 was collected. How many students attended?

Psychometrists and Mathematics



Psychometrists measure mathematically the speed and accuracy of mental processes. By interpreting statistically the results of large numbers of tests on learning, motivation, memory, and other psychological variables, psychometrists gain information about the range and distribution of human behavior.

In 1905 a French psychologist named Alfred Binet perfected a scale of normal (average) intelligence at various age levels. The Binet scale uses the fact that children in each age group normally possess a certain kind of knowledge. The Binet scale groups together in a series of tests the items of knowledge common to each age level. The most difficult test passed by an individual determines his mental age.

Many of you may know your I.Q., but do you know how this figure was derived? An intelligence quotient (I.Q.) is computed on the work pad. This is a simple but much used application of mathematics in the field of psychometrics. The mental age (MA), as determined by a Binet test, is divided by the chronological age (CA), and the quotient is multiplied by 100. You can see that an individual whose mental and chronological ages are the same has an I.Q. of 100 (or 100%), whereas someone whose mental age is less than his chronological age has an I.Q. of less than 100. On the other hand, the boy in the photograph was found to have a mental age of 9 yrs. 10 mos. at a chronological age of 7 yrs. 3 mos. and his I.Q., shown on the work pad, is 136.

Just for Fun

The Day and the Date

Do you know on what day of the week you were born? You can find out without asking your family and without consulting a calendar. In fact, you can find out the day of the week for any date in history.

To find Valentine's day, February 14, 1896, proceed as follows:

1.	Write these numbers, one beneath the other:	
	The last two digits of the year	96
	One fourth of these two digits. (There is no remainder; 1896 was a leap year.)	24
	The key number for February	3
	The number of the day of the month	14
	The key number for the century	2
2.	Find the sum of these numbers	139
3.	Divide the sum by 7	$139 \div 7 = 19\frac{6}{7}$

The remainder indicates the day of the week. A remainder of 1 means Sunday; 2, Monday, and so on through Friday; and 0, or no remainder, means Saturday. The remainder is 6; so you know that Valentine's Day was a Friday in 1896.

Table B shows two key numbers for the eighteenth century, because in 1752 the calendar was reformed, or brought back in line with the motions of the heavenly bodies.

Table A

January	0
(leap years, 0)	
February	4
(leap years, 3)	
March	4
April	0
May	2
June	5
July	0
August	3
September	6
October	1
November	4
December	6

Table B

Twentieth century	0
Nineteenth century	2
Eighteenth century	
Sept. 14, 1752 to Jan. 1, 1800	4
Jan. 1, 1700 to Sept. 2, 1752	1
Seventeenth century	2
Sixteenth century	3
Fifteenth century	4
Fourteenth century	5
Thirteenth century	6
Twelfth century	7
Eleventh century	8
Tenth century	9
(Add 1 for each century you go back.)	

Extra for Experts

Systems of Numeration

In the decimal system 1234 means $1 \times 10^3 + 2 \times 10^2 + 3 \times 10 + 4$. The value of the same sequence of digits in numeral systems whose bases are 7 and 5 are shown in this table. Note that in the 7 system each place has a value 7 times the place at its right, and in the 5 system each place has a value 5 times the place at its right.

1 2 3 4	
BASE 7	BASE 5
$1 \times 7^3 + 2 \times 7^2 + 3 \times 7 + 4$	$1 \times 5^3 + 2 \times 5^2 + 3 \times 5 + 4$
$1 \times 343 + 2 \times 49 + 3 \times 7 + 4$	$1 \times 125 + 2 \times 25 + 3 \times 5 + 4$
$343 + 98 + 21 + 4$	$125 + 50 + 15 + 4$
466	194
Write: $1234_{\text{VII}} = 466_{\text{X}}$	**Write:** $1234_{\text{V}} = 194_{\text{X}}$
Read: 1–2–3–4, base 7 = 4–6–6, base 10.	**Read:** 1–2–3–4, base 5 = 1–9–4, base 10.

To convert 466_{X} to base 7, divide repeatedly by 7, noting the remainders on the right. Continue the process until the quotient is 0.

$$\frac{466}{7} = 66 \text{ with remainder } 4$$

$$\updownarrow$$

$$\frac{66}{7} = 9 \text{ with remainder } 3$$

$$\updownarrow$$

$$\frac{9}{7} = 1 \text{ with remainder } 2$$

$$\updownarrow$$

$$\frac{1}{7} = 0 \text{ with remainder } 1$$

What do the numerals in red signify? The 1 signifies that 466 can be divided by 7^3 once. The 2 signifies that the remainder of $466 - 1 \cdot 7^3$ can be divided by 7^2 twice. The 3 signifies that the remainder of $466 - 1 \cdot 7^3 - 2 \cdot 7^2$ can be divided by 7^1 three times. The 4 signifies that $466 - 1 \cdot 7^3 - 2 \cdot 7^2 - 3 \cdot 7 = 4$, which is less than 7. Together, these are the coefficients of $1 \cdot 7^3 + 2 \cdot 7^2 + 3 \cdot 7^1 + 4$ or 1234_{VII}.

Similarly, to convert 194 to base 5, divide by 5 until the quotient is 0.

Numerals in the decimal system are successions of basic digits 0, 1, 2, . . ., 9, each less than 10. Since each basic numeral in the 7 system is a remainder in a division by 7, the basic numerals in this system are 0, 1, 2, 3, 4, 5, 6. Similarly, in the 5 system the basic numerals are 0, 1, 2, 3, 4, and in a base n system they are 0, 1, 2, . . . $(n - 1)$.

Do you know that many modern digital computers use the binary (base 2) system? Since numerals in the binary system are successions of the digits 0 and 1, the machine can denote numbers by a sequence of electronic devices, each either on or off. A 1 is registered by an ON state, while a 0 is recorded by an OFF state. As the table shows, it takes four digits, 1010_{II}, to write 10_{X}.

BASE 2 EQUIVALENCE TABLE				
2^3	2^2	2^1	1	Base 10
0 ●	0 ●	0 ●	0 ●	0
0 ●	0 ●	0 ●	1 ●	1
0 ●	0 ●	1 ●	0 ●	2
0 ●	0 ●	1 ●	1 ●	3
0 ●	1 ●	0 ●	0 ●	4
0 ●	1 ●	0 ●	1 ●	5
0 ●	1 ●	1 ●	0 ●	6
0 ●	1 ●	1 ●	1 ●	7
1 ●	0 ●	0 ●	0 ●	8
1 ●	0 ●	0 ●	1 ●	9
1 ●	0 ●	1 ●	0 ●	10

Questions

1. Find the largest digit in a numeral system with the given base.

a. 6 **b.** 4 **c.** 2 **d.** h

2. Convert the given numbers to the decimal system.

a. 1357_{IX} **b.** 1021_{III} **c.** 10101_{II} **d.** 246_{VII}

3. Convert the given numbers to the system with the base indicated.

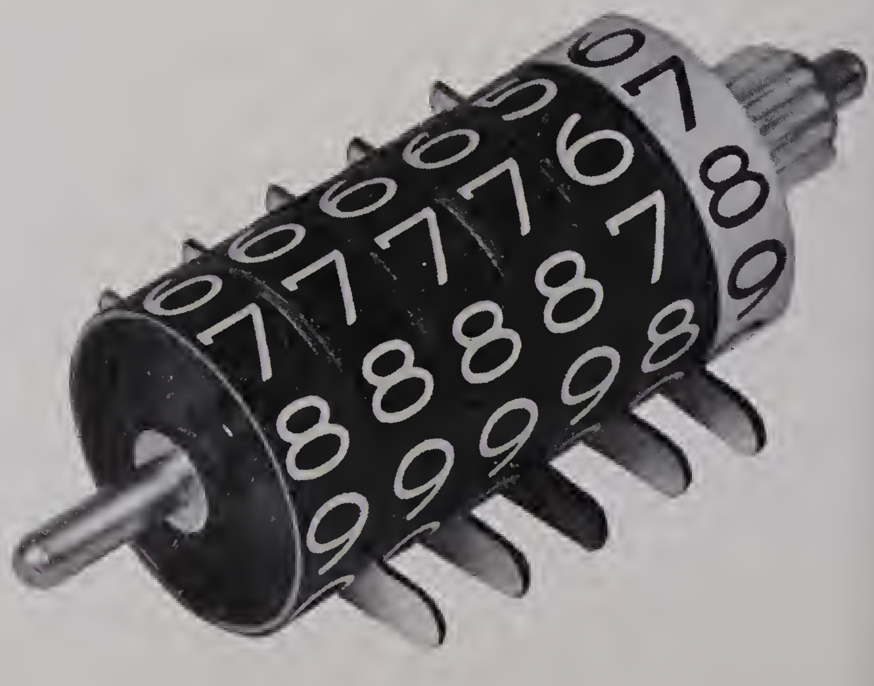
a. 345_{X}; VII **b.** 20_{X}; II **c.** 500_{X}; V **d.** 701_{X}; VI

4. Extend the binary equivalency table above to 40_{X}, using 1 and 0.

5. **a.** How can you tell the difference between odd and even numbers in the binary system? **b.** Why does $2_{\text{X}} = 10_{\text{II}}$?

6. In any numeral system with base n, why does $n = 10_n$?

CHAPTER 6



Working with Polynomials

In your study of arithmetic, you learned basic combinations, and rules for finding answers using the combinations in appropriate patterns. In studying polynomials, you will learn similar operations and patterns and the reasons for them.

Patterns occur in many situations. For example, the odometer (lower photo), which records an automobile's mileage, and the abacus (upper photo) illustrate the same pattern of "carrying in addition." As a figure on the odometer changes from 9 to 0, the figure to its left moves up one numeral. On an abacus, when all ten beads on one wire are counted and moved to the zero position, one bead on the wire to the left is counted.

ADDITION AND SUBTRACTION OF POLYNOMIALS

See T.M. pg. 22 (1).

6–1 Adding Polynomials

Each of the terms 7, x, $5y^2$, and $-2xy$ is called a *monomial* (mo-**no**-me-ell). A **monomial** is a term which is either a numeral (7), a variable (x), or a product of a numeral and one or more variables.

A monomial or the indicated sum of monomials is called a **polynomial** (pol′e-**no**-me-ell). For example, $5y^2 + 7$ is a polynomial of two terms, called a **binomial** (by-**no**-me-ell), and $a^2 - 2ab + b^2$ is a polynomial of three terms, called a **trinomial** (try-**no**-me-ell).

The **degree of a monomial in a variable** is the number of times that the variable occurs as a factor in the monomial. For example, $3x^2y^5z$ is of degree 2 in x, 5 in y, and 1 in z. The **degree of a monomial** is the sum of the degrees in each of its variables. The degree of $3x^2y^5z$ is 8, because $2 + 5 + 1 = 8$. If a monomial other than 0 contains no variables, its degree is zero. The monomial 0 has no degree.

The **degree of a polynomial** is the same as the greatest of the degrees of its terms. The degrees of the terms of $15x^4 - 7x^3y^2 + 2xy$ are, in order, 4, 5, and 2. Thus, the degree of this polynomial is 5.

To add two polynomials, you use the commutative, associative, and distributive properties to combine similar terms. For example:

$$\begin{aligned}(5x^2y + 15) + (7x^2y - 3x + 2) &= (5 + 7)x^2y - 3x + (15 + 2)\\ &= 12x^2y - 3x + 17\end{aligned}$$

Pupils have already done simpler exercises of this type.

To add two polynomials, combine their similar terms.

You may also do the addition vertically:

$$\begin{array}{rcrcr} 5x^2y & & & + & 15 \\ 7x^2y & - & 3x & + & 2 \\ \hline 12x^2y & - & 3x & + & 17 \end{array}$$

Why can you align the similar terms beneath one anot
See T.M. pg. 23 (2).

You can check the addition of polynomials by assigning a numerical value to each variable and evaluating the expressions. For example,

Check: Let $x = 2$ and $y = 3$.

$$\begin{array}{rcrcrcrcrcrcrcl} 5x^2y & & & + & 15 & \rightarrow & 5\cdot4\cdot3 & & & + & 15 & = & 75 \\ 7x^2y & - & 3x & + & 2 & \rightarrow & 7\cdot4\cdot3 & - & (3\cdot2) & + & 2 & = & \underline{80} \\ \hline 12x^2y & - & 3x & + & 17 & \rightarrow & 12\cdot4\cdot3 & - & (3\cdot2) & + & 17 & \overset{?}{=} & 155 \\ & & & & & & & & & & 155 & = & 155\ \checkmark \end{array}$$

A successful check suggests but does not prove correct computation.

Though you choose small numbers for checking, why should you not select 1 or 0?

In working with polynomials, arrange the terms in order of either decreasing degree or increasing degree in a particular variable.

In order of decreasing degree in a: $a^3 + 6a^2 + 12a + 8$

In order of increasing degree in b: $9 - 6b^2 + b^4$

In order of decreasing degree in r: $32r^5 + 7r^4d - 2r^2d^2 - 18d^3$

For every positive integer n, $1^n = 1$ and $0^n = 0$. $\therefore$ $1^2 = 1^3$ and $0^3 = 0$! Also for every n, $0\cdot n = 0$.

ORAL EXERCISES

Give each of the following indicated sums. What is the degree of each sum?

1. $\begin{array}{r} 2x + 3 \\ \underline{3x - 1} \end{array}$

$5x + 2;\ 1$

2. $\begin{array}{r} 6a - 7 \\ \underline{4a + 3} \end{array}$

$10a - 4;\ 1$

3. $\begin{array}{r} a^2 + 1 \\ \underline{a^2 - 1} \end{array}$

$2a^2;\ 2$

4. $\begin{array}{r} x - y^2 \\ \underline{x + y^2} \end{array}$

$2x;\ 1$

5. $\begin{array}{r} x^2 - y^2 + z^2 \\ \underline{2x^2 - 3y^2 - 2z^2} \end{array}$

$3x^2 - 4y^2 - z^2;\ 2$

6. $\begin{array}{r} 2n^2 - 5k + 6 \\ \underline{n^2 - k - 8} \end{array}$

$3n^2 - 6k - 2;\ 2$

7. $\begin{array}{r} a^2 + b - c \\ \underline{a^2 \qquad - c} \end{array}$ $2a^2 + b - 2$

8. $\begin{array}{r} 3b^3 + 4d \\ \underline{2c^2 + 3d} \end{array}$ $3b^3 + 2c^2 +$

9. $\begin{array}{r} -x^3 - y^2 + 8 \\ x^3 + y^2 - 6 \\ \underline{-x^3 + y^2 + 2} \end{array}$

$-x^3 + y^2 + 4;\ 3$

10. $(-2a + 3) + (-2a - 3)$ $-4a;\ 1$

11. $(-5x - 6y) + (-5x + 6y)$ $-10x;\ 1$

12. $(2x^3 + 9) + (y^2 + 1)$ $2x^3+y^2+10;\ 3$

13. $(3x^3 - 5y^2 + 8) + (-x - 10)$ $3x^3-5y^2-x-2;\ 3$

14. $(3x^2 + b) + (3x^2 - b)$ $6x^2;\ 2$

15. $(5x - 4y^2) + (7x - 3y^2)$ $12x-7y^2;\ 2$

16. $(x^2 - 2x) + (-3x^3 + 6)$ $-3x^3+x^2-2x+6;\ 3$

17. $(y^2 + 4y) + (-y - 4)$ $y^2+3y-4;\ 2$

18. $(4a^4 - 10a^2) + (-10a^2 + 25)$ $4a^4-20a^2+25;\ 4$

19. $(b^4 - 6b^2) + (5b^2 - 30)$ $b^4-b^2-30;\ 4$

20. $(x^2 - 3x + 7) + (2x^2 + 5x - 9)$ $3x^2+2x-2;\ 2$

21. $(a^2 + 5a - 9) + (3a^2 - 10a - 10)$ $4a^2-5a-19;\ 2$

22. $(a^2 - 2ab^2 + b^2) + (a^2 + 2ab^2 + b^2)$ $2a^2+2b^2;\ 2$

23. $(-2z^3 + 5z) + (z^4 - 5z^2 + 4)$ $z^4-2z^3-5z^2+5z+4;\ 4$

Read each of the following polynomials, giving the terms in order of decreasing degree in a variable.

SAMPLE. $3r + 2r^2 + 4 + r^3$ *What you say:* $r^3 + 2r^2 + 3r + 4$

24. $4x^2 + 1 + 4x$ $4x^2+4x+1$

25. $49y^2 + 1 + 14y$ $49y^2+14y+1$

26. $49 + n^2 - 14n$ $n^2-14n+49$

27. $64 + x^2 - 16x$ $x^2-16x+64$

28. $s^3 + 3s + 3s^2 + 1$ s^3+3s^2+3s+1

29. $2r^3 + 6r^2 + 2 + 6r$ $2r^3+6r^2+6r+2$

30. $2a^3 - 2 - 6a^2 + 6a$ $2a^3-6a^2+6a-2$

31. $b^3 + 3b - 1 - 3b^2$ b^3-3b^2+3b-1

32. $x^2 + y^2 - 2xy$ $x^2-2xy+y^2$

33. $4x^2 + y^2 + 4xy$ $4x^2+4xy+y^2$

34. $x^3 + y^3 + 3x^2y + 3xy^2$ $x^3+3x^2y+3xy^2+y^3$

35. $m^3 - n^3 + 3mn^2 - 3m^2n$ $m^3-3m^2n+3mn^2-n^3$

WRITTEN EXERCISES

Find each of the sums indicated.

A

1.
$$\begin{array}{r} -2.5n - 1.6h \\ -\ .5n - 2.4h \\ \hline \end{array}$$

2.
$$\begin{array}{r} -4.1r + 2.1s \\ -2.9r - 4.2s \\ \hline \end{array}$$

3.
$$\begin{array}{r} x^3 - y^3 \\ \frac{1}{2}x^3 + \frac{1}{3}y^3 \\ \hline \end{array}$$

4.
$$\begin{array}{r} m^2 + n^2 \\ \frac{1}{4}m^2 - \frac{1}{8}n^2 \\ \hline \end{array}$$

5.
$$\begin{array}{r} -\frac{1}{3}m + \frac{1}{3}n + \frac{1}{6} \\ m - n - 1 \\ \hline \end{array}$$

6.
$$\begin{array}{r} -\frac{1}{2}x - \frac{1}{2}y - \frac{1}{4} \\ x - y + 1 \\ \hline \end{array}$$

7.
$$\begin{array}{r} \frac{3}{4}x - y \\ \frac{1}{4}x - y \\ \hline \end{array}$$

8.
$$\begin{array}{r} -z + \frac{1}{2}y \\ z - \frac{1}{2}y \\ \hline \end{array}$$

9.
$$\begin{array}{r} 4yz - 6 \\ -7yz + 16 \\ \hline \end{array}$$

10.
$$\begin{array}{r} 10ab - 13 \\ -23ab + 17 \\ \hline \end{array}$$

11.
$$\begin{array}{r} x + 5 \\ 4x - 7 \\ 5x - 8 \\ \hline \end{array}$$

12.
$$\begin{array}{r} z - 7 \\ 3z - 5 \\ 9z + 15 \\ \hline \end{array}$$

13.
$$\begin{array}{r} 6\ xy - .9 \\ -\ .3xy + 8 \\ -5\ xy + 1.7 \\ \hline \end{array}$$

14.
$$\begin{array}{r} -\ .5rs + 4 \\ 2\ rs + .3 \\ -\ .6rs + 9 \\ \hline \end{array}$$

15.
$$\begin{array}{rrr} a & +\ b & +\ .6 \\ -a & -\ b & +\ .7 \\ \hline \end{array}$$

16.
$$\begin{array}{rrr} 3n & +\ .5t & -\ .7 \\ 6n & -\ .8t & +\ .9 \\ \hline \end{array}$$

17.
$$\begin{array}{rrr} 9h^2 & -\ 1.3dh & +\ 2d^2 \\ 3h^2 & -\ 3hd & -\ 1.6d^2 \\ \hline \end{array}$$

18.
$$\begin{array}{rrr} -x^2 & -\ yx & -\ 2.3y^2 \\ x^2 & -\ xy & +\ 2.2y^2 \\ \hline \end{array}$$

19.
$$\begin{array}{rrrr} 8x^3 & -\ 40x^2 & +\ 50x & \\ & -\ 20x^2 & +\ 100x & -\ 125 \\ \hline \end{array}$$

20.
$$\begin{array}{rrrr} 2y^3 & -\ 6y^2 & +\ 2y & \\ & -\ y^2 & +\ 3y & -\ 1 \\ \hline \end{array}$$

Check each of the following additions. When values are given for the variables, use them. When an addition is incorrect, do it correctly.

21.
$$\begin{array}{rr} 5x & -\ 6 \\ 2x & +\ 1 \\ \hline 7x & -\ 5 \end{array}$$
Let $x = 3$.

22.
$$\begin{array}{rr} 3y & +\ 7 \\ 2y & -\ 80 \\ \hline 5y & -\ 87 \end{array}$$
Let $y = 2$.

23.
$$\begin{array}{rr} 3r & -\ 8 \\ -2r & +\ 1 \\ -r & +\ 3 \\ \hline 4r & -\ 4 \end{array}$$
Let $r = 2$.

24.
$$\begin{array}{rrr} 5x & -\ y & +\ 2 \\ -x & -\ y & -\ 3 \\ 7x & +\ 2y & -\ 8 \\ \hline 11x & & -\ 3 \end{array}$$
Let $x = 5$, $y = 2$.

25.
$$\begin{array}{rrr} x^2 & +\ x & +\ 6 \\ 3x^2 & -\ x & -\ 8 \\ \hline 4x^2 & & -\ 2 \end{array}$$
Let $x = 2$.

26.
$$\begin{array}{rrr} a^2 & -\ a & +\ 5 \\ 2a^2 & +\ 2a & -\ 5 \\ \hline 3a^2 & +\ a & \end{array}$$
Let $a = 3$.

27.
$$\begin{array}{rrr} 5x & -\ y & +\ 7 \\ x & +\ y & +\ 9 \\ & -\ y & +\ 3 \\ \hline 6x & -\ y & +\ 19 \end{array}$$

28.
$$\begin{array}{rrr} 2x & +\ 7 & -\ y \\ 3x & -\ 8 & -\ y \\ x & & +\ y \\ \hline 6x & -\ 1 & -\ y \end{array}$$

29.
$$\begin{array}{rrr} x^2 & +\ 5x & -\ 1 \\ 3x^2 & -\ x & -\ 8 \\ \hline 4x^2 & +\ 4x & -\ 9 \end{array}$$

B

30. $(10x^3 + 5x^2 - 3x - 11) + (8 + 3x - x^2 + 2x^3)$

31. $(5 + 4y - 3y^2 - y^3) + (y^3 + 8y^2 - 14y - 3)$

32. $(2t^4 - t^2 + t - 1) + (t^5 - t^3 + 3 + 2t)$

33. $(7 - s^3 + 5s - s^2) + (s^2 + 5s + 8 - s^4)$

6–2 Subtracting Polynomials

Remind pupils of the property of the opposite of a sum, page 121. No really new idea is here. See 6-2 T.M. pg. 23.

To subtract polynomials, use the same procedure as with binomials: add the opposite of each term of the subtrahend to the minuend. For example,

$$(3a + 4) - (a - 2) = 3a + 4 + (-a) + 2 = 2a + 6$$

EXAMPLE 1. $(15x^2 - 3x + 9) - (2x^2 + x - 3)$

Solution 1:
$$\begin{aligned} & (15x^2 - 3x + 9) - (2x^2 + x - 3) \\ &= 15x^2 - 3x + 9 - 2x^2 - x + 3 \\ &= 13x^2 - 4x + 12 \end{aligned}$$

Solution 2:

$$\begin{array}{r} 15x^2 - 3x + 9 \\ \underline{2x^2 + x - 3} \\ 13x^2 - 4x + 12 \end{array}$$

To subtract one polynomial from another, add to the minuend the opposite of each term of the subtrahend.

Symbols of inclusion may appear in an equation to indicate addition or subtraction of a polynomial.

EXAMPLE 2. Solve for x:

$$x^2 + 4x + (x - 16 + 2x) = 5x - (2 + 5x - x^2)$$

Solution: $x^2 + 4x + (x - 16 + 2x) = 5x - 2 - 5x + x^2$

Simplify parentheses → $x^2 + 4x + 3x - 16 = -2 + 0 + x^2$

Add like terms → $x^2 + 7x - 16 = x^2 - 2$

Subtract x^2 from each member → $7x - 16 = -2$

Add 16 to each member → $7x = 14$

Divide each member by 7 → $x = 2$

Check:

$$2^2 + 4 \cdot 2 + (2 - 16 + 2 \cdot 2) \stackrel{?}{=} 5 \cdot 2 - (2 + 5 \cdot 2 - 2^2)$$

$$4 + 8 + (2 - 16 + 4) \stackrel{?}{=} 10 - (2 + 10 - 4)$$

$$12 + (-10) \stackrel{?}{=} 10 - (8)$$

$$2 = 2 \checkmark$$

∴ The solution set is $\{2\}$, Answer.

Note: In the solution use was made of the fact that adding the same polynomial to each member of an open sentence produces an equivalent sentence.

ls should be able to explain why each step of the simplification produces an equivalent
ıtion, as in Chapter 5.

ORAL EXERCISES

Subtract the lower polynomial from the one above it.

1. $\begin{array}{r} 2a + b \\ \underline{a - b} \end{array}$ a+2b

2. $\begin{array}{r} 7a + 8b \\ \underline{-2a + b} \end{array}$ 9a+7b

3. $\begin{array}{r} 6x - 3y \\ \underline{-3x + 2y} \end{array}$ 9x-5y

4. $\begin{array}{r} 3x + y \\ \underline{x - y} \end{array}$ 2x+2y

5. $\begin{array}{r} 3a + 2b - 4 \\ \underline{-3a + 4b \phantom{{}-4}} \end{array}$ 6a-2b-4

6. $\begin{array}{r} 2x - y + 3 \\ \underline{- 2y + 5} \end{array}$ 2x+y-2

7. $\begin{array}{r} 3.7b + 4a \\ \underline{-1.5b + 3a} \end{array}$ 5.2b+a

8. $\begin{array}{r} 5.4r^2 - 8rd \phantom{{}-5} \\ \underline{3.1r^2 + 5rd - 5} \end{array}$ 2.3r²-13rd+5

Perform the following subtractions.

Avoid referring to Exercises 9-18 as "practice in removing parentheses preceded by a minus sign." Encourage pupils to think in terms of number properties, not formal procedures.

SAMPLE. $(5z + 6) - (2z - 8)$

What you say: $5z + 6 - 2z + 8 = 3z + 14$

9. $(5a - 6) - (3a - 2)$ 2a-4

10. $(6b + 6) - (b - 1)$ 5b+7

11. $(x + y) - (x + y)$ 0

12. $(x - y) - (x + y)$ -2y

13. $(a - b) - (2a - b)$ -a

14. $(x + y) - (x - y)$ 2y

15. $(5a - 6b + c) - (3a + b - c)$ 2a-7b+2

16. $(3x - 2y - 4) - (2x + 4y - 2)$ x-6y-

17. $(22n - k) - (n - k)$ 21n

18. $(7x - y) - (x - y)$ 6x

WRITTEN EXERCISES

Subtract. Check by evaluation, using $x = 3$, $y = 2$, $a = 5$, $b = 4$.

Ⓐ

1. $\begin{array}{r} 21 - 14x \\ 12 - 6x \\ \hline \end{array}$

2. $\begin{array}{r} 3x + 2y \\ x - y \\ \hline \end{array}$

3. $\begin{array}{r} 2x^2 + 2x + 7 \\ x^2 + 3x - 8 \\ \hline \end{array}$

4. $\begin{array}{r} 3b^2 + b - 1 \\ 2b^2 + 6b + 5 \\ \hline \end{array}$

5. $\begin{array}{r} .4x^2 \qquad - .7 \\ .5x^2 - 3x - .8 \\ \hline \end{array}$

6. $\begin{array}{r} 2 - 3a + 2a^2 \\ 5 - 6a + a^2 \\ \hline \end{array}$

7. $\begin{array}{r} x^2 \quad - 1 \\ x + 2 \\ \hline \end{array}$

8. $\begin{array}{r} a^2 + a \\ a - 1 \\ \hline \end{array}$

9. $\begin{array}{r} .3x^2 + 7x - 1.8 \\ x^2 - 9x - .2 \\ \hline \end{array}$

Solve each of the equations.

10. $2y + (5 - y) = 18$

11. $5w - (2w + 3) = 12$

12. $13z - (z + 21) = 39$

13. $3x - (5 - 2x) = 35$

14. $7x + (8x + 54) = 9$

15. $9y + (7y + 67) = 3$

16. $5x + (2 - 7x) = 30$

17. $y + (9 - 8y) = 65$

18. $9n - (3n - 6) = 42$

19. $6z - (4z - 8) = 24$

20. $(3x + 4) - (x + 8) = 18$

21. $(5y + 7) - (y + 2) = 17$

22. $7n - (n^2 - n - 9) = 17 - n^2$

23. $19x - (1 - 2x - x^2) = 22 + x^2$

24. $(2x + 4) - (x - 7) = (4 - 3x) + (3x + 1)$

25. $(3y + 6) - (2y - 2) = (17 - 2y) + (2y - 13)$

26. $10 - (y + 6) + (3y - 2) = -5y + 9 - (-5y + 3)$

27. $12 - (z + 8) - (5z - 7) = -6z + 16 - (-6z + 7)$

28. $x - (.5x + 1) - (.3x + 2) = .5x - (.5x - 3)$

29. $.5n + (n - 1) - (.2n + 10) = .2n - (.2n - 28)$

Ⓑ

30. $2a - [5a - (6a + 2)] = 10 - a$

31. $7x - [x - (2x + 8)] = 3x - 2$

Assign at least one of the Exercises 22, 23, 32, and 33 to prepare for problems in whi
equations containing the square of a variable occur. These equations reduce to linear
equations, of course.

32. $20x - [2x - (x + 2) + x^2 - 6] = 28 - x - x^2$
33. $5y - [y - (2y + 8) + 3 - y^2] = y^2 - y - 5$
34. $-[-5t - (4 - t) + 7] - (t - 2) = 4t - 1$
35. $-[-(7 - 2s) - 8 + 3s] - (3s + 1) = -3s - 1$

Ⓒ **36.** $2a + \{5a - [3 - (a + 2)] + 6\} = -(2a + 5)$
37. $5n + \{n - [2 - (2 + 3n)] + n\} = -(2n - 6)$
38. $1.2x - \{10x + [2 - .6x - (3 + \overline{x + .2})] + 13\} = -1$
39. $1.4y - \{9y + [3y - .3 - (1.2y + \overline{y - 3})] + 1.7\} = -3$

Note the bar used as a symbol of inclusion.

MULTIPLICATION OF POLYNOMIALS

6–3 The Product of Powers

You will recall that b^4 (read b exponent four or b fourth) stands for $b \cdot b \cdot b \cdot b$. Therefore,

$$b^4 \cdot b^2 = \overbrace{\underbrace{(b \cdot b \cdot b \cdot b)}_{4\text{ factors}} \cdot \underbrace{(b \cdot b)}_{2\text{ factors}}}^{6\text{ factors}} = b^6 = b^{4+2}$$

In multiplying these two powers of the same base, you could have found the exponent by retaining the base and adding the exponents of the factors. In general, for positive integral exponents m and n,

$$b^m \cdot b^n = \overbrace{\underbrace{(b \cdot b \cdot \ldots b)}_{m\text{ factors}}\underbrace{(b \cdot b \cdot \ldots b)}_{n\text{ factors}}}^{m + n\text{ factors}} = b^{m+n}$$

The associative property is used here. See T.M. pg. 23 (3).

This result is usually stated as the ***rule of exponents for multiplication:***

For all positive integers m and n: $b^m \cdot b^n = b^{m+n}$

To multiply monomials, you may use this rule of exponents together with the commutative and associative properties to determine the numerical factor and the variable factors of the product:

$$(2rz^3)(-3r^6z^3) = (2 \cdot -3)(r^1 \cdot r^6)(z^3 \cdot z^3) = -6 \cdot r^{1+6} \cdot z^{3+3} = -6r^7z^6$$

Note that this rule of exponents applies only when the bases of the powers are the *same*. You cannot use it, for example, to simplify the product of x^8y^9 because the bases of the powers x^8 and y^9 are *different*.

WRITTEN EXERCISES

Watch for the common error, $b^m \cdot b^n = b^{mn}$.

Find each of the following products. See T.M. pg. 23 (4).

A
1. $(-3rs)(5rs^2)r^4$
2. $(-6xy)(8x^2y)y^3$
3. $(-3a)(-3a)(3a)$
4. $(-4a)(3ab)(-5b)$
5. $(-5x)(-6y)(-z)$
6. $(6n)(-6n)(-6n)$
7. $(6x)(-3xy)(-4y)$
8. $(-7a)(-b)(-2c)$
9. $\frac{1}{2}gt(\frac{1}{2}gt)$
10. $(8rs^2t)(-6r^2s)$
11. $\frac{1}{4}tx(\frac{1}{4}tx)$
12. $rs(rs)(-r)$

13. $ab(-b)(ab)$
14. $-rt(r^2t)(-rt)$
15. $-3x(4x^3y^2)(-y^4x^2)$
16. $-5a(2a^2b^5)(-ba^6)$
17. $(.2^2r^3s^7)(.4^2rs^8)$
18. $(.3^2zv^5)(9^2z^3v^4)$
19. $(-c^2d^4f^3)(-dc^7f^3)$
20. $(-w^2x^5y^4)(-x^7y^2w^3)$
21. $(-a)^2(a^3b)(-b)^3$
22. $(-k)^3(k^4h^2)(-h)^2$
23. $(-2m)^3(dm)(.3d)^2$
24. $(.4b)^2(cb)(-2c)^3$

Simplify by finding each indicated product and then combining similar terms.

B
25. $(-7a^2)(5b)(-2b) + (4ab)(-2ab)$
26. $(5p^2)(-3q)(-qp^2) - (7qp^3)(7pq)$
27. $(-3rst)(4r^2st) + (-6s^2)(2rt^2)(-r^2)$
28. $(9v^2wz^3)(-2vw^2z) + (-3z^2)(-6v^3w^2)(wz^2)$
29. $-(2^2c^5d^3)(5cd^7)(-5e) + (3^2d^6c^3e)(-d^2c^2)(cd^2)$
30. $(6^2m^3n^2p)(-2m^4p^4n^5) - (-p^3m^2n^3)(8^2m^5n^4p^2)$

C
31. $x^3(2x + 1) + 3x^2(x^2 + x)$
32. $3r^2(r^2 - r) - r^3(2r - 1)$
33. $10^m(10^n + 10^r)$
34. $a^m(a^n + b^m) + b^m(a^m + b^n)$

6–4 The Power of a Product

Consider $2b^3$ and $(2b)^3$; they are not equal unless $b = 0$.

$$2b^3 = 2 \cdot b \cdot b \cdot b$$
$$(2b)^3 = 2b \cdot 2b \cdot 2b = 2^3 \cdot b^3 = 8b^3$$

In general, for every positive integral exponent m, you can see that:

$$(ab)^m \quad \overbrace{(ab)(ab)\ldots(ab)}^{(ab)\text{ is a factor } m \text{ times}}$$
$$\therefore (ab)^m \quad \underbrace{(a \cdot a \cdot \ldots a)}_{m \text{ factors}}\underbrace{(b \cdot b \cdot \ldots b)}_{m \text{ factors}} = a^m b^m$$

Emphasize the use of commutative and associative properties.

These results may be summarized as the ***rule of exponents for a power of a product:***

For every positive integer m: $(ab)^m = a^m b^m$

See note on distributivity, T.M. pg. 24 (5).

For example:

$$(-3x)^4 = (-3)^4 x^4 = 81x^4, \text{ and } (4rs)^2 = 4^2 r^2 s^2 = 16r^2 s^2$$

Of course, the product whose power you are to find may itself be a power:

$$(b^2)^3 = b^2 \cdot b^2 \cdot b^2 = b^{2+2+2} = b^6 = b^{2\cdot 3}$$

In general,

$$(b^m)^n = \overbrace{(b^m)(b^m)\ldots(b^m)}^{b^m \text{ is a factor } n \text{ times}}$$

$$\therefore (b^m)^n = \overbrace{b^{m+m+\ldots+m}}^{n \text{ terms}} = b^{mn}$$

This result gives you the ***rule of exponents for a power of a power:***

For all positive integers m and n: $(b^m)^n = b^{mn}$

You use both of these rules in the following illustration:

$$(-7w^7v^3)^3 = (-7)^3(w^7)^3(v^3)^3 = -343w^{21}v^9$$

ORAL EXERCISES

Give each of the following indicated powers.

1. $(2xy)^3$ $8x^3y^3$
2. $(-3ab)^3$ $-27a^3b^3$
3. $(-2rst)^2$ $4r^2s^2t^2$
4. $(3klm)^2$ $9k^2l^2m^2$
5. $(2^2)^3$ $2^6 = 64$
6. $(2^3)^2$ $2^6 = 64$
7. $(b^3)^3$ b^9
8. $(c^4)^4$ c^{16}
9. $(-.2a^2)^3$ $-.008a^6$
10. $(-.3b^3)^2$ $.09b^6$
11. $(-5x^4)^2$ $25x^8$
12. $(-4y^5)^2$ $16y^{10}$
13. $(-a^2b)^3$ $-a^6b^3$
14. $(-cd^2)^4$ c^4d^8
15. $(6m^7p)^1$ $6m^7p$
16. $(7k^4n^3)^1$ $7k^4n^3$

What is the square of each of the following?

17. $-.3xy^2$ $.09x^2y^4$
18. $.2a^2b$ $.04a^4b^2$
19. $7v^3w^2$ $49v^6w^4$
20. $-5r^2s^3$ $25r^4s^6$

If a student's answer is incorrect, have him explain the meaning of the term, rather than merely quote a rule. See T.M. pg. 24 (6).

WRITTEN EXERCISES

Find each of the following indicated products.

A

1. $(-2m^2)^3$
2. $(-3s^2)^3$
3. $-3x(4xy)^2$
4. $-6n(5mn)^2$
5. $(-b)(-3b)^3$
6. $(-r)(-6r)^3$
7. $(8m)(2mn)^3$
8. $(3y)(5xy)^3$
9. $(5a)(-2a^2b)^2$
10. $(-7b)(3b^2c)^2$
11. $(-5x)^2(.2y^2x)^3$
12. $(-.3a)^2(-b^2a)^3$
13. $(.2b^2d)^4(-3db)^2$
14. $(-.5k^2p)^3(-2kp)^3$
15. $(3st^2v)^2(-3s^2tv^3)^2$

Simplify each of the following expressions.

B

16. $(3xy)^2 + (-5x)(-4xy)(-y)$
17. $(-2c)(-3cd)^3 + (3c)^2(cd)^2(-6d)$
18. $(-a)^2a + (2a^2)(-2a^2)^3 - (2a^2)^4$
19. $(2b)^2(-25b^4) + (-4b)(-b)^2 + (-10b^3)^2$
20. $(-zt)^3(.2z^2t) + (t^2z)^2(.3z^3) + (t^2z)(z^2t)^2$
21. $(-mk)^5(.3mk^2) + (m^2k)^3(.2k^2)^2 + (-m^3k)^2(-k)^4k$

C

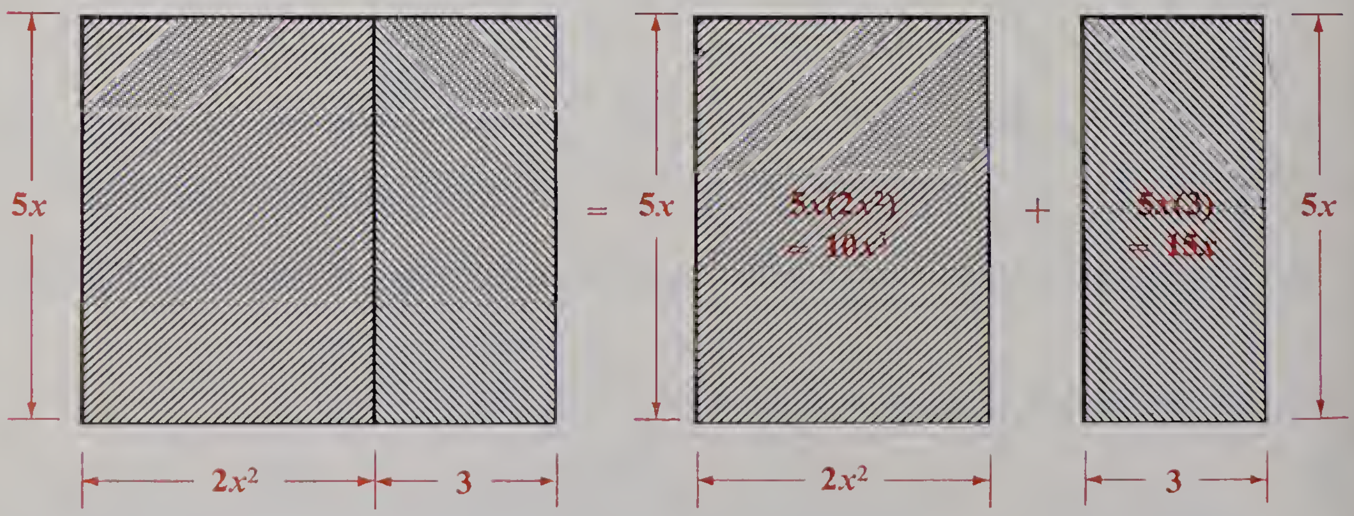
22. $2x^3y(x^2y + 3y^3) - (xy)^2(x^3 - xy^2)$
23. $(-5c^2d)(1 - c^4d^3) + (-2cd)(-3c + c^5d^3)$
24. $3k^2m(m^2k - 1) - .2km[k + (km)^2]$
25. $(ab)^3(a^2 - ab^2) + (-.3b^2a)[(-ba)^3 + a^4b]$

6–5 Multiplying a Polynomial by a Monomial

The distributive property of numbers, together with the rules of exponents for multiplication, enables you to multiply any polynomial by a monomial:

$$5x(2x^2 + 3) = 5x \cdot 2x^2 + 5x \cdot 3 = 10x^3 + 15x.$$

This result is illustrated in the figure.



The largest rectangle is made up of the other two. Its area is equal to the sum of the areas of the other two rectangles.

You may multiply either horizontally or vertically, as shown here.

$$-7y^2(3y^3 - 2y^2 + 1) = -21y^5 + 14y^4 - 7y^2 \qquad \begin{array}{r} 3y^3 - 2y^2 + 1 \\ -7y^2 \\ \hline -21y^5 + 14y^4 - 7y^2 \end{array}$$

To multiply a polynomial by a monomial, use the distributive property: multiply each term of the polynomial by the monomial, and then add the products.

ORAL EXERCISES

Read the product of each indicated multiplication.

1. $7(a - 2)$ $7a-14$
2. $15(2x - 1)$ $30x-15$
3. $-.5(x + y)$ $-.5x-.5y$
4. $-.3(2a + b)$ $-.6a-.3b$
5. $8(2x - 3y)$ $16x-24y$
6. $9(5m - 2n)$ $45m-18n$
7. $a^2(a - .3)$ $a^3-.3a^2$
8. $x^3(x - .7)$ $x^4-.7x^3$
9. $-3x^4(-2 + x)$ $6x^4-3x^5$
10. $-5a^5(-3 + a)$ $15a^5-5a^6$
11. $-1(8 + 2b)$ $-8-2b$
12. $-1(3x + 5)$ $-3x-5$
13. $-3a(-2a - 4)$ $6a^2+12a$
14. $-5r(-7 - 3r)$ $35r+15r^2$
15. $(2n - 4)(-1)$ $-2n+4$
16. $(5r - s)(-1)$ $-5r+s$
17. $\begin{array}{l} 2x - 3y \\ .6x \\ \hline \end{array}$ $1.2x^2-1.8xy$
18. $\begin{array}{l} 4x - 7y \\ .5y \\ \hline \end{array}$ $2xy-3.5y^2$
19. $\begin{array}{l} -3x^3 + 4x - 1 \\ -3x \\ \hline \end{array}$ $9x^4-12x^2+3x$
20. $\begin{array}{l} -y^2 + 2y + 5 \\ -4y \\ \hline \end{array}$ $4y^3-8y^2-20y$

WRITTEN EXERCISES

Find each of the following products.

A

1. $-5(x^2 - 3x + 7)$
2. $-6(a^2 - 2a + 1)$
3. $a(a^2 - 2ab + b^2)$
4. $x(x^2 - 2xy + y^2)$
5. $b^2(-a + b + c)$
6. $r^2(-r - s + t)$
7. $-x^2(5x^2 - x + 2)$
8. $-y^2(7 - 3y - y^3)$
9. $(2 - 3v - 4v^2)(-2v^4)$
10. $(x^2 - 6x + 9)(-3x^3)$
11. $3x^2y(5 - 2xy^4 + 3x^2y^3 - y^5)$
12. $5rs^2(3 + 4r^5s - r^3s^4 - 7rs^7)$
13. $-2a^3b^2(a^4b - a^3b^2 + 2a^2b^4 - b^5)$
14. $-3c^2d^4(c^5d + 4c^4d^3 - c^2d^4 - 2d^5)$

Solve each of the following equations.

15. $3x + 5(x - 3) = 9$

16. $8n + 6(-n - 2) = 16$

17. $-6y - .5(10 - y) = 17$

18. $-7x + .8(7 - x) = 18$

19. $(5x + 3) - 3(x - 2) = 1$

20. $(8x + 7) - 5(x - 8) = 44$

21. $12(x + 1) + 3(4x - 2) = 18$

22. $7(8 - x) + 5(2 + 2x) = 96$

Ⓑ **23.** $2x - 2(x + 21) = 3x - 3(7 - x)$

24. $5n - (n - 8) = 14n - 7(2n + 8)$

25. $.5(3y - 7) + 13 = .2(3y - 12) - (6y - 5)$

26. $.8x - (-x + 5) = -(x - 3) - .4(2 + 2x)$

27. $5 - 3[2n - 2(5 - 3n)] = 4(2 - 3n)$

PROBLEMS

Give each answer as a polynomial or as a directed number.

Ⓐ **1.** A man travels for 2 hours at $(45 + x)$ miles per hour, then at $(200 + x)$ miles per hour for 2 hours. Represent the total distance traveled.

2. An airplane flew at x miles per hour for 2 hours, then at $(x + 25)$ miles per hour for another 4 hours. The entire distance was 1150 miles. Find its rate during the four-hour period.

3. A plane traveled at 190 miles per hour for h hours. After refueling, it traveled at 210 miles per hour for $(h + 1)$ hours. The entire trip was 1210 miles. Find the number of hours in the second part of the trip.

4. Mr. Macy drove 303 miles one day. During the first x hours, he averaged 34 miles per hour, but during the next $(x + 3)$ hours, his average speed was 47 miles per hour. How long did the trip take?

5. Harry mowed a rectangular lawn l feet long and 50 feet wide. Bob mowed one $(l + 20)$ feet long and 60 feet wide. Bob mowed 1500 square feet more than Harry. Find the length of each lawn.

6. Mr. Bernard's and Mr. Cayne's houses stand on rectangular lots of equal depth d. Mr. Bernard's lot is 30 feet wider than Mr. Cayne's. Find the depth of the lots if their areas differ by 3600 square feet.

7. Charles earned d dollars on each of two days and $(d - 1)$ dollars on each of the next three days. What were his daily earnings if he earned \$19.50 during the five days?

8. Mr. Reynolds traveled for 2 hours on a jet airplane and for one-half hour more on a piston-driven aircraft. The average speed of the jet was 4.6 miles per minute more than that of the piston-driven craft. If Mr. Reynolds traveled 1467 miles on this trip, find each plane's speed.

9. A factory produced x dresses daily for six days and $(110 + x)$ dresses daily for the next five days. During the eleven days it completed 6600 dresses. How many dresses were produced each day during this time?

10. A machine which caps 1500 bottles an hour operated 2 hours longer than one which caps 1200 bottles an hour. If a total of 16,500 bottles were capped, how long did each machine operate?

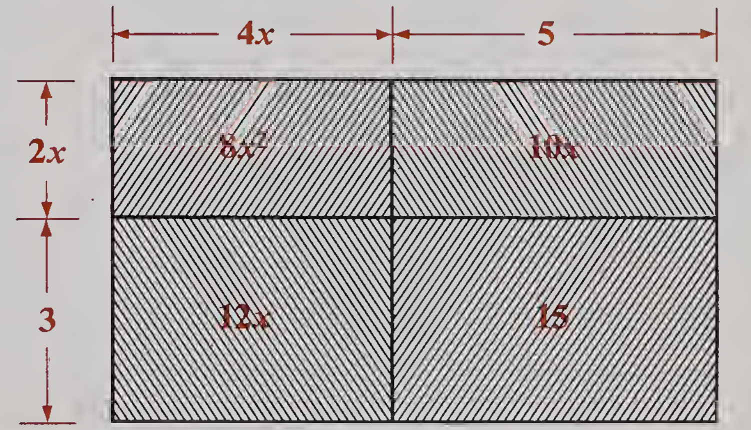
6–6 Multiplying Two Polynomials

To simplify the product $(2x + 3)(4x + 5)$, first treat $(4x + 5)$ as a number to be multiplied by the binomial $2x + 3$. By the distributive property, you find

$$\begin{aligned}(2x + 3)(4x + 5) &= \underbrace{(2x)(4x + 5)}_{} + \underbrace{3(4x + 5)}_{}\\ &= (2x)(4x) + (2x)(5) + 3(4x) + 3(5)\\ &= 8x^2 + \underbrace{10x + 12x}_{} + 15\\ &= 8x^2 + 22x + 15\end{aligned}$$

mphasize the *twofold*
;e of the distributive
operty.

The figure illustrates this product.



To multiply one polynomial by another, use the distributive property: multiply each term of one polynomial by each term of the other, and then add the products.

Usually it is convenient to set up the multiplication of polynomials in vertical form, and to work from left to right, thus:

$$\begin{array}{rl} & 4x + 5\\ & \underline{2x + 3}\\ \text{This is } 2x(4x + 5) \rightarrow & 8x^2 + 10x\\ \text{This is } 3(4x + 5) \rightarrow & \underline{\qquad\quad 12x + 15}\\ \text{This is } (2x + 3)(4x + 5) \rightarrow & 8x^2 + 22x + 15\end{array}$$

To check the accuracy of your multiplication, evaluate the factors and their product, using any numbers except 0 and 1.

S 6-6 T.M. pg. 24 for the multiplication of integers in arithmetic as an illustration of multiplying
P ynomials.

WRITTEN EXERCISES

Give each of the following products in its simplest form.

Ⓐ

1. $(x + 7)(x + 2)$
2. $(a + 8)(a + 7)$
3. $(y - 9)(y + 5)$
4. $(n - 9)(n + 4)$
5. $(x - 7)(x - 6)$
6. $(y - 3)(y - 8)$
7. $(2x + 3)(5x + 1)$
8. $(7n + 4)(8n + 9)$
9. $(7x - 3)(5x + 2)$
10. $(12w + 5)(5w - 2)$
11. $(3y - 4)(4y - 6)$
12. $(6x - 5)(7x - 8)$
13. $(a + b)(a + b)$
14. $(x - y)(x - y)$
15. $(x - y)(x + y)$
16. $(c + d)(c - d)$
17. $(m - 5)(m - 5)$
18. $(x + 9)(x + 9)$
19. $(m + 3n)(m + 3n)$
20. $(m - 4n)(m - 4n)$
21. $(r + 3s)(r - 3s)$
22. $(3a - b)(3a + b)$
23. $(2a - .3b)(a + 5b)$
24. $(2a - .5b)(a + 3b)$

Ⓑ

25. $(m - .3)(m^2 - 3m + .4)$
26. $(2n - .3)(n^2 - .6n - 1)$
27. $(a^2 + ab + b^2)(a - b)$
28. $(a^2 - ab + b^2)(a + b)$
29. $(a^2 - 8)(a^2 + 1)$
30. $(y^2 + 40)(y^2 - 5)$
31. $(xy - .7)(xy - .8)$
32. $(ab - .1)(ab + .3)$

Ⓒ

33. $(3x^2 - x + 2)(x^2 + 2x + 1)$
34. $(5y^2 - y + 3)(y^2 - 3y - 2)$
35. $(2a^2 + a - .1)(a^2 + .3a + 4)$
36. $(.4b^2 - b + 3)(b^2 + 2b + .5)$
37. $(2a - 4b - 3c)(4a + 4b + 5c)$
38. $(3m - m^2)(mp + p - p^2)$
39. $(x - 1)(x^4 + x^3 + x^2 + x + 1)$
40. $(y + 1)(y^4 - y^3 + y^2 - y + 1)$

By using the properties of numbers, prove that each of the following statements is true for all values of the variables.

41. $(a + b)(c + d) = ac + bc + ad + bd$
42. $(a - b)(c + d) = ac - bc + ad - bd$
43. $(x + y)(x - y) = x^2 - y^2$
44. $(x + y)(x + y) = x^2 + 2xy + y^2$

6–7 Problems about Areas

The solution of problems concerned with areas requires the ability to multiply polynomials. In analyzing such problems, you will find sketches especially useful.

EXAMPLE. **To support the weight of a building, an engineer determines that its foundation must have an area of 496 square feet bearing on the soil. The building is to be rectangular, and its foundation, 2 feet thick. If the inside length of the foundation is two times the inside width, find these dimensions.**

Solution:

1 **Let w = number of feet in width of inner rectangle.**

Then $2w$ = number of feet in length of inner rectangle.

2

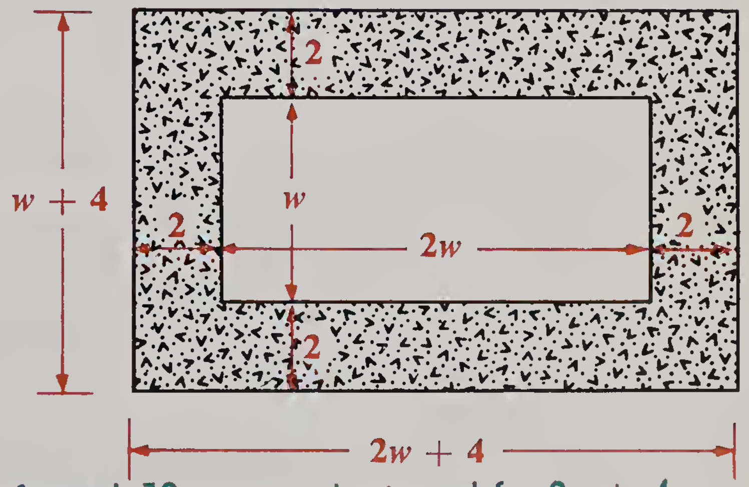
$$\underbrace{\text{Total Area}} \; \overset{}{=} \; \underbrace{\text{Area of Concrete}} \; + \; \underbrace{\text{Inner Area}}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$(2w + 4)(w + 4) = 496 + 2w \cdot w$$

3 **Solve the equation:**

$$\begin{aligned} (2w + 4)(w + 4) &= 496 + 2w^2 \\ 2w^2 + 8w + 4w + 16 &= 496 + 2w^2 \\ 12w + 16 &= 496 \\ 12w &= 480 \\ w &= 40 \\ 2w &= 80 \end{aligned}$$



Ask students to tell how the equation was transformed; 12w was substituted for 8w + 4w and 2w² was subtracted from each member.

4 **(1) Is the length twice the width? $80 = 2(40)$ ✓**

(2) Is the area of the top of the concrete 496 square feet?

$$\underbrace{\text{Area of Concrete}} = \underbrace{\text{Total Area}} - \underbrace{\text{Inner Area}}$$

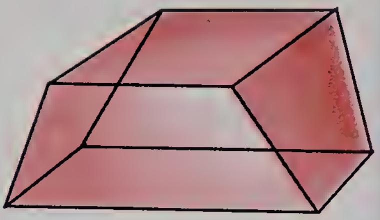
$$\begin{aligned} 496 &\overset{?}{=} 84 \cdot 44 - 80 \cdot 40 \\ 496 &\overset{?}{=} 3696 - 3200 \\ 496 &= 496 \checkmark \end{aligned}$$

Inside width of foundation: 40 feet
Inside length of foundation: 80 feet } **Answer.**

PROBLEMS

Make a sketch for each problem, and solve.

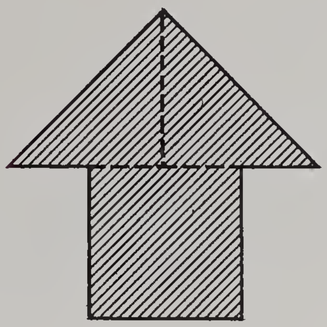
A



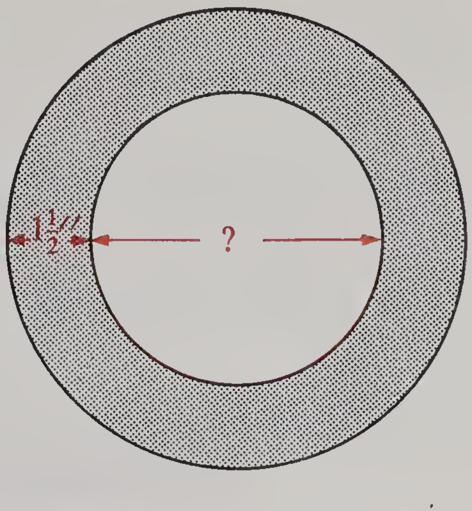
1. One end of a casting is a square. The other is a rectangle whose length is 3 inches greater than a side of the square end and whose width is 2 inches less than a side of the square end. The areas of the two ends are equal. Find a side of the square end.
2. Two hi-fi loud-speaker enclosures are a square and a rectangle, respectively. The length of the rectangle is 1 inch greater than a side of the square, and its width is 3 inches less than a side of the square. The area of the rectangle is 45 square inches less than the area of the square. How long is a side of the square?
3. The distance from home plate to first base in softball is 30 feet less than the corresponding distance in baseball. The square bounded by the bases of a softball diamond is 4500 square feet less in area than that of a baseball diamond. Find the distance from one base to the next in softball.
4. The distance from first to second base in indoor baseball is 33 feet less than the corresponding distance in softball. The square bounded by the bases of an indoor baseball diamond is 2871 square feet less in area than that bounded by the bases of a softball diamond. Find the distance from one base to the other in indoor baseball, using only the information given in this problem.
5. The Taylors plan to replace their croquet court by a badminton court. The badminton court will require 920 square feet less than the croquet court and will be 16 feet shorter and 10 feet narrower. If the croquet court is twice as long as it is wide, find the dimensions of the badminton court.
6. The standard basketball court for college women is twice as long as it is wide. The standard basketball court for college men is 4 feet longer, 5 feet wider, and 650 square feet greater in area than the women's court. Find the dimensions of the men's court.
7. A field is used for both lacrosse and field hockey. The length of the lacrosse field is 75 feet greater than its width. The length and width of the hockey field are 60 feet and 105 feet less than the length and width of the lacrosse field, and its area is 38,700 square feet less. Find the dimensions of the lacrosse field.
8. A rectangular field is planted in wheat. It is 500 feet longer than it is wide. In planting, a strip 10 feet wide is left unplanted on two adjacent sides. The area left unplanted is 44,900 square feet. What are the dimensions of the field?

B **9.** The cross section of a rivet is in the shape shown. The base of the triangle is 3 inches longer, and its height is 2 inches shorter, than a side of the square, and its area is one-half that of the square. Find a side of the square.

Ex. 9



Ex. 11



10. Two crates are in the form of cubes. An edge of one crate is 1 foot longer than that of the other, and it requires 54 square feet more lumber to build. Find an edge of the smaller crate. (Disregard the thickness of the wood.)

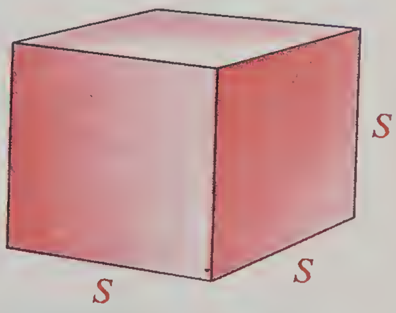
11. A cross section (see diagram) of a circular water main is to have a wall $1\frac{1}{2}$ inches thick and an area of $478\frac{1}{2}$ square inches. Find its inner diameter. (Use $\pi = \frac{22}{7}$.)

12. A circular concrete conduit has walls 2 inches thick, and the cross-sectional area of the concrete is 88 square inches. Find the inner diameter of the conduit.

13. The height of a box is 2 feet more than its length, and its width is 2 feet less than its length. The volume of the box is 80 cubic feet less than that of a cube having the same length. What are the dimensions of the box?

14. An open box is to be made from a piece of cardboard 8 inches long and 6 inches wide by cutting out a square from each corner and turning up the sides. Find the dimensions of the box, if the area of its base is 8 square inches less than the total area cut from the cardboard.

6–8 Powers of Polynomials



The area A of a square is found by using the equation $A = s^2$, and the volume V of a cube, by using $V = s^3$. The figure shows a cube in which each edge s is given as $(2x - 3)$. The base of this cube, then, has an area of $(2x - 3)^2$, read *the square of* $2x - 3$. The volume is $(2x - 3)^3$, read *the cube of* $2x - 3$.

In each of these expressions, the exponent shows how many times the polynomial is to be used as a factor. To find the product of these factors, expressed as a sum of monomials, is to **expand the expression.** To expand the expression $(2x - 3)^2$, find the product $(2x - 3)(2x - 3)$. To expand $(2x - 3)^3$, find the product $(2x - 3)[(2x - 3)(2x - 3)]$.

$$\begin{array}{rrrr}
2x & -\ 3 & & \\
2x & -\ 3 & & \\
\hline
4x^2 & -\ 6x & & \\
 & -\ 6x & +\ 9 & \\
\hline
4x^2 & -\ 12x & +\ 9 & = (2x - 3)^2
\end{array}
\qquad
\begin{array}{rrrrr}
4x^2 & -\ 12x & +\ 9 & & \\
2x & -\ 3 & & & \\
\hline
8x^3 & -\ 24x^2 & +\ 18x & & \\
 & -\ 12x^2 & +\ 36x & -\ 27 & \\
\hline
8x^3 & -\ 36x^2 & +\ 54x & -\ 27 & = (2x - 3)^3
\end{array}$$

WRITTEN EXERCISES

Expand each of the following expressions. Check your work by evaluating the factors and their products in each case.

Watch for the common error, $(a+b)^2 = a^2 + b^2$. See 6-8 T.M. pg. 24.

Ⓐ
1. $(a + b)^2$
2. $(a - b)^2$
3. $(a - b)^3$
4. $(a + b)^3$
5. $(3x + 2)^2$
6. $(5x + 2)^2$
7. $(7y + 3z)^2$
8. $(2m + 6n)^2$
9. $(r - 8s)^2$
10. $(c - 4d)^2$
11. $(3x - 1)^3$
12. $(2y - 1)^3$

Ⓑ
13. $(a + b + c)^2$
14. $(a - b + c)^2$
15. $(.2x^2 + x - .3)^2$
16. $(.3y^2 - y + .2)^2$
17. $(x - y)^4$
18. $(x + y)^4$
19. $(2x^2 + x - 2)^3$
20. $(3y^2 - y + 2)^3$
21. $(.8x^2 - .9x)^3$

PROBLEMS

Solve and check each of the following problems.

Ⓐ
1. A side of the square opening of a ventilating duct is 3 inches longer than a side of another such opening. Their areas differ by 81 square inches. Find the length of a side of each opening.
2. Two square pieces of asbestos are used to insulate a corner stove. One piece is 10 inches shorter than the other, and its area is 1100 square inches less. What are the dimensions of the smaller piece?
3. The squares of two consecutive integers differ by 103. Find the integers.
4. The difference between the squares of two consecutive integers is 95. Find the integers.
5. A square picture is in a frame 1 inch wide. If the area of the frame is 24 square inches, find the width of the picture.

6. An inlaid checkerboard has a mahogany border 2 inches wide. The area of the border is 176 square inches. Find the length of the board.

7. The product of two consecutive integers exceeds the square of the smaller integer by 13. Find the integers.

8. The product of two consecutive integers is 17 less than the square of the larger integer. Find the integers.

B 9. An adjoining living room and bedroom have the same width, but the living room is 5 feet longer. If the bedroom is square and contains 75 square feet less area than the living room, what are the dimensions of the living room?

10. A rectangle is 5 feet longer than it is wide. A square with the same length has an area 105 square feet larger. Find the dimensions of the rectangle.

11. A circular skating rink is to be 28 feet larger in diameter than an adjoining circular restaurant. The area of the rink will be 3080 square feet larger than that of the restaurant. Find (a) the diameter of the rink and (b) the cost of linoleum for the restaurant floor at 35 cents per square foot. (Use $\pi = \frac{22}{7}$.)

12. A round iron rod 60 inches long has a diameter 1 inch less than that of another round rod of the same length, and its volume is 45π cubic inches less. Find (a) both diameters, (b) both volumes, and (c) the weights of both rods. (Iron weighs $\frac{5}{18}$ pounds per cubic inch.)

DIVISION OF POLYNOMIALS

6–9 The Quotient of Powers

Recall (p. 139) that dividing by a number is the same as multiplying by the reciprocal of the number. Therefore, another way to write the quotient $\frac{xy}{cd}$ is $xy \cdot \frac{1}{cd}$. You also know (p. 139) that $\frac{1}{cd}$ equals $\frac{1}{c} \cdot \frac{1}{d}$. Putting these facts together, you have

$$\begin{aligned} \frac{xy}{cd} &= xy\left(\frac{1}{c} \cdot \frac{1}{d}\right) && \textbf{Substitution principle} \\ &= \left(x \cdot \frac{1}{c}\right)\left(y \cdot \frac{1}{d}\right) && \textbf{Commutative and associative properties} \\ &= \frac{x}{c} \cdot \frac{y}{d} && \textbf{Meaning of division} \end{aligned}$$

Stress the statement and proof of this property. It is fundamental in fraction work.

This gives the ***property of quotients:***

$$\frac{xy}{cd} = \frac{x}{c} \cdot \frac{y}{d}$$

for every replacement of x and y by numbers, and c and d by nonzero numbers.

In particular, if $c = 1$, you have: $\frac{xy}{d} = x \cdot \frac{y}{d}$;

and if $x = 1$, you have: $\frac{y}{cd} = \frac{1}{c} \cdot \frac{y}{d}$.

Emphasize these special cases. Pupils often make mistakes like: $\frac{15\cdot 35}{35} = 3\cdot 7$; $\frac{xy}{d} = \frac{x}{d} \cdot \frac{y}{d}$.

For example,

$$\frac{25 \cdot 28}{5 \cdot 14} = \frac{25}{5} \cdot \frac{28}{14} = 5 \cdot 2 = 10$$

$$\frac{7 \cdot 15}{5} = 7 \cdot \frac{15}{5} = 7 \cdot 3 = 21$$

$$\frac{9}{2 \cdot 3} = \frac{1}{2} \cdot \frac{9}{3} = \frac{1}{2} \cdot 3 = \frac{3}{2}$$

This property of quotients is helpful in simplifying quotients of powers; consider $\frac{b^7}{b^4}$:

$$\frac{b^7}{b^4} = \frac{b^3 \cdot b^4}{b^4} = b^3 \cdot \frac{b^4}{b^4} = b^3 \cdot 1$$

Recall: $\frac{b^4}{b^4} = 1$, because 1 is the root of $x \cdot b^4 = b^4$

$$\therefore \frac{b^7}{b^4} = b^3 = b^{7-4}$$

Notice that you could have found the exponent in the quotient by retaining the base and subtracting the exponent in the denominator from the exponent in the numerator.

Similarly, whenever $m > n$, you may write

$$\frac{b^m}{b^n} = \frac{b^{m-n} \cdot b^n}{b^n} = b^{m-n} \cdot \frac{b^n}{b^n} = b^{m-n}$$

On the other hand,

$$\frac{b^2}{b^9} = \frac{b^2}{b^7 \cdot b^2} = \frac{1}{b^7} \cdot \frac{b^2}{b^2} = \frac{1}{b^7} \cdot 1$$

$$\therefore \frac{b^2}{b^9} = \frac{1}{b^7} = \frac{1}{b^{9-2}}$$

Whenever $m < n$,

$$\frac{b^m}{b^n} = \frac{b^m}{b^{n-m} \cdot b^m} = \frac{1}{b^{n-m}} \cdot \frac{b^m}{b^m} = \frac{1}{b^{n-m}}$$

These are two important ***rules of exponents for division:*** See 6-9 T.M. pg. 24.

For positive integers m and n and nonzero b,

$$\text{if } m > n, \text{ then } \frac{b^m}{b^n} = b^{m-n}$$

$$\text{if } m < n, \text{ then } \frac{b^m}{b^n} = \frac{1}{b^{n-m}}$$

When dividing monomials, you use these rules together with the property of quotients obtained above:

$$\frac{15x^3y^4}{-5xy^3} = \frac{15}{-5} \cdot \frac{x^3}{x} \cdot \frac{y^4}{y^3} = -3 \cdot x^{3-1} \cdot y^{4-3} = -3x^2y$$

$$\frac{-7a^{10}c^3}{-42a^8c^5} = \frac{-7}{-42} \cdot \frac{a^{10}}{a^8} \cdot \frac{c^3}{c^5} = \frac{1}{6} \cdot a^{10-8} \cdot \frac{1}{c^{5-3}} = \frac{a^2}{6c^2}$$

ORAL EXERCISES

Give each of the following quotients.

1. $\frac{y^{10}}{y^4}$ y^6
2. $\frac{z^8}{z^2}$ z^6
3. $\frac{u^7}{u^{14}}$ $\frac{1}{u^7}$
4. $\frac{v^8}{v^{16}}$ $\frac{1}{v^8}$
5. $\frac{p^7}{-p^2}$ $-p^5$
6. $\frac{q^6}{-q^3}$ $-q^3$
7. $\frac{30x^2}{2x}$ $15x$
8. $\frac{20r^2}{5r}$ $4r$
9. $\frac{14x^3y}{-2x}$ $-7x^2y$
10. $\frac{21ab^2}{-3b}$ $-7ab$
11. $\frac{-22m^2n}{-2m}$ $11mn$
12. $\frac{-121bc^2}{-11c}$ $11bc$
13. $\frac{-45x^2}{5x}$ $-9x$
14. $\frac{-170z^2}{5z}$ $-34z$
15. $\frac{-5a^3b}{-15a^2b^4}$ $\frac{a}{3b^3}$
16. $\frac{-7r^3s}{-21rs^5}$ $\frac{r^2}{3s^4}$
17. $\frac{(-1.5xyz)^2}{-1.5xy}$ $-1.5xyz^2$
18. $\frac{(-.6abc)^2}{-.6ac}$ $-.6ab^2c$
19. $\frac{36a^2b}{-12a^4b^2}$ $\frac{-3}{a^2b}$
20. $\frac{20xy^2}{-10x^2y^3}$ $\frac{-2}{xy}$
21. $\frac{-5m^2n}{10m^2n^2}$ $\frac{-1}{2n}$
22. $\frac{7c^2d^3}{-28c^3d^3}$ $\frac{-1}{4c}$
23. $\frac{-.3r^2s}{-3rs^2}$ $\frac{r}{10s}$
24. $\frac{-11ab^2}{-.11a^2b}$ $\frac{100b}{a}$

WRITTEN EXERCISES

Find each of the following quotients.

Ⓐ

1. $\dfrac{64a^2b}{8a}$
2. $\dfrac{48b^2c}{6c^3}$
3. $\dfrac{-40d^2e}{-5de^2}$
4. $\dfrac{-96x^2y}{-8xy^2}$
5. $\dfrac{45y^2z}{-5y^4}$
6. $\dfrac{54r^4s}{-9r^2}$
7. $\dfrac{-84s^2t}{7st^2}$
8. $\dfrac{-132m^2n}{11mn^2}$
9. $\dfrac{-8a^2b^7}{-56a^{12}b^3}$
10. $\dfrac{-13g^2h^3}{-39g^3h}$
11. $\dfrac{.3x^2y^7}{-5x^8y}$
12. $\dfrac{.6rs^7}{-8r^2s^4}$
13. $\dfrac{-5a^2b}{21(abc)^2}$
14. $\dfrac{-4m^2n}{(4mnp)^3}$
15. $\dfrac{-9abc}{-6c^2}$
16. $\dfrac{-6xyz}{-9x^2}$
17. $\dfrac{2.7p^5r^2}{(-3)^3pr^4}$
18. $\dfrac{(-2)^4vw^9}{1.6v^2w^7}$
19. $\dfrac{(-4)^3x^3z^5}{8x^{12}z^{10}}$
20. $\dfrac{(-3)^4d^{11}h^4}{9^2d^8h^4}$

Ⓑ

21. $\dfrac{-3a^xb^y}{2ab}$
22. $\dfrac{-2x^ay^b}{6x^by^a}$
23. $\dfrac{32(ab)^m}{8a^mb^n}$
24. $\dfrac{51(a^mb)^n}{-13(ab^n)^m}$

6–10 Zero as an Exponent (Optional)

If you use the rule of exponents to evaluate $\dfrac{b^m}{b^m}$, you obtain two expressions:

$$\frac{b^m}{b^m} = b^m \cdot \frac{1}{b^m} = b^{m-m} \cdot 1 = b^0 \quad \text{and} \quad \frac{b^m}{b^m} = b^m \cdot \frac{1}{b^m} = \frac{1}{b^{m-m}} = \frac{1}{b^0}.$$

These two expressions seem to indicate that b^0 should represent a number that is its own reciprocal. The only number with this property is 1.

From these considerations, you make this definition of b^0 (read *b* exponent zero): $\mathbf{b^0}$ is 1 for every nonzero *b*. No meaning is assigned to the expression 0^0. Emphasize that 5^0, for example, has no meaning until we ma
convenient definition.

ORAL EXERCISES

Simplify each of the following. Assume that 0 is not a member of the replacement set of any of the variables.

1. $(1000)^0$ 1
2. $(-49)^0$ 1
3. $(-\frac{7}{4})^0$ 1
4. $(\frac{1}{1000})^0$ 1
5. $a^5 \div a^0$ a^5
6. $c^6 \div c^0$ c^6
7. $f^0 \div f^9$ $\frac{1}{f^9}$
8. $g^0 \div g^4$ $\frac{1}{g^4}$

9. $(7b)^0$ 1
10. $(-3y)^0$ 1
11. $b^0 \div b^m$ $\frac{1}{b^m}$
12. $b^m \div b^0$ b^m
13. $-5a^0$ -5
14. $-8d^0$ -8
15. $(-7)^0 \cdot 0^2$ 0
16. $(-15)^0 \cdot 0^3$ 0
17. $b^0 \cdot b^m$ b^m
18. $(-a)^0 \cdot a^k$ a^k
19. $[-(c^0)]^5$ -1
20. $(m^0)^3$ 1

WRITTEN EXERCISES

Simplify each of the following.

A

1. $\frac{3x^4}{x} - \frac{4x^2}{x^2} + 7 - 4x(3x^2)$

2. $-\frac{4y^5}{y^3} - 2y(-3y) + \frac{8y^3}{-2y^3} + 4$

3. $\frac{-70de^7}{-7de} + (5e^3)^2 - (.10e^6)^0$

4. $\frac{56rs^8}{-8rs^4} + (-7s^2)^2 + (.7s^2)^0$

5. $\frac{-81x^2y^6z^4}{(3xy^3z^2)^2} + (-3)^0$

6. $(-2)^0 + \frac{(-2a^3b^2z^4)^2}{-16a^6b^4z^8}$

7. $\frac{x(x^2+1)}{x} - (x^2+1)^0$

8. $\frac{y^2(1-y^2)}{y^2} + (y^2)^0$

9. $\frac{x^3(3x-5)z}{x^3z} - (3x-5)^0$

10. $\frac{y^5(4-y)t}{y^5t} + (y-4)^0$

6–11 Dividing a Polynomial by a Monomial

One way to simplify the numerical expression $(93 + 48) \div 3$ is to use the distributive property:

Stress the use of the distributive property to combat the common error, $\frac{42+3}{3} = 42 + 1$.

$$\frac{\mathbf{93 + 48}}{\mathbf{3}} = \frac{1}{3}(93 + 48) = \frac{1}{3}(93) + \frac{1}{3}(48) = 31 + 16 = 47.$$

Similarly, you may simplify the algebraic expression $(ax + ay) \div a$:

$$\begin{aligned}\frac{ax + ay}{a} &= \frac{1}{a}(ax + ay)\\ &= \frac{1}{a}(ax) + \frac{1}{a}(ay)\\ &= \left(\frac{1}{a}\cdot a\right)x + \left(\frac{1}{a}\cdot a\right)y\\ &= 1x + 1y = x + y\end{aligned}$$

The effect of this procedure is that of dividing each term of the polynomial $ax + ay$ by the monomial a.

EXAMPLE 1. $\dfrac{3a^3 + 15a^2 - 9a}{3a}$

Solution:

$$\frac{3a^3 + 15a^2 - 9a}{3a}$$

$$= \frac{3a^3}{3a} + \frac{15a^2}{3a} - \frac{9a}{3a}$$

$$= a^2 + 5a - 3$$

EXAMPLE 2. $\dfrac{ay^2 - y + a}{ay}$

Solution:

$$\frac{ay^2 - y + a}{ay}$$

$$= \frac{ay^2}{ay} - \frac{y}{ay} + \frac{a}{ay}$$

$$= y - \frac{1}{a} + \frac{1}{y}$$

> To divide a polynomial by a monomial, use the distributive property: divide each term of the polynomial by the monomial.

WRITTEN EXERCISES

Find each of the following indicated quotients.

A

1. $\dfrac{12x + 15y}{3}$
2. $\dfrac{5x + 10y}{5}$
3. $\dfrac{2x + 3x}{x}$
4. $\dfrac{6y + 16y}{y}$
5. $\dfrac{5a + b}{a}$
6. $\dfrac{8m - n}{n}$
7. $\dfrac{r^2 - 7r}{r}$
8. $\dfrac{2s^2 + 4s}{s}$
9. $\dfrac{4c^2 + 3c}{3}$
10. $\dfrac{3b^2 + 2b}{2}$
11. $\dfrac{8a^3 - 4a}{2a}$
12. $\dfrac{7c^3 + 14c}{7c}$
13. $\dfrac{12x^3 + 6x}{6x}$
14. $\dfrac{15y^2 - 5y}{5y}$
15. $\dfrac{24n^3 - 12n^2 + 15n}{3n}$
16. $\dfrac{50r^3 + 10r^2 - 35r}{5r}$
17. $\dfrac{8x^3 - 4x^2 - 2x}{-x}$
18. $\dfrac{2n^4 - 3n^3 - 4n^2}{-n}$
19. $\dfrac{-60m^3n^3 + 36m^2n^2 - 6mn}{-6mn}$
20. $\dfrac{12x^3y^3 - 6x^2y^2 + 18xy}{-6xy}$
21. $\dfrac{5x^4 - 15x^3 + 45x^2 - 10x}{5x}$
22. $\dfrac{35a^4 - 28a^3 - 56a^2 - 14a}{7a}$
23. $\dfrac{2.4a^2b^2 + .6ab^2 + 30a^2b}{.3ab}$
24. $\dfrac{3.2st^2 + .8s^2t + 40s^2t^2}{.4st}$

Ⓑ **25.** $\dfrac{32a^5 - 6a^4b + 24a^3b^2}{-8a^2}$

26. $\dfrac{12h^4 - 24h^3k - 6h^2k^2}{-36h^2}$

27. $\dfrac{30x^3y^3 - 45x^2y^2 + 15xy}{75xy^2}$

28. $\dfrac{12r^2t^2 + 18r^2t^3 - 6r^3t^3}{21r^2t^4}$

29. $\dfrac{21m^3n - 28m^2n^2 + 35mn^3}{-35mn^3}$

30. $\dfrac{9r^2s + 18rs^2 - 27s^3}{-27rs^2}$

Ⓒ **31.** If $\dfrac{a + b}{b} = 3.5$, find $\dfrac{a}{b}\cdot$

32. If $\dfrac{n^2 - x^2}{x^2} = -.91$, find the positive value of $\dfrac{n}{x}\cdot$

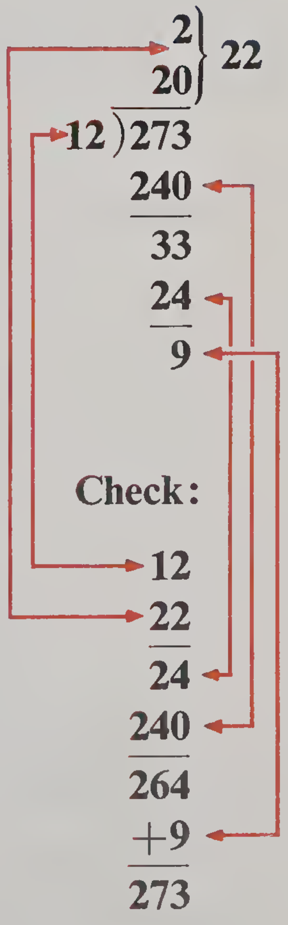
33. Solve the equation: $\dfrac{5a^2 + 2a}{a} = 3a - 8.$

34. Solve the equation: $\dfrac{3x^2 - 7x}{x} = x - 13.$

6–12 Dividing a Polynomial by a Polynomial

See 6-12 T.M. pg. 25.

The adjoining example illustrates a division problem in arithmetic. Can you recognize these points? **1.** The method uses repeated subtraction; first 20(12), then 2(12) is subtracted from 273. **2.** The distributive property helps shorten the number of steps; without it you subtract 12 from 273 twenty-two times. **3.** The check is a transformation of the division equation, 273 − (22)(12) = 9; the check is (22)(12) + 9 = 273. In general form, these statements are:



Dividend − Quotient × Divisor = Remainder
Dividend = Quotient × Divisor + Remainder

Both equations are equivalent to a third:

$$\frac{\textbf{Dividend}}{\textbf{Divisor}} = \boxed{\textbf{Quotient}} + \frac{\textbf{Remainder}}{\textbf{Divisor}}$$

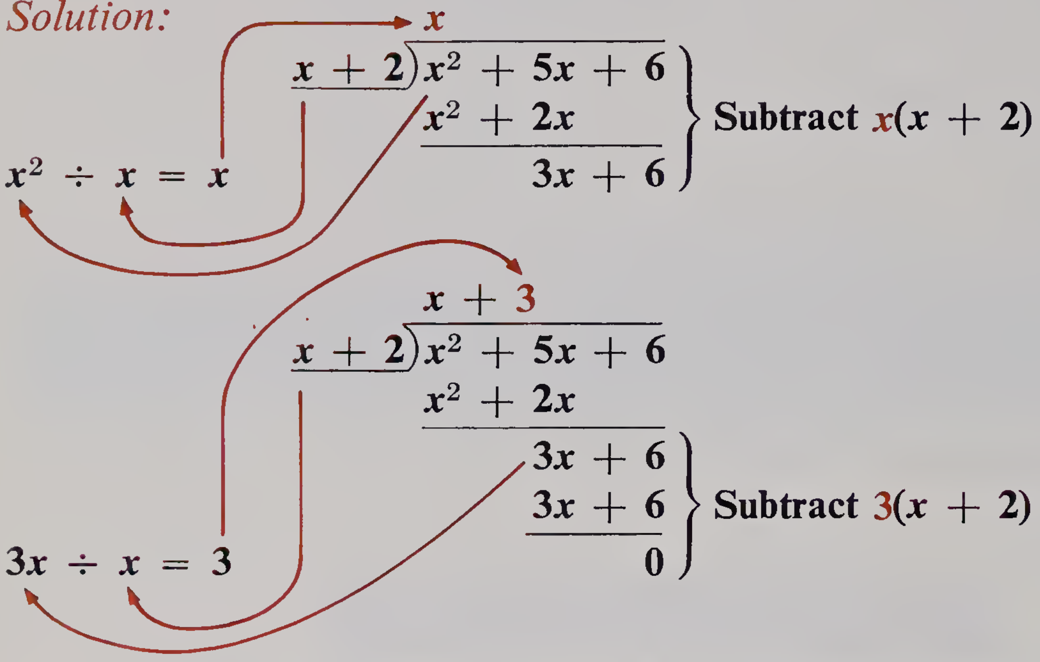
which for the example is $273 \div 12 = 22\frac{9}{12}$. It is this last form which gives the complete quotient $22\frac{3}{4}$.

This is sometimes called the "partial quotient" to distinguish it from the entire right member, which is the "complete quotient."

When dividing polynomials, you follow a similar pattern, after the terms of divisor and dividend have been written in order of decreasing degree in a variable:

EXAMPLE 1. $x + 2\overline{\big)x^2 + 5x + 6}$

Solution:



$$\begin{array}{rl} & x \\ x+2 & \overline{\big)x^2 + 5x + 6} \\ & \underline{x^2 + 2x} \\ & \quad\;\; 3x + 6 \end{array} \left.\vphantom{\begin{array}{l}a\\a\\a\end{array}}\right\} \text{Subtract } x(x+2)$$

$$x^2 \div x = x$$

$$\begin{array}{rl} & x + 3 \\ x+2 & \overline{\big)x^2 + 5x + 6} \\ & \underline{x^2 + 2x} \\ & \quad\;\; 3x + 6 \\ & \quad\;\; \underline{3x + 6} \\ & \qquad\quad\;\; 0 \end{array} \left.\vphantom{\begin{array}{l}a\\a\\a\end{array}}\right\} \text{Subtract } 3(x+2)$$

$$3x \div x = 3$$

Check: $(x + 2)(x + 3) + 0 = x^2 + 5x + 6$

In the following example the steps in the division are shown compactly. Example 2 also shows how to insert missing terms in a dividend, using 0 as a coefficient.

EXAMPLE 2. $m^2 + 2m + 1\overline{\big)m^3 - 7}$

Solution:

$$\begin{array}{rrrrr} & & & m & -\ 2 \\ m^2 + 2m + 1 \big) & \overline{m^3} & \overline{+\ 0m^2} & \overline{+\ 0m} & \overline{-\ 7} \\ & \underline{m^3} & \underline{+\ 2m^2} & \underline{+\ m} & \\ & & -\ 2m^2 & -\ m & -\ 7 \\ & & \underline{-\ 2m^2} & \underline{-\ 4m} & \underline{-\ 2} \\ & & & 3m & -\ 5 \end{array}$$

Check: $(m^2 + 2m + 1)(m - 2) + (3m - 5) = m^3 - 7$

When do you stop dividing?

You stop when the remainder is *zero* or the *degree of the remainder* is less than that of the divisor.

WRITTEN EXERCISES

Divide the polynomials as indicated. Express each answer as a complete quotient, and check.

A

1. $\dfrac{x^2 + 5x + 6}{x + 2}$
2. $\dfrac{x^2 + 3x + 2}{x + 1}$
3. $\dfrac{x^2 + 7x + 12}{x + 3}$
4. $\dfrac{x^2 + 11x + 28}{x + 7}$
5. $\dfrac{y^2 - 13y + 42}{y - 6}$
6. $\dfrac{x^2 - 15x + 56}{x - 7}$
7. $\dfrac{r^2 - 5r - 7}{r + 1}$
8. $\dfrac{n^2 - 7n - 9}{n + 1}$
9. $\dfrac{4 - 8n + 3n^2}{3n - 2}$
10. $\dfrac{64 - 16z + z^2}{z - 8}$
11. $\dfrac{16x^2 - 49}{4x - 7}$
12. $\dfrac{25x^2 - 81}{5x + 9}$
13. $\dfrac{81 + a^2}{a + 9}$
14. $\dfrac{x^2 + 9}{3 + x}$
15. $\dfrac{y^2 - 6yz - 27z^2}{y - 9z}$
16. $\dfrac{w^2 - 11wx - 102x^2}{w - 17x}$
17. $\dfrac{3x^2 - 14xy + 25y^2}{x - 3y}$
18. $\dfrac{5x^2 - 13xy + 16y^2}{x - 2y}$
19. $\dfrac{6y^2 + 11y - 10}{2y + 5}$
20. $\dfrac{10n^2 + 7n - 12}{2n + 3}$
21. $\dfrac{8x^3 - y^3}{2x - y}$
22. $\dfrac{216b^3 + a^3}{a + 6b}$
23. $\dfrac{343y^3 + x^3}{x + 7y}$
24. $\dfrac{2x^2 + 11x - 18}{2x - 3}$
25. $\dfrac{12x^2 + 4x - 18}{2x + 3}$

B

26. $x + 3\overline{)3x^3 + 11x^2 + 11x + 15}$
27. $x - 5\overline{)2x^3 - 7x^2 - 17x + 10}$
28. $2x - 5\overline{)6x^3 + x^2 - 18x - 33}$
29. $2n - 3\overline{)2n^3 - 5n^2 + 21n - 14}$
30. $x^2 - 2x + 3\overline{)x^4 - 4x^3 + 10x^2 - 12x + 9}$
31. $y^2 + 2y - 1\overline{)y^4 + 4y^3 + 2y^2 - 4y + 1}$
32. One factor of $x^3 + 1$ is $x + 1$. Find the other factor.
33. One factor of $y^3 - 1$ is $y - 1$. Find the other factor.
34. Is $2y + 3$ a factor of $6y^3 + 2y^2 + y - 9$? Justify your answer.
35. Is $3z - 2$ a factor of $9z^3 - 4z^2 - 5z + 8$? Justify your answer.
36. If 718 is divided by a certain number, the quotient is 59, and the remainder is 10. Find the number.
37. If 395 is divided by a certain number, the quotient is 28, and the remainder is 3. Find the number.

THE HUMAN
EQUATION

Child of the Moon

Some 3600 years ago, the Pharaoh of Egypt had a prime minister name Joseph, of whom you surely have heard. Joseph was not the only subject of th Pharaoh whose name is known today. A certain Aahmesu, whose name mear "born of the moon," is remembered also. His position in life was much humble than Joseph's; probably he was a scribe. Nowadays he is known as Ahmes, th writer of the *Ahmes Papyrus*.

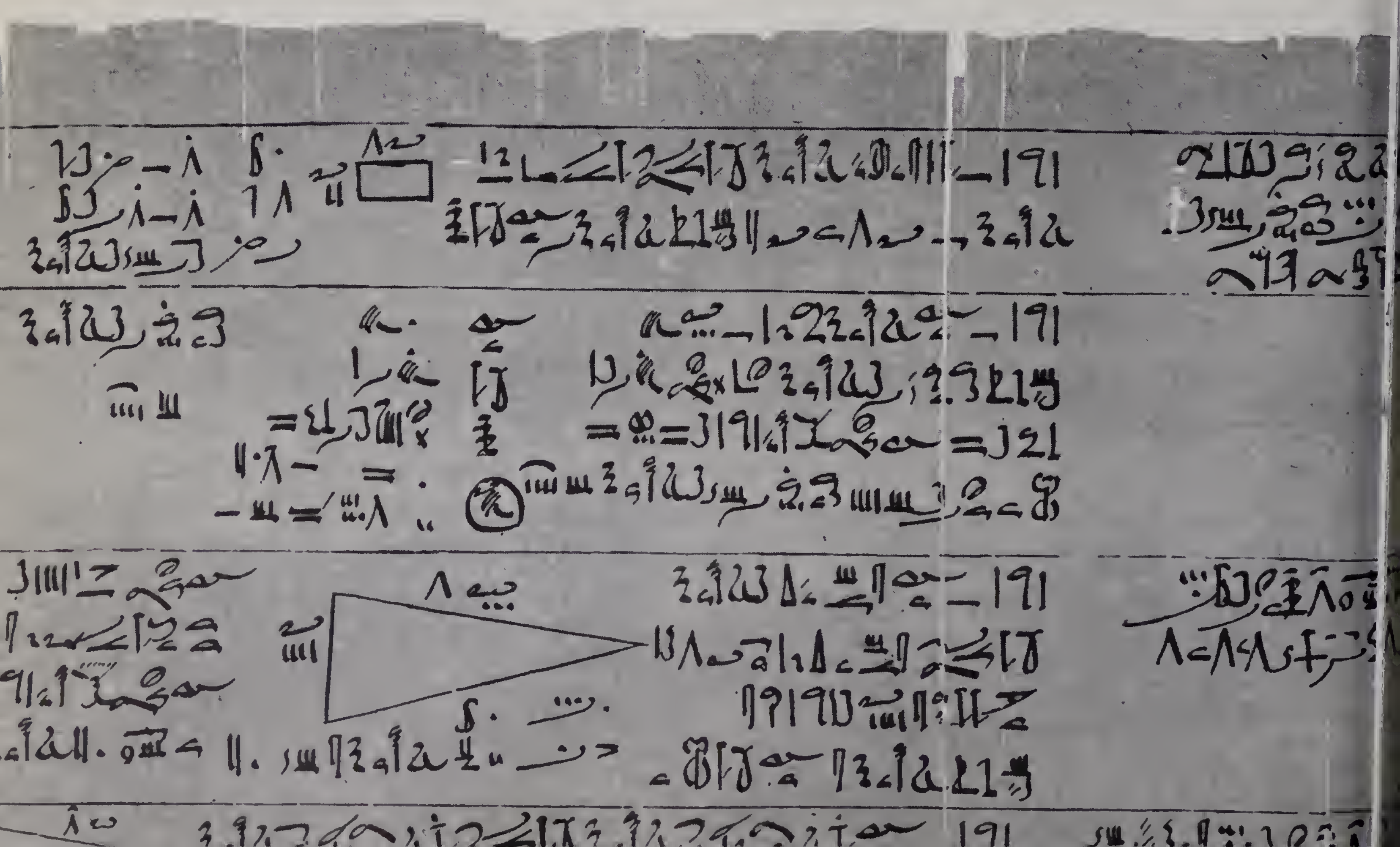
The *Ahmes Papyrus* is an ancient handbook of mathematics. In it there ar eighty algebra problems, each with its solution. Many of these problems were the "find-a-number" type. Here is one of Ahmes' number problems:

> Heap, its two-thirds, its one-half, and its three-sevenths, added together, becomes thirty-three. What is the quantity?

Other problems dealt with everyday affairs, with bread and beer, with feedin livestock and storing grain. Some of these were practical; some were clearl just for fun. Here is one of the latter sort:

> There are seven houses; in each are seven cats. Each cat kills seven mice. Each mouse would have eaten seven ears of spelt [wheat]. Each ear of spelt will produce seven hekats of grain. What is the total of these? [That is, how much grain was saved?]

A portion of the Ahmes Papyrus. *This ancient book is now in the British Museu*



Chapter Summary

Inventory of Structure and Method

1. To **add** polynomials, combine their similar terms. To **subtract** a polynomial from another, add to the minuend the opposite of each term of the subtrahend.
   You can **check** your work with polynomials by substituting a particular value (except 0 and 1) for each variable and evaluating each expression.
2. To **multiply two powers** with the **same base,** use the rule: $b^m \cdot b^n = b^{m+n}$ for all positive integers m and n.
   To **multiply monomials,** multiply the numerical factors and the variable factors. When the base of a power is a product, the rule $(ab)^m = a^m b^m$ applies for every positive integer m. When the base is itself a power, then for all positive integers m and n, $(b^m)^n = b^{mn}$.
3. To **multiply a polynomial by a monomial,** apply the distributive property, multiplying each term of the polynomial by the monomial. To **multiply a polynomial by a polynomial,** use the distributive property to multiply one polynomial by each term of the other. Expand a power of a polynomial by using the polynomial as a factor as many times as shown by the exponent and by performing the indicated multiplications.
4. To **divide** two **powers** with the **same base,** use the rule: for positive integers m and n and nonzero b, if $m > n, \frac{b^m}{b^n} = b^{m-n}$, and if $m < n, \frac{b^m}{b^n} = \frac{1}{b^{n-m}}$.
   To **divide monomials,** divide the numerical factors and the variable factors. To **divide a polynomial by a monomial,** apply the distributive property, dividing each term of the polynomial by the monomial. To **divide a polynomial by a polynomial,** arrange the terms of divisor and dividend in order of decreasing degree in one variable, and then proceed as in arithmetic division. The process stops when the remainder is 0, or when its degree is *less* than that of the divisor. In general,

$$\frac{\textbf{Dividend } (D)}{\textbf{divisor } (d)} = \textbf{Quotient } (Q) + \frac{\textbf{Remainder } (R)}{\textbf{divisor } (d)}$$

5. (Optional) By definition, for any nonzero number b, $b^0 = 1$.

Vocabulary and Spelling

monomial (*p. 197*)
polynomial (*p. 197*)
binomial (*p. 197*)
trinomial (*p. 197*)
degree (monomial) (*p. 197*)
degree (polynomial) (*p. 197*)
expand (an expression) (*p. 214*)

Chapter Test

6–1 Find each of the following sums.

1. $(13x + 61) + (56x - 16)$

2. $$\begin{array}{r} 3a^2 - 5a - 6 \\ \underline{-4a^2 + 6a - 7} \end{array}$$

Check each addition by letting $n = 2$ and $r = 3$.

3. $(5n - 6) + (2n - 80) + (3n + 17) = 10n - 69$

4. $(r^2 + r + 6) + (2r^2 - 2r - 5) = 3r^2 - r + 1$

6–2 In each case, subtract the lower polynomial from the one above it. Check your work by letting $x = 2$, $y = 3$, and $z = 4$.

5. $$\begin{array}{r} 2x - 3y - 6z \\ \underline{-5x + 7y - 2z} \end{array}$$

6. $$\begin{array}{r} x^2 + y^2 - z^2 \\ \underline{-x^2 - y^2 - z^2} \end{array}$$

Solve each of the following equations.

7. $30a + (6 - 2a) = 34$

8. $2n - [n - (2n + 8) - 1] = 0$

6–3 Find each product.

9. $(3mn^3t^2)(-m^2t^3)$

10. $(-r)^3(2rs^2)(3r^2s^4)$

6–4 Simplify each expression.

11. $(-4m^2n^3)^3$

12. $(-3x)^2(2xy^2z^5)^3$

Find each product.

6–5 **13.** $-4x(15x - 22y)$

14. $-t^2(6 + 2t - 3t^3)$

6–6 **15.** $(c + 8)(c - 2)$

16. $(8a - 5y)(2a - 4y)$

6–7 **17.** A square garden is bordered by a path that is 2 feet wide. The area of the path is 184 square feet. How long is the garden?

6–8 **18.** A number is 5 more than another, and the difference of their squares is 135. Find the larger number.

6–9 **19.** Find the quotient. $-48d^4h^3k^2 \div -3d^4hk^4$

6–10 (Optional) Simplify. **20.** **a.** $(-5)^0$ **b.** $3c^0$; $(c \neq 0)$

Find each quotient.

6–11 **21.** $\dfrac{12a^3 - 3a}{3a}$

22. $\dfrac{16bc - 8b^2c^2 - c^3d}{-4bc^3d}$

6–12 **23.** $\dfrac{4n^2 - 29n + 45}{n - 5}$

24. $(10x^3 - 21x^2 + 14x + 12) \div (5x - 3)$

Chapter Review

6–1 Adding Polynomials *Pages 197–200*

1. The terms $-3n$ and $\frac{5m}{2}$ are each ___?___.

2. The polynomial $3x + 2y + 5$ is a ___?___.

3. The degree of $8x + 4x^2y + 5x^3y^2$ is ___?___.

4. When a polynomial of second degree is added to a polynomial of third degree, the sum is a polynomial of the ___?___ degree.

Find each sum, and arrange in order of decreasing degree in n.

5. $(3n^3 + 5 - 2n) + (n^2 - 6n - 8)$

6. $(2m^3n - 3m^2n^2) + (4m^2n^2 - mn^3) + (-mn^3 - 7m^3n)$

7. It is best not to use ___?___ or ___?___ as a replacement for a variable in checking.

Check each addition, letting a = 3, b = 2.

8.
$$\begin{array}{rrr} 3a & +\ b & +\ 4 \\ 2a & -\ 3b & -\ 2 \\ -6a & -\ 2b & -\ 5 \\ \hline a & & -\ 3 \end{array}$$

9.
$$\begin{array}{rrr} 2a^2b & -\ 3ab^2 & +\ b^3 \\ -a^2b & +\ ab^2 & -\ 2b^3 \\ -a^2b & +\ 2ab^2 & +\ b^3 \\ \hline & 0 & \end{array}$$

6–2 Subtracting Polynomials *Pages 200–203*

10. To subtract a polynomial, add the ___?___ of each of its terms.

Do each subtraction, and check, using suitable values for the variables. In a vertical arrangement, subtract the lower polynomial from the one above it.

11. $(3t^2 - 2t + 1) - (t^2 - 2t + 2)$

12.
$$\begin{array}{rrr} 5 & -\ 2c & +\ c^2 \\ 3 & +\ 2c & -\ c^2 \\ \hline \end{array}$$

13.
$$\begin{array}{rrr} 4h^2 & & -\ 1 \\ 5h^2 & -\ 2h & \\ \hline \end{array}$$

14. Subtract $2a + b + 7$ from $a - 2b + 3$.

Solve, and check.

15. $7k - (2k + 3) = 3k - 9$

16. $3 - [2n - (5 + n) + 3] = 2 + (6n + 3)$

6–3 The Product of Powers *Pages 203–204*

17. The product of 2^9 and 2^9 has the base ___?___ with the exponent ___?___.

18. $c^3 \cdot c = ?$

19. $-3a^2b^5(a^3bc^2)(-a^6b^3c^5) = ?$

6-4 The Power of a Product *Pages 204–206*

20. $(2ab)^3 = (?)^3 \cdot (a)^? \cdot (?)^? = ?$

21. In the expression $(b^n)^m$ the base __?__ is to be taken as a factor m times.

22. $(3p^3r^4)^2(-p^2r^3s)^5 = ?$

6-5 Multiplying a Polynomial by a Monomial *Pages 206–209*

23. $2a(5a + 3b) =$ __?__ $(5a) + 2a(?)$

24. $-3yz(3yz^2 - 2y^2z) = ?$

Solve, and check.

25. $-2x + 4(x - 2) = -7$

26. $8a - 5(a - 3) = 18a - 6(3a + 1)$

27. A rectangle has a base represented by $(2n - 5)$ feet and an altitude, by n feet. Express its area as a binomial.

6-6 Multiplying Two Polynomials *Pages 209–210*

28. $(2a - 3)(4a + 5)$ indicates that $(4a + 5)$ is to be multiplied by __?__ and then by __?__. The final product is __?__.

In Exercises 29–31, find the products, and check by evaluation, using $a = 2, x = 3$.

29. $(5x + 2)(3x + 7)$

30.
$$\begin{array}{r} 7a - 3x \\ \underline{3a - x} \end{array}$$

31.
$$\begin{array}{r} a + 9 \\ \underline{2a - 4} \end{array}$$

6-7 Problems about Areas *Pages 211–213*

Items 32–35 refer to this problem: The length of a rectangle is 3 times the width. If the length were made 4 inches smaller and the width, 2 inches longer, the area would be unchanged.

32. If you let s represent the width of the original rectangle, then the length is __?__ and the area is __?__.

33. Write another expression for the area of the new rectangle.

34. Write an equation by which you can solve this problem.

35. The dimensions of the original rectangle are __?__ by __?__ inches.

36. A rectangular enclosure is 30 feet longer than it is wide. It consists of a concrete walk, 5 feet in uniform width, around a pool. If the area of the walk is 1200 square feet, find the dimensions of the pool.

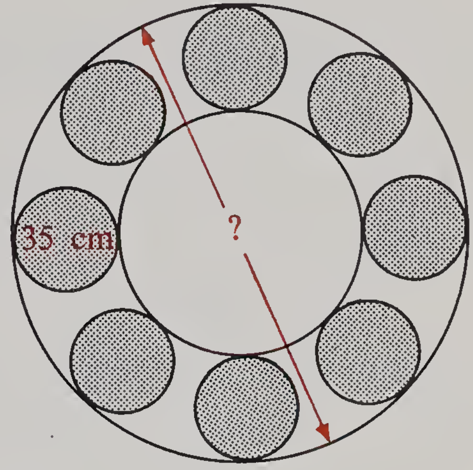
6–8 Powers of Polynomials

Pages 213–215

37. $(a + 1)^2 = ?$

38. $(x^2 - 5x - 1)^3 = ?$

39. A wheel of a roller skate is to be designed so that its cross section is as represented in the diagram. The diameter of each ball bearing is to be .35 centimeters. The difference between the areas of the outer and inner rings is 5.115 square centimeters. Find the outside diameter of the wheel. (Use $\pi = \frac{22}{7}$.)



6–9 The Quotient of Powers

Pages 215–218

40. $\dfrac{-6m^3n^4}{15m^6n} = ?$

41. $(3r^2s^3t^4) \div (6r^2s^4t^6) = ?$

6–10 Zero as an Exponent (Optional)

Pages 218–219

42. $(-\frac{2}{3})^0 = ?$

43. $(a - b)^0 = ?; (a \neq b)$

6–11 Dividing a Polynomial by a Monomial

Pages 219–221

44. $\dfrac{5a - 3}{-3} = ?$

45. $\dfrac{5m^2n + 10mn - 25mn^2}{-50m^3n^3} = ?$

6–12 Dividing a Polynomial by a Polynomial

Pages 221–223

46. Before dividing one polynomial by another, arrange the terms of each polynomial in order of __?__ or __?__ degree in a given variable.

Find each quotient. Check either by multiplication or by evaluation.

47. $\dfrac{6a^2 - 25a + 14}{3a - 2}$

48. $\dfrac{8c^3 - 12c^2 + 6c - 1}{2c - 1}$

49. $\dfrac{2a^3 - 13ab^2 - a^2b - 6b^3}{a + 2b}$

50. $\dfrac{9r^3 + 6r^2s - 11rs^2 + s^3}{3r - 2s}$

Cumulative Review: Chapters 1-6

Complete each of the following statements.

1. $\{12, 3, 9\}$ is a ___?___ of $\{3, 6, 9, 12\}$.

2. ___?___ is the graph of the solution set of the inequality $2 < x \leq 5$.

3. The value of $(2a + b)(2a - b)$, when $a = 3$ and $b = 4$, is ___?___.

4. $\{2, 4, 6, 8, 10 \ldots\}$ is ___?___ under the operations of addition and ___?___.

5. $5r + r = 6r$ by the ___?___ property.

6. $2(x - 1) = 3$ and $2y = 5$ are ___?___ equations, since they have the ___?___ solution set.

7. $53 + (47 + 25) = (53 + 47) + 25$ by the ___?___ property.

8. If the replacement set of x is $\{-2, -1, 0, 1, 2\}$, the solution set of $3x \neq -3$ is ___?___.

Express the following numbers in algebraic symbols.

9. Twice the cube of a number n.

10. The sum of a number n and its square.

11. The difference of a number n and its reciprocal.

12. The sum of the squares of the two numbers a and b.

13. Simplify: $5(m^3 - n^2) - 9m^3 + 5n^2$

Solve each of the following open sentences. Check your answers.

14. $.6y + 51 = 3$

15. $3n - \frac{2}{3}(5n - 6) = 2$

16. $4(3x + 5) - 2x = 3(4x - 5) + 25$

17. $3t - [2t - (3t + 5) - 7] = 0$

Solve each inequality, and graph its solution set. (Optional)

18. $-6 \leq 2x - 3 < 5$

19. $|2r - 3| \geq 5$

20. $\frac{5}{4}m + 6 \geq \frac{m}{2} + 3$

In each case select the correct answer.

21. If $b = -3$, then b^3 is (-9) (27) (-27).

22. An exponent indicates how many times a certain number is to be used as a (*factor*) (*addend*) (*quotient*).

23. In $-ab$ the coefficient of ab is (0) (1) (-1).

24. $2a - b$ subtracted from 0 equals $(b - 2a)$ $(2a + b)$ $(2a - b)$.

25. 0 subtracted from $2a - b$ equals $(b - 2a)$ $(2a + b)$ $(2a - b)$.

26. The reciprocal of $-p$ is (p) $\left(\frac{1}{p}\right)$ $\left(-\frac{1}{p}\right)$.

27. If $12a^2 - 15ab$ is divided by $-3a$, the result is

$$(5b - 4a) \quad (-4a - 5b) \quad (4a - 5b).$$

28. $(-2r^2s^3)^3$ equals $(-8r^5s^6)$ $(8r^6s^9)$ $(-8r^6s^9)$.

Write the simplest expression which completes each statement correctly.

29. The sum of $(a^3 + b^3)$ and $(a^3 - b^3)$ is ___?___.

30. The difference of $(a^3 + b^3)$ and $(a^3 - b^3)$ is ___?___.

31. The product of $(a^3 + b^3)$ and $(a^3 - b^3)$ is ___?___.

32. $n^2 - 2n(4 - n) - 3 =$ ___?___.

33. The result of multiplying $(3ax^2 - 15a^2x + 2a^3)$ by $-2ax$ is ___?___.

34. $4m^2(m^3 - 3n^4) - 5n^2(m^4n - 2m^3n^2) =$ ___?___.

35. $(-6x^2y^2 + 3xy^2 - 2x^2y)$ divided by $-6x^2y^2 =$ ___?___.

36. $(x^2 - 5x + 2)$ multiplied by $(3x - 2) =$ ___?___.

37. $(8x^2 - 22x + 15)$ divided by $(4x - 5) =$ ___?___.

Answer by writing an algebraic expression in simplest form.

38. Expand $(2r - 3s)^3$.

39. Divide $(10 - x^2 + 6x^3 - 27x)$ by $(3x - 5)$.

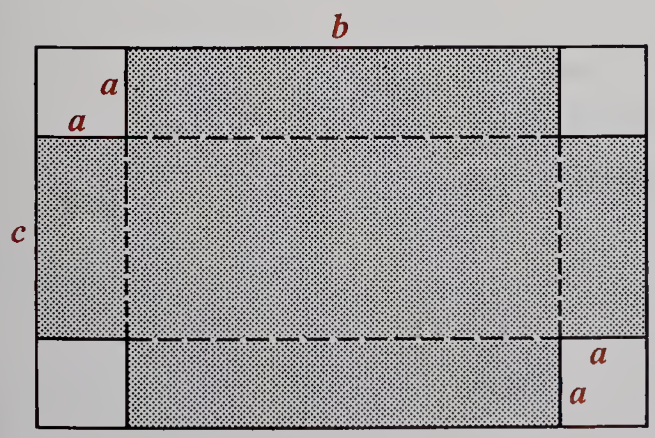
40. Express the perimeter of the shaded part of the figure; below left.

41. Give the area of the shaded straight-line figure as the sum of three algebraic terms.

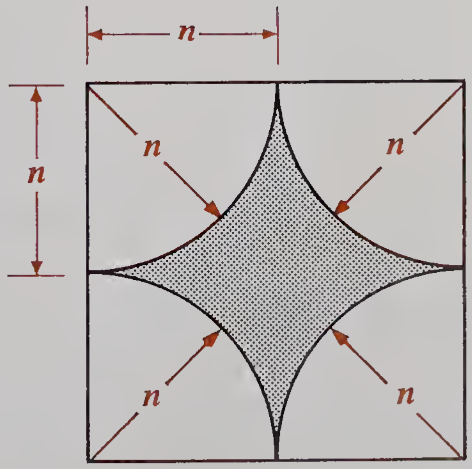
42. Find the perimeter of the shaded part of the figure at the right below. (Leave the answer in terms of π.)

43. Express the area of the shaded curved-line figure as the difference of two algebraic terms. (Leave the answer in terms of π.)

Ex. 40-41



Ex. 42-43



Translate the rule into algebraic symbols by completing the open sentence.

44. To approximate the number of tons T of one size of coal in a bin $l \times w \times h$ feet, divide 3 times the volume of the bin by 94; $T = \underline{\ ?\ }$.

45. Solve for k, then evaluate: $L = \dfrac{mt - k}{t}$; $L = 99$, $m = 100$, $t = 4$.

46. Solve for C, then evaluate: $F = \dfrac{9C}{5} + 32$; $F = 40°$

47. Solve for x: $\dfrac{x}{m} = \dfrac{1}{m}$

48. If l is the length, w the width, and d the depth of a rectangular bin, all expressed in feet, then the number of bushels b of corn which the bin can hold is its capacity (volume) divided by 2.5. Translate this rule into an open sentence whose left member is b.

Solve each of the following problems.

49. Two jet planes left Chicago at the same time, one for the east and the other traveling 25 miles per hour faster for the west. After four hours, the planes were 4900 miles apart. Find their rates.

50. Divide 75 into two parts such that three times the smaller is 5 less than twice the larger.

51. The sum of three numbers is 360. The second number is three times the first, and the third is 20 more than the first. Find the numbers.

52. Shirley has only nickels and dimes in her bank. There are 23 coins in all. If the value of all of the dimes exceeds the value of all of the nickels by 95¢, how many coins of each kind are there?

53. The length of a rectangular flower bed is twice its width. A concrete walk 3 feet wide surrounds the flower bed. The area of the walk is 216 square feet. Find the dimensions of the flower bed.

54. A dealer has two blends of tea, the orange pekoe being worth $.75 a pound more than the black pekoe. He mixes 20 pounds of the better tea with 40 pounds of the cheaper tea and sells the new blend at $1.50 a pound. How much is a pound of each original blend worth?

Extra for Experts

Negative Exponents

To apply the rule $b^m \div b^n = b^{m-n}$ when $m < n$, that is, when $m - n$ represents a negative number, you must understand the meaning of a negative exponent. Observe the following example:

$$\frac{b^5}{b^2} = b^{5-2} = b^3 \qquad \frac{b^2}{b^5} = b^{2-5} = b^{-3}.$$

Since $\frac{b^5}{b^2}$ and $\frac{b^2}{b^5}$ are reciprocals, then b^3 and b^{-3} also must be reciprocals; $b^3 = \frac{1}{b^{-3}}$ and $b^{-3} = \frac{1}{b^3}$. In general,

$$b^n = \frac{1}{b^{-n}} \text{ and } b^{-n} = \frac{1}{b^n}\,;\ (b \neq 0)$$

The other rules of exponents may now be extended to include negative exponents. Study the following illustrations.

1. $\frac{a^3}{n^2} = a^3 \cdot \frac{1}{n^2} = a^3n^{-2}$

2. $3x^{-1}y^2 = 3 \cdot \frac{1}{x^1} \cdot y^2 = \frac{3y^2}{x}$

3. $10^{-3} = \frac{1}{10^3} = .001$

4. $\left(\frac{2}{3}\right)^{-2} = \frac{1}{(\frac{2}{3})^2} = \frac{1}{\frac{4}{9}} = \frac{9}{4}$

5. $-5m^{-3} \cdot 2m^{-2} = -10m^{-5}$

6. $(3 \times 10^5)(4 \times 10^{-2}) = 12 \times 10^3 = 12{,}000$

7. $\frac{8 \times 10^7}{4 \times 10^9} = 2 \times 10^{-2} = .02$

8. $(-2n^2)^{-3} = \frac{1}{(-2n^2)^3} = \frac{1}{-8n^6}$

Questions

Write with only positive exponents.

1. $a^{-1}b^2$ **2.** $2a^{-3}$ **3.** $\frac{4h^3}{k^{-2}}$ **4.** $\frac{3m^{-1}}{2n^{-4}}$

Express each fraction as a product of powers.

5. $\frac{a}{b^2}$ **6.** $\frac{2r^{-2}}{s^{-5}}$ **7.** $\frac{x^2}{y^3z^{-4}}$ **8.** $\frac{p^{-2}}{q^{-1}t^{-3}}$

Evaluate.

9. $(\frac{1}{4})^{-3}$ **10.** $(-4)^{-3}$ **11.** 10^{-5} **12.** $(-\frac{3}{4})^{-2}$

Simplify.

13. $(3c^{-4})^2$ **14.** $\frac{6ab^{-2}}{-2a^{-3}b^{-6}}$ **15.** $\frac{4.8 \times 10^{-8}}{2.0 \times 10^{-10}}$

Just for Fun

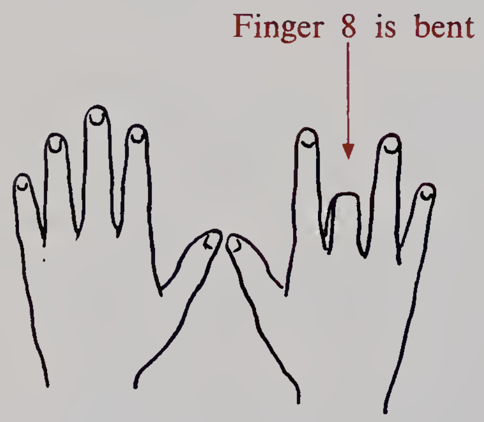
A Handy Way to Multiply by Nine

As a small child, you probably counted on your fingers, but did you ever multiply on them? Finger multiplication works very well when the multiplier is 9.

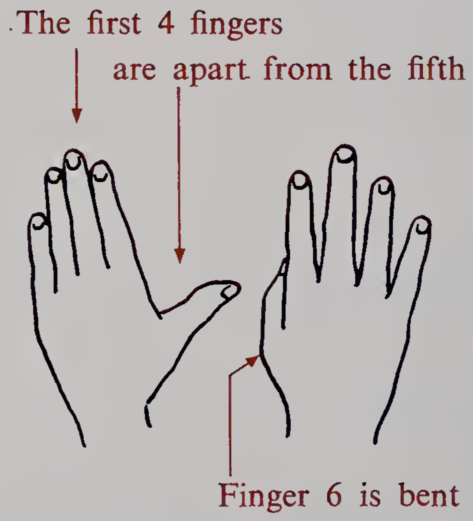
Suppose you wish to multiply 8 by 9. Spread your hands on the table, fingers outstretched. Bend the eighth finger from the left. Now you have your product, seven fingers to the left of the bent finger, two fingers to its right, or 72. (See Figure 6–1.)

You can multiply by 9 any number from 1 to 10 in this way. Try the other one-digit numbers.

Don't think that multiplying by 9 on your fingers is limited to such small numbers. Here is how to proceed when the number to be multiplied has two digits; for example, 46. Start with the 6 of the 46; that is, begin by bending the sixth finger. Now, count off the 4 of 46 by putting the first four fingers together, as shown in Figure 6–2. Your fingers are now in three groups — four together at the far left, one alone just to the left of the bent finger, and four to the right of the bent finger. The product is 414.



· *Figure 6–1* ·



· *Figure 6–2* ·

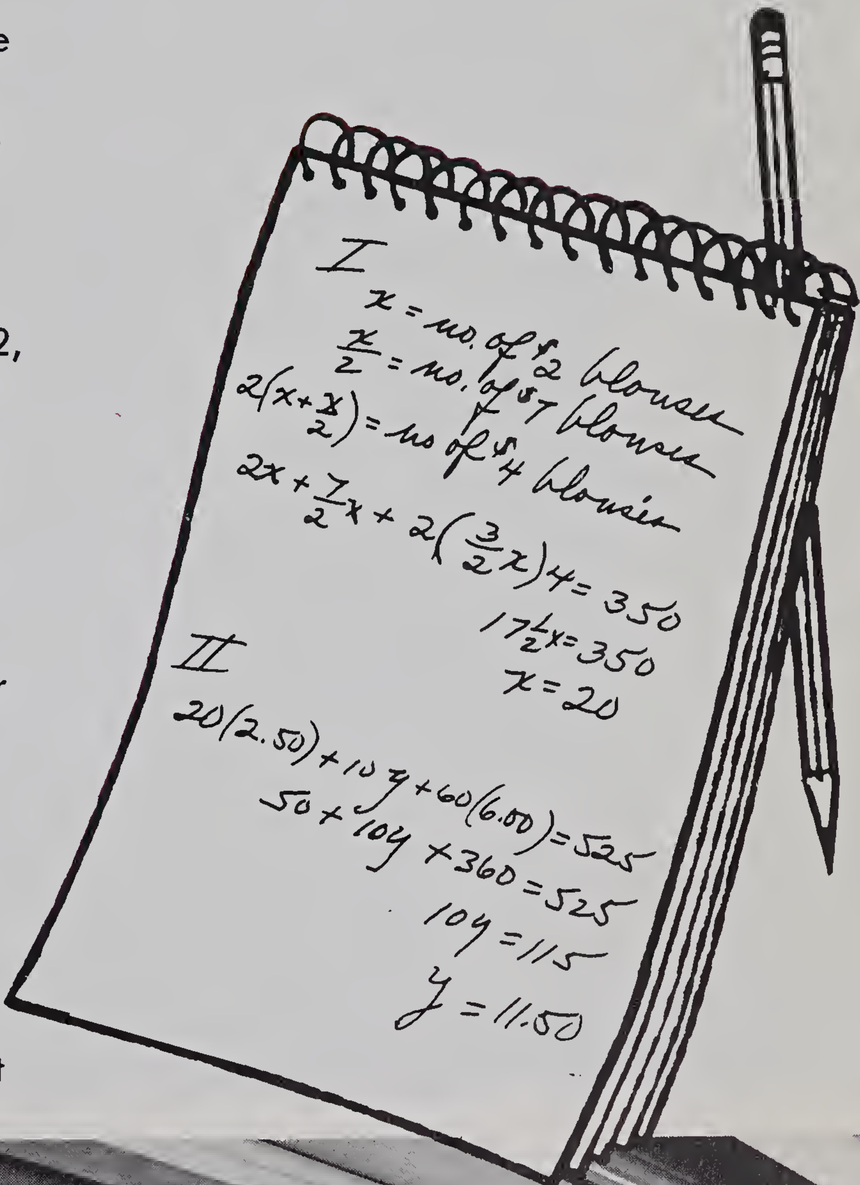
The handy way of multiplying by 9 may be used for many numbers between 11 and 90, but not for all of them. It is up to you to discover which numbers cannot be multiplied this way, and why!

Merchandisers and Mathematics

Merchandisers purchase stock for retail and wholesale stores. The young lady in the photo, an assistant buyer in a department store, is checking price tags against an inventory list. Keeping a large stock in order, estimating the potential profit it represents, and calculating the amount of capital in merchandise, require a good grasp of mathematics.

Illustrated on the work pad are two such problems. The merchandiser has selected three styles of blouses wholesale priced at \$2, \$4, and \$7. Estimating that the demand will be greatest for the \$4 blouses and least for the \$7 blouses, she decides to buy x number of \$2 blouses, $\frac{x}{2}$ number of \$7 blouses, and $2\left(x + \frac{x}{2}\right)$ number of \$4 blouses. Since her store has allotted about \$350 for the purchase, she sets up the equation shown in Part I, and finds that she will buy 20, 10, and 60 blouses, respectively.

The buyer must also decide on the retail selling prices for this merchandise. The store would like a total markup of about 50%, but the per cent of markup usually varies for differently priced items. The buyer decides on a 25% markup for the lower-priced blouses, and a 50% markup for the medium-priced blouses. If the store is to realize a total markup of 50%, the higher-priced blouses must sell at \$11.50, as shown in Part II.



I

$x = \text{no. of \$2 blouses}$

$\frac{x}{2} = \text{no. of \$7 blouses}$

$2\left(x + \frac{x}{2}\right) = \text{no of \$4 blouses}$

$2x + \frac{7}{2}x + 2\left(\frac{3}{2}x\right)4 = 350$

$17\frac{1}{2}x = 350$

$x = 20$

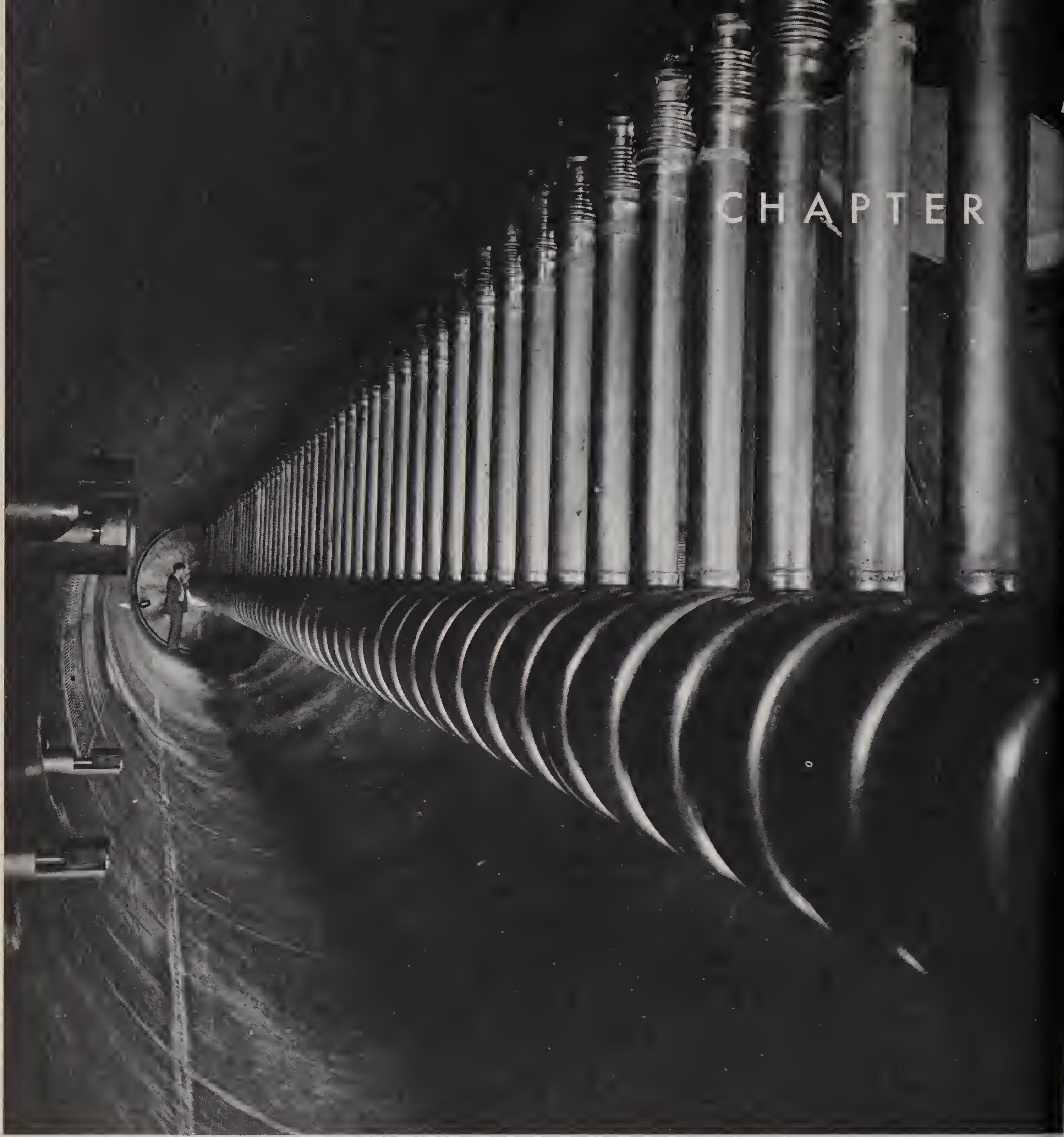
II

$20(2.50) + 10y + 60(6.00) = 525$

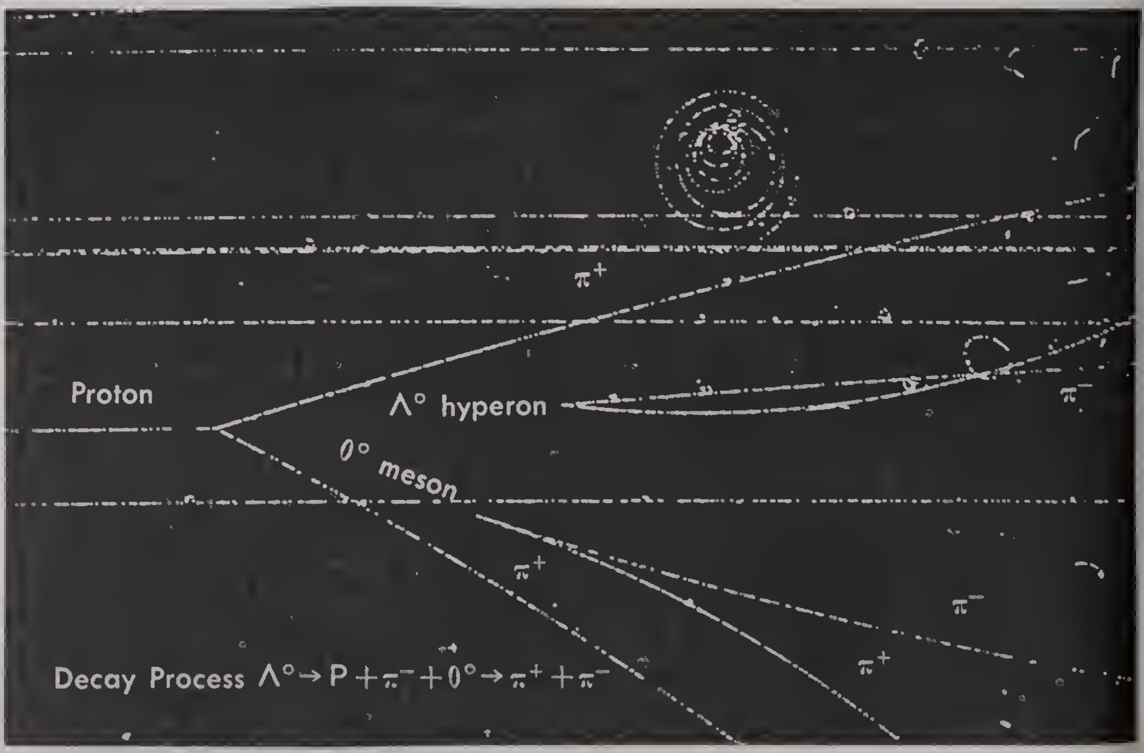
$50 + 10y + 360 = 525$

$10y = 115$

$y = 11.50$



CHAPTER


π^+
Proton
Λ° hyperon
π^-
θ° meson
π^+
π^-
π^+
Decay Process $\Lambda^\circ \rightarrow P + \pi^- + \theta^\circ \rightarrow \pi^+ + \pi^-$

Special Products and Factoring

"*I wonder what makes it tick?*" This question has presented a challenge to many boys and girls and has led many scientists to new discoveries. Scientists and mathematicians, young or old, also share a curiosity about putting things together.

With the linear accelerator (upper photo) physicists attempt to determine the patterns of disintegration and combination of the elements they study (lower photo). Products like synthetic rubber represent chemists' attempts to duplicate natural products.

Like scientists, mathematicians analyze and combine the elements with which they work. These elements, such as numbers and sets, can be identified (or analyzed) more easily than the atom particles of physics. Also, they are not as complex as the compounds which the chemist tries to synthesize. Yet, the careful determination of relationships and the tracing of their patterns in order to explain their properties are the same. In this chapter you will learn to analyze and combine algebraic elements into forms with which you can work easily.

THE DISTRIBUTIVE PROPERTY IN FACTORING

7–1 Factoring in Algebra

It is often necessary to know how a number can be written as a product of two or more other numbers. Because

$$90 = 9 \cdot 10 \quad \text{and} \quad 90 = 5 \cdot 18,$$

9 and 10, and 5 and 18 are called factors of 90. Because $90 = \frac{1}{2} \cdot 180$ you may think that $\frac{1}{2}$ and 180 also can be called factors of 90. However, if fractions generally were allowed as factors, any nonzero fraction would be a factor of every number. Therefore, you usually specify a particular set of numbers to be used as factors.

Finding numbers belonging to a given set of numbers and having their product equal to a given number is called **factoring the number over the given set**. Hereafter, integers will be factored over the set of integers, unless another set is specified.

Emphasize that you *factor over a set*. See 7-1 T.M. pg. 26.

An important subset of the integers that often is chosen as a set of possible factors is the set of *prime numbers.* A **prime number** is an integer *greater than one*, having no positive integral factor other than itself and one. The first prime numbers are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$$

See Extra for Experts, Chapter 9 for a discussion the sieve of Eratosthenes.

To factor 90 over the set of primes, write

$$90 = 2 \cdot 3 \cdot 3 \cdot 5 = 2 \cdot 3^2 \cdot 5.$$

The prime factors of 90 are 2, 3, and 5, with 3 occurring twice.

To express an integer as a product of primes, you usually can proceed in several ways. For example,

$$\begin{aligned} 392 &= 7 \cdot 56 \\ &= 7 \cdot 7 \cdot 8 \\ &= 7 \cdot 7 \cdot 2 \cdot 2 \cdot 2 \\ &= 7^2 \cdot 2^3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 392 &= 2 \cdot 196 \\ &= 2 \cdot 2 \cdot 98 \\ &= 2 \cdot 2 \cdot 2 \cdot 49 \\ &= 2 \cdot 2 \cdot 2 \cdot 7 \cdot 7 \\ &= 2^3 \cdot 7^2 \end{aligned}$$

See T.M. pg. 26(1) for a note on unique factorization.

In the second method you look systematically for the smallest prime factor of the number still to be factored at each stage. That is, you first try 2, and try it again and again until it no longer can be used; then you try 3, then 5, then 7, and so on until all the factors are prime numbers.

Once you know the prime factors of an integer, it is easy to list all its positive factors. The factors of 392, for example, are 1 and all possible products of one or both of the primes 2 and 7, each with an exponent less than or equal to its exponent in 392. These factors are

Call attention to the Just for Fun in this chapter.

$$1, 2, 2^2, 2^3, 7, 2 \cdot 7, 2^2 \cdot 7, 2^3 \cdot 7, 7^2, 2 \cdot 7^2, 2^2 \cdot 7^2, 2^3 \cdot 7^2$$

or

$$1, 2, 4, 8, 7, 14, 28, 56, 49, 98, 196, 392$$

By factoring integers into products of primes, you can determine the largest integral factor of both of two integers. To find the greatest common factor of 392 and 1260, notice that

$$392 = 2^3 \cdot 7^2 \quad \text{and} \quad 1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7,$$

so the largest power of 2 common to 392 and 1260 is 2^2 and the largest common power of 7 is 7. The greatest common factor is, therefore, $2^2 \cdot 7$ or 28.

In algebra you often need to express a polynomial as a product.

Transforming a given polynomial into a product of other polynomials is called **polynomial factoring**. For example, each term of the polynomial $5x + 5y$ contains 5 as a factor; therefore, by the distributive property

$$5x + 5y = 5(x + y).$$

Both 5 and $x + y$ are factors of $5x + 5y$.

When factoring polynomials whose numerical coefficients are integers, you look for factors that are either integers or polynomials with integral coefficients. Some of the factors of $6x^2y$ are 1, 2, 3, 6, $2x^2$, and $3xy$. You see that $2x^3$ is not a factor of $6x^2y$, since there is no polynomial by which you can multiply $2x^3$ to obtain $6x^2y$.

ORAL EXERCISES

Tell why each statement is true or why it is false.

SAMPLE 1. 7 is a prime factor of 42.

What you say: True, because 7 is a prime and $42 = 7 \cdot 6$.

SAMPLE 2. 9 is a factor of 12 over the set of integers.

What you say: False, because there is no integer a such that $12 = 9a$.

1. 6 is a factor of 48 over the set of integers. T; $48 = 6 \cdot 8$
2. 14 is a factor of 28 over the set of integers. T; $28 = 14 \cdot 2$
3. The smallest prime factor of 246 is 3. F; 2 a prime < 3, $246 = 2 \cdot 123$
4. 13 is a prime factor of 52. T; 13 prime, $52 = 13 \cdot 4$
5. 17 is a prime factor of 68. T; 17 prime, $68 = 17 \cdot 4$
6. 1 is not a factor of 6. F; $6 = 1 \cdot 6$
7. 10 is a factor of 10. T; $10 = 10 \cdot 1$
8. 2^3 is a factor of 144. T; $144 = 2^3 \cdot 18$
9. 3 is a prime factor of 123. T; 3 prime, $123 = 3 \cdot 41$
10. 3 is a prime factor of 1234. F; no integer a such that $1234 = 3a$
11. 9 is not a factor of 801. F; $801 = 9 \cdot 89$
12. 9 is not a factor of 802. T; no integer a such that $802 = 9a$
13. 3 is a factor of 0. T; $0 = 3 \cdot 0$

Name the monomial with the largest numerical coefficient and the greatest degree in each variable that is a factor of both monomials in each pair.

SAMPLE 3. $12x^2y^3$, $42x^4y$ *What you say:* $6x^2y$.

14. 42, $56p^2$ 14
15. 72, $30v^3$ 6
16. $3x$, $18xy$ $3x$
17. $5b$, $40bc$ $5b$
18. $60m^3n$, $48m^2n$ $12m^2n$
19. $70rs^4$, $105rs^3$ $35rs^3$
20. $-21v^3w^2$, $14v^2w^5$ $7v^2w^2$
21. $75y^2z^4$, $-80y^7z^2$ $5y^2z^2$
22. $23xy$, $32a$ 1

WRITTEN EXERCISES

Factor each integer over the set of primes.

Ⓐ **1.** 210
2. 182
3. 2310
4. 1155
5. 1500
6. 2000
7. 13
8. 17

Find all the positive integral factors of each number.

9. 35
10. 91
11. 63
12. 52
13. 100
14. 441
15. 726
16. 1690

Find the greatest common factor of each pair of integers.

17. 144, 630
18. 231, 294
19. 180, 1368
20. 4200, 3850

Give the second factor for each monomial.

21. $5x^2y = 5x(?)$
22. $-24a^2b^3 = -8ab^2(?)$
23. $20ab^2c^3 = 4bc(?)$
24. $-14cd^3e^2 = -2cde(?)$
25. $18mn^4p^3 = 6n^2p^2(?)$
26. $-21r^5s^3t^2 = 7r^2s^2t^2(?)$
27. $38p^4d^3q^5 = -2pdq(?)$
28. $-39u^6v^4w^2 = 3u^4v^3w^2(?)$
29. $28k^2j^2m^2 = -7mjk^2(?)$
30. $-25m^8n^7 = 5n^7m^5(?)$

For each pair of monomials, find the highest power of the first monomial that is a factor of the second.

SAMPLE. $2a;\ 12a^3b^2$

Solution: $(2a)^2 = 4a^2$, and $12a^3b^2 = (4a^2)(3ab^2)$.
$(2a)^3 = 8a^3$, and 8 is *not* a factor of 12.
$\therefore$ $(2a)^3$ is not a factor of $12a^3b^2$, but $(2a)^2$ is.
$(2a)^2$, **Answer.**

Ⓑ **31.** $x;\ 7x^5y^2$
32. $y;\ -3x^2y^7$
33. $b^2;\ 15b^5c^4$
34. $h^2;\ 34a^5h^7$
35. $5r;\ 250r^4t^3$
36. $3v;\ 27v^6w^2$
37. $3rp;\ 192r^4p^6$
38. $7rq^3;\ 98r^3q^7$
39. $2x^2y^2;\ 6x^6y^6$
40. $3u^3v^3;\ 9u^9v^9$
41. $5a^2b^3;\ 625a^8b^7$
42. $4r^3s^2;\ 256r^{12}s^6$
43. $2pq^4;\ 16p^4q^{16}$
44. $7m^3n;\ 343m^9n^3$
45. $2c^5d^3e;\ 8c^{20}d^{12}e^4$

7–2 Identifying Common Factors

As each term of $4ab + 6a$ has $2a$ as a factor, using the distributive property, you have

$$4ab + 6a = 2a(2b + 3).$$

See 7-2 T.M. pg. 26.

You see, $2a$ is a monomial, and $4ab + 6a$ is a polynomial. Thus, $2a$ is a *monomial* factor of the polynomial $4ab + 6a$. When you factor a polynomial, first see whether each term has the same monomial as a factor. A monomial is a **common monomial factor** of a polynomial if it is a factor of *every* term of the polynomial.

Be sure to use the *greatest common monomial factor*. The **greatest common monomial factor** of a polynomial is the common monomial factor having the greatest numerical coefficient and the greatest degree.

Examine this chart, observing how each polynomial is factored.

Emphasize the use of the distributive property.

Given Polynomial	Factors		Factored Expression
$5x^2 - 3x$	x	$5x - 3$	$x(5x - 3)$
$5y^2 + 35y$	$5y$	$y + 7$	$5y(y + 7)$
$6z^4 + 36z^3 + 60z^2$	$6z^2$	$z^2 + 6z + 10$	$6z^2(z^2 + 6z + 10)$
$12cm - 12cn^2$	$12c$	$m - n^2$	$12c(m - n^2)$

The associative and commutative properties for addition, together with the distributive property, enable you to factor a polynomial by grouping.

$$\begin{aligned} ax + by + ay + bx &= (ax + bx) + (ay + by) \\ &= (a + b)x + (a + b)y \\ &= (a + b)(x + y) \end{aligned}$$

In the last step, $(a + b)$ is treated as a single term in applying the distributive property.

Here is another polynomial that can be factored readily when you group the terms appropriately:

$$\begin{aligned} 2vw - 15st - 10vt + 3sw &= (2vw - 10vt) + (3sw - 15st) \\ &= 2v(w - 5t) + 3s(w - 5t) \\ &= (2v + 3s)(w - 5t) \end{aligned}$$

pupils omit the last step, point out that in factoring a polynomial they express it as product, *not* as a sum of products.

Of course, there can be more than one convenient way to group terms:

$$\begin{aligned} 2vw - 15st - 10vt + 3sw &= (2vw + 3sw) + (-15st - 10vt) \\ &= w(2v + 3s) + (-5t)(3s + 2v) \\ &= w(2v + 3s) + (-5t)(2v + 3s) \\ &= (w - 5t)(2v + 3s). \end{aligned}$$

Because multiplication is commutative, this result is the same as the preceding one.

ORAL EXERCISES

Give the greatest common monomial factor of each polynomial.

SAMPLE. $8s^2 - 24s$ *What you say:* $8s$.

1. $2a^2 + 12a$ 2a
2. $9b^2 - 81b$ 9b
3. $12c^2 - 6$ 6
4. $9d^2 + 27$ 9
5. $e^2 + 9$ 1
6. $2f^2 - 7$ 1
7. $7g^2 - 28g$ 7g
8. $13h^2 + 26h$ 13h
9. $i^2 - 1$ 1
10. $4j^2 - 1$ 1
11. $b^2 + 2bx + x^2$ 1
12. $x^2 + 10xy + 25y^2$ 1
13. $3x^2 - 12x + 18$ 3
14. $18n^2 - 27n + 9$ 9
15. $6x^2 - 24y^2$ 6
16. $8k^2 - 72$ 8
17. $mn^2t - m^2nt^2$ mnt
18. $a^2bx^2 + ab^2x$ abx
19. $60m^3n + 48m^2n$ $12m^2$
20. $70rs^4 - 105s^3r$ $35rs^3$
21. $-21v^3w^2 + 14v^2w^5$ 7v
22. $-80y^7z^2 + 75y^2z^4$ 5y
23. $2x^4 + 6x^3 - 10x^2$ 2x
24. $9y^5 - 6y^4 + 3y^3$ $3y^3$

WRITTEN EXERCISES

Write in factored form.

SAMPLE. $25x^2 - 15x$ *Solution:* $25x^2 - 15x = 5x(5x - 3)$

A

1. $3x^2 + 12y^2$
2. $7a^2 + 14b^2$
3. $18x^2 - 12x$
4. $27r^2 - 90r$
5. $x^2 + 7x$
6. $n^2 + 13n$
7. $3x^2 - 21x^3$
8. $5b^2 - 70b^3$
9. $6x^2 - 4x$
10. $8a^2 + 12a$
11. $b^3 + b^2 + b$
12. $a^3 - a^2 - a$
13. $a^2b + ab^2$
14. $xy^2 - x^2y$
15. $15a^2c - 3c$
16. $12x^2y + 2y$
17. $5r^2s - 10rs^2$
18. $3a^2b - 9ab^2$
19. $t + 2mt$
20. $3nx - x$
21. $-12x^2 - 6x$

Write each expression in factored form.

SAMPLE. $x^2(x+2) + 7(x+2)$

Solution: $x^2(x+2) + 7(x+2) = (x^2+7)(x+2)$

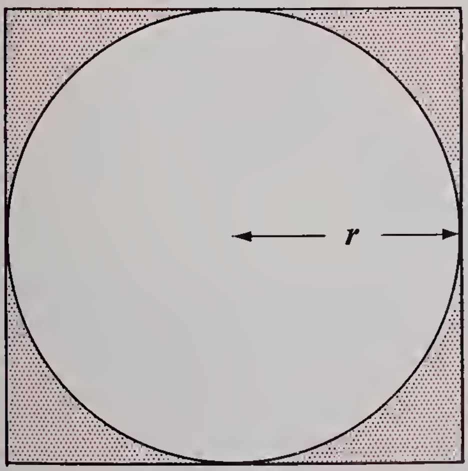
22. $y(y-1) + 2(y-1)$
23. $a(a-8) + 9(a-8)$
24. $(4c+5d)x - (4c+5d)y$
25. $(x+1)(2x+3) - (x+1)$
26. $(x-y)^2 + (x+y)(x-y)$
27. $2m(m-n) - (m+n)(m-n)$
28. $(b-3c)x^2 + (b-3c)y^2$
29. $k^2 + 2k + kt + 2t$
30. $x^2 - x + xy - y$
31. $dg + dm - fg - fm$
32. $rs - rt - ks + kt$
33. $3b^2 + 2b + 12b + 8$

34. $12x^4 - 8x^2 + 20x$
35. $18y^4 + 30y^3 - 42y$
36. $5r^2t - 10rt + 5rt^2$
37. $18a^3c^2 - 9a^2c + 6ac^2$
38. $3x^2 - 2x + 6x - 4$
39. $10y^2 - 3 + 15y - 2y$
40. $x^2 + x(a+b)$
41. $3a^2 + 6a(x-y)$
42. $m^2 + 14a + 2m + 7am$
43. $2p^2 + 2a + 4ap + p$
44. $x^3 - 21 - 3x^2 + 7x$
45. $r^3 - s^3 - sr^2 + rs^2$

PROBLEMS

Write an algebraic expression in factored form for the shaded area A of each figure on page 244.

SAMPLE.

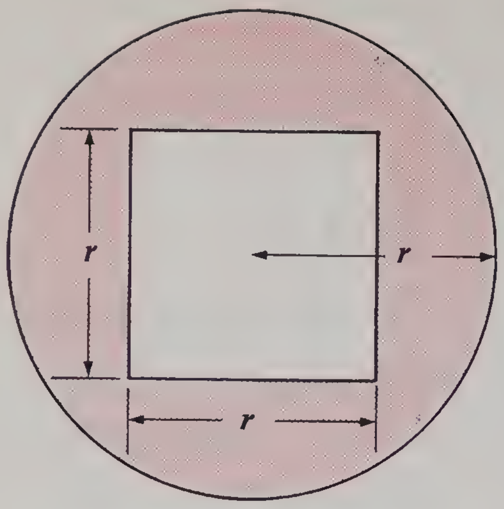


Solution:

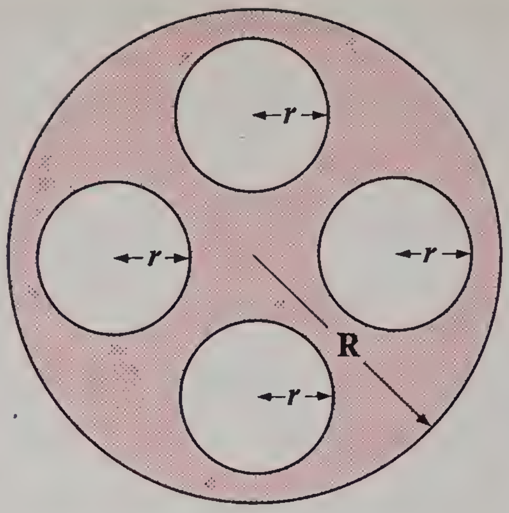
$$\begin{aligned} A &= \text{area of square} - \text{area of circle} \\ A &= (2r)^2 - \pi r^2 \\ &= 4r^2 - \pi r^2 \\ &= r^2(4-\pi), \textbf{ Answer.} \end{aligned}$$

A

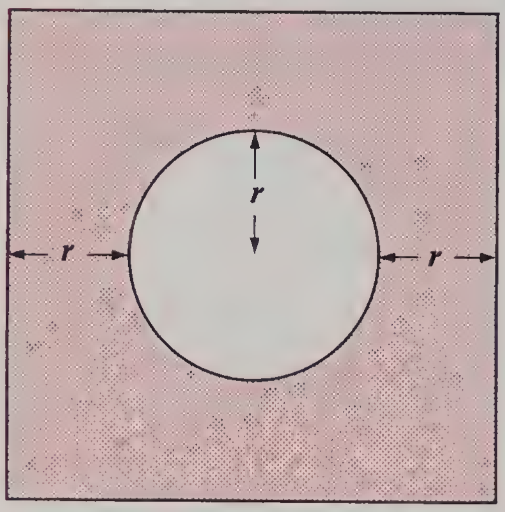
1.



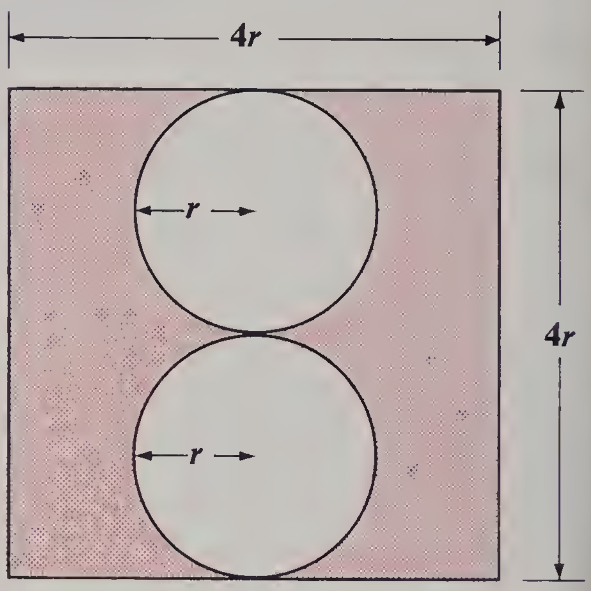
5.



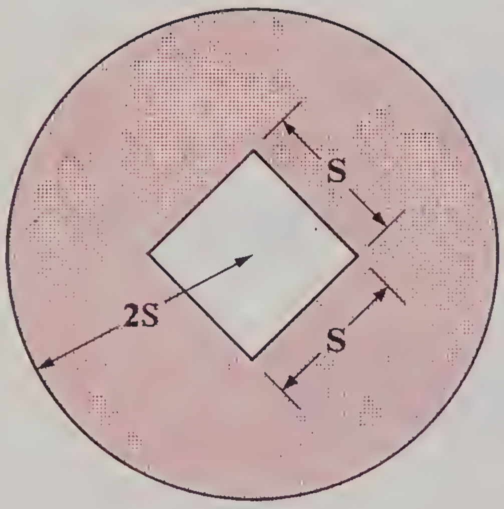
2.



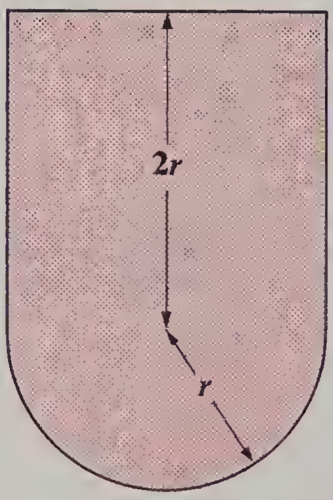
6.



3.



7.



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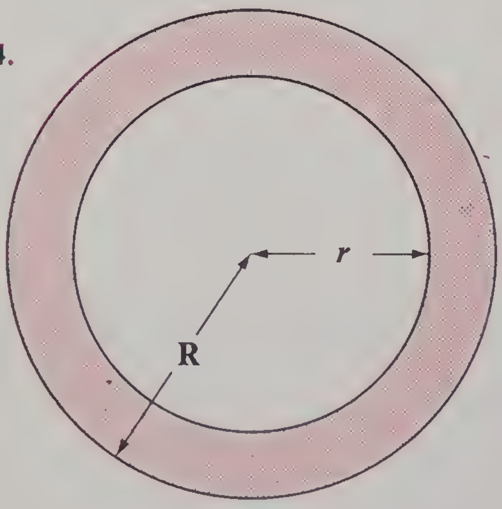

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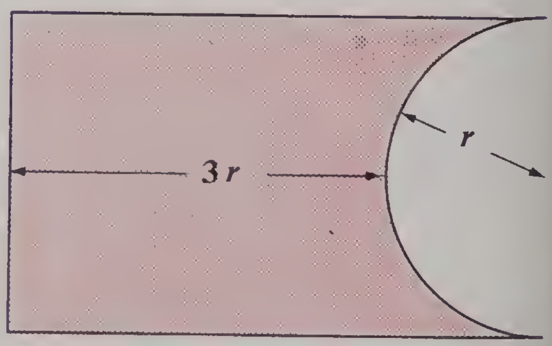


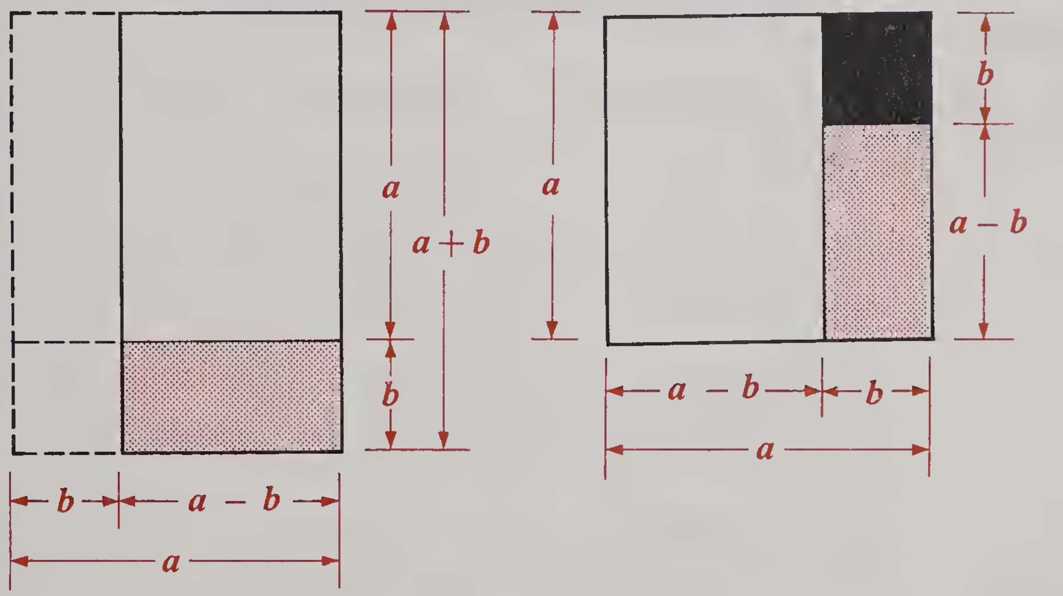
7-3 Multiplying the Sum and Difference of Two Numbers

Certain products occur so often that you should recognize them at sight. Study each of the three examples below:

$$\begin{array}{rrr} x & +3 & \\ x & -3 & \\ \hline x^2 & +3x & \\ & -3x & -9 \\ \hline x^2 & & -9 \end{array} \qquad \begin{array}{rrr} 2y & -5z & \\ 2y & +5z & \\ \hline 4y^2 & -10yz & \\ & 10yz & -25z^2 \\ \hline 4y^2 & & -25z^2 \end{array} \qquad \begin{array}{rrr} a & +b & \\ a & -b & \\ \hline a^2 & +ab & \\ & -ab & -b^2 \\ \hline a^2 & & -b^2 \end{array}$$

The examples illustrate this rule: the product of the sum and difference of two numbers is the square of the first number minus the square of the second number,

$$(a + b)(a - b) = a^2 - b^2.$$



$(a + b)\,(a - b) = a^2 - b^2$

· *Figure 7–1* ·

ORAL EXERCISES

Square each monomial.

SAMPLE. $-4n^3$ *What you say:* $16n^6$

1. $5x$ $25x^2$ **3.** $2x^3$ $4x^6$ **5.** $-3n^2$ $9n^4$ **7.** $8rt$ $64r^2t^2$ **9.** $-2sm^2$ $4s^2m^4$

2. $3a$ $9a^2$ **4.** $7b^2$ $49b^4$ **6.** $-6x^3$ $36x^6$ **8.** $12xy$ $144x^2y^2$ **10.** $-3tv^2$ $9t^2v^4$

Pupils enjoy such arithmetic short cuts.

Find each product.

11. $(x + y)(x - y)$ $x^2 - y^2$
12. $(c - d)(c + d)$ $c^2 - d^2$
13. $(a - 1)(a + 1)$ $a^2 - 1$
14. $(b + 6)(b - 6)$ $b^2 - 36$
15. $(3a + b)(3a - b)$ $9a^2 - b^2$
16. $(1 + 5b)(1 - 5b)$ $1 - 25b^2$
17. $(2m - n)(n + 2m)$ $4m^2 - n^2$
18. $(r + 3s)(3s - r)$ $9s^2 - r^2$
19. $(x^2 - 9)(x^2 + 9)$ $x^4 - 81$
20. $(y^3 - 4)(y^3 + 4)$ $y^6 - 16$
21. $(st - 8)(st + 8)$ $s^2t^2 - 64$
22. $(mn - 7)(mn + 7)$ $m^2n^2 - 49$
23. $(x - \frac{1}{4})(x + \frac{1}{4})$ $x^2 - \frac{1}{16}$
24. $(z + \frac{1}{2})(z - \frac{1}{2})$ $z^2 - \frac{1}{4}$
25. 29×31 or $(30 - 1)(30 + 1)$ 89
26. 18×22 or $(20 - 2)(20 + 2)$ 39
27. 45×55 2475
28. 76×84 6384

WRITTEN EXERCISES

Multiply (a) by using the sum and difference of two numbers and (b) by using the distributive property.

SAMPLE. (83)(77)

Solution: **a.** $(83)(77) = (80 + 3)(80 - 3) = 6400 - 9 = 6391$

b. $(83)(77) = 80(77) + 3(77) = 6160 + 231 = 6391$

Ordinary multiplication, which uses the distributive property, also can be used for this part.

Ⓐ

1. (9)(11)
2. (51)(49)
3. (22)(18)
4. (58)(62)
5. (33)(27)
6. (67)(73)
7. (36)(44)
8. (84)(76)
9. (55)(45)
10. (95)(85)
11. (46)(34)
12. (94)(106)

Ⓑ

13. (1010)(990)
14. (520)(480)
15. $(5\frac{1}{2})(4\frac{1}{2})$
16. (1130)(1070)
17. (640)(560)
18. $(8\frac{1}{3})(9\frac{2}{3})$
19. (1150)(1250)
20. (360)(240)
21. $(4\frac{1}{4})(3\frac{3}{4})$

7–4 Factoring the Difference of Two Squares

By the symmetric property of equality, the relation $(a + b)(a - b) = a^2 - b^2$ is reversible;

$$a^2 - b^2 = (a + b)(a - b).$$

This shows you how to factor an algebraic expression consisting of the difference of two squares.

$$m^2 - 16 = (m + 4)(m - 4)$$
$$36x^2 - y^2 = (6x + y)(6x - y)$$
$$9m^2 - 49t^4 = (3m + 7t^2)(3m - 7t^2)$$
$$-r^2s^2 + v^2w^2 = v^2w^2 - r^2s^2 = (vw + rs)(vw - rs)$$

If, as in $49x^2y^4$, the degree in *each* variable is even and the numerical coefficient is the square of an integer, then the monomial is a square; $49x^2y^4 = (7xy^2)^2$. Sometimes it is difficult to tell at sight whether a numerical coefficient is a square. However, Table 3 in the Appendix enables you to tell whether or not each integer from 1 to 10,000 is the square of an integer.

ORAL EXERCISES

Tell whether or not each of the following is the difference of two squares. If it is, give its factors.

$-9)$ **1.** $r^2 - 81$
$-d)$ **2.** $c^2 - d^2$
$1-a)$ **3.** $121 - a^2$
$-6)$ **4.** $s^2 - 36$
$-y)$ **5.** $x^2 - y^2$
$-1)$ **6.** $z^2 - 1$

7. $x^2 - 100$ $(x+10)(x-10)$
8. $16 - y^2$ $(4+y)(4-y)$
9. $56x^2 - y^2$ no
10. $64x^2 - 10y^2$ no
11. $-m^2 + n^2$
12. $-1 + c^2$ $(c+1)(c-1)$

13. $-81x^2 - y^2$ no
14. $25r^2 - 50s^2$ no
15. $r^2 - 9s^2$ $(r+3s)(r-3s)$
16. $a^2b^2 - c^2d^2$
17. $-r^2s^2 + r^2$
18. $-c^2 + c^2d^2$ $(cd+c)(cd-c)$ or $c^2(d+1)(d-1)$

19. $x^3y^2 - z^2$ no
20. $-ab^2 - cd^2$ no
21. $x^2 - 1$ $(x+1)(x-1)$
22. $-x^2 + 1$ $(1+x)(1-x)$
23. $x^4y^2 - x^4z^2$ $(x^2y+x^2z)(x^2y-x^2z)$ or $x^4(y+z)(y-z)$
24. $a^2b^2 - a^4c^2$

11. $(n+m)(n-m)$

$+cd)(ab-cd)$

24. $(ab+a^2c)(ab-a^2c)$ or $a^2(b+ac)(b-ac)$

17. $(r+rs)(r-rs)$ or $r^2(1+s)(1-s)$

WRITTEN EXERCISES

Factor, and check by multiplication. You may refer to Appendix Table 3. See 7-4 T.M. pg. 26.

1. $x^2 - 16$
2. $r^2 - 9$
3. $x^2 - 4y^2$
4. $a^2 - 4b^2$
5. $R^2 - r^2$
6. $a^2 - b^2y^6$
7. $16a^2 - b^4$
8. $256x^4 - y^2$
9. $4x^4 - z^2$
10. $m^2 - 16n^4$
11. $196b^2 - 121x^2$
12. $289x^2 - 676y^2$
13. $-9 + 4r^2t^2$
14. $-144 + m^2n^2$
15. $1 - 9n^2$
16. $25m^2 - 1$
17. $.09a^2 - 4$
18. $.04b^2 - 49$
19. $c^2 - .64$
20. $.81 - d^2$
21. $x^2 - \frac{1}{9}$

22. $\frac{1}{16} - y^2$

23. $a^2 - \dfrac{b^2}{4}$

24. $-\dfrac{c^2}{25} + \dfrac{d^2}{36}$

Factor, and check by multiplication.

SAMPLE. $2a - 8a^3$ Encourage the habit of looking first for a common monomial factor.

Solution: $2a - 8a^3 = 2a(1 - 4a^2) = 2a(1 + 2a)(1 - 2a)$

B

See 7-4 T.M. pg. 26.

25. $-x^4 + 25x^6$
26. $16s^2 - s^4$
27. $a^2 - a^4$
28. $c^3 - c^5$
29. $x^3y - xy^5$
30. $a^6b - a^2b^3$
31. $2p^3q^4 - 72pq^2$
32. $147x^2y - 3x^4y^3$
33. $2a^4 - 32$
34. $n - n^5$
35. $-a^{2n} + 1$
36. $-n^{2x} + t^2$
37. $-m^{5a} + m^a$
38. $-b^{2x} + b^{6x}$
39. $.18r^4 - .08s^6t^2$

C

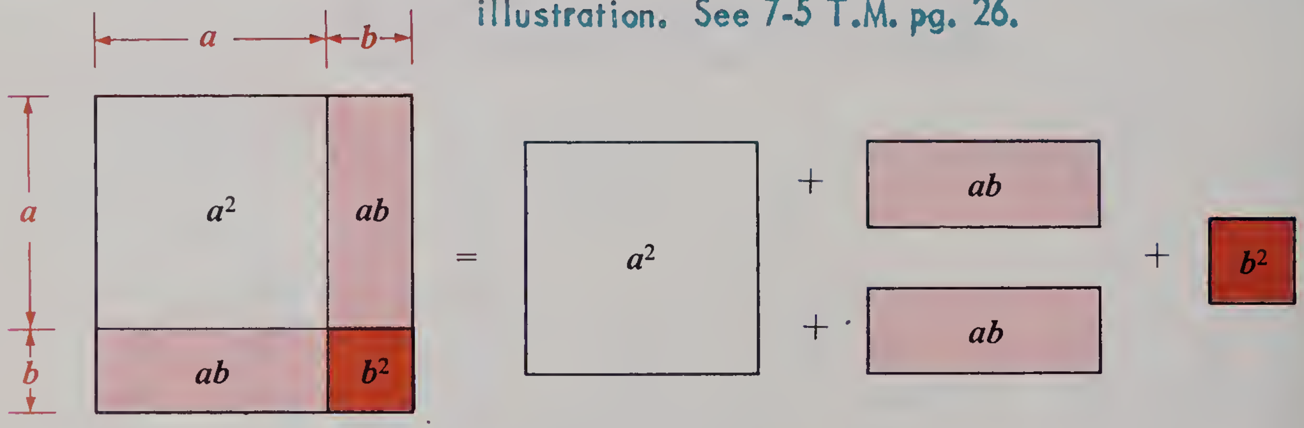
40. $(a + 2b)^2 - x^4$
41. $(3a - 1)^2 - y^6$
42. $-1 + (x - 1)^2$
43. $-9 + (x + 3)^2$

44. Show that the difference between the squares, **a.** of two consecutive integers equals the sum of the integers; **b.** of two consecutive odd integers equals twice the sum of the integers.

QUADRATIC TRINOMIALS

7–5 Squaring a Binomial: Plateau Section*

Each side of the square in Figure 7–2 is $(a + b)$ units in length. You can consider the square as being made up of four areas, as shown. The total area can be expressed as a square of a binomial $(a + b)^2$, or as a trinomial square $(a^2 + 2ab + b^2)$. Call attention to the geometric illustration. See 7-5 T.M. pg. 26.



$$(a + b)^2 = a^2 + 2ab + b^2$$

Figure 7–2

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*In a Plateau Section, skills you have learned are applied in new ways; no new ideas are introduced.

The binomial $(a + b)$ is squared at the right by the usual method of multiplication. Notice how each term in the product is obtained.

$$\begin{array}{rrr} a & + & b \\ a & + & b \\ \hline a^2 & + & ab \\ & & ab + b^2 \\ \hline a^2 & + & 2ab + b^2 \end{array}$$

1. Square the first term in the binomial.

2. Double the product of the two terms.

3. Square the second term in the binomial.

Now examine the square of a binomial difference. The binomial $(a - b)$ is squared by multiplication at the right. Again, notice how each term is obtained.

$$\begin{array}{rrr} a & - & b \\ a & - & b \\ \hline a^2 & - & ab \\ & - & ab + b^2 \\ \hline a^2 & - & 2ab + b^2 \end{array}$$

1. Square the first term in the binomial.

2. Double the product of the two terms.

3. Square the second term in the binomial.

Whenever you square a binomial, the product is a trinomial square, whose terms show this pattern. Pupils should learn these patterns and recognize them as true for all numbers *a* and *b*.

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2.$$

Knowing these relationships, you can write the square of a binomial without performing long multiplication.

$$(x + 1)^2 = x^2 + 2x + 1 \qquad (2n + 3)^2 = 4n^2 + 12n + 9$$
$$(y - 1)^2 = y^2 - 2y + 1 \qquad (4m^2 - n^3)^2 = 16m^4 - 8m^2n^3 + n^6$$
$$(-ab^2 + c^3)^2 = (c^3 - ab^2)^2 = c^6 - 2ab^2c^3 + a^2b^4$$

ORAL EXERCISES

Read each of the following as a trinomial square.

$mh+$ 1. $(m + h)^2$

2. $(c + d)^2$ $c^2+2cd+d^2$

$+9$ 3. $(x + 3)^2$

4. $(x + 5)^2$ $x^2+10x+25$

5. $(x - y)^2$ $x^2-2xy+y^2$

6. $(m - n)^2$ $m^2-2mn+n^2$

7. $(a - 4)^2$ $a^2-8a+16$

8. $(y - 8)^2$ $y^2-16y+64$

9. $(2a + 1)^2$ $4a^2+4a+1$

10. $(3b + 1)^2$ $9b^2+6b+1$

11. $(3x + 5)^2$

12. $(3n + 7)^2$ $9n^2+42n+49$

11. $9x^2+30x+25$

13. $(5x + b)^2$ $25x^2+10bx+b^2$

14. $(8a + b)^2$ $64a^2+16ab+b^2$

15. $(3a - 2)^2$ $9a^2-12a+4$

16. $(2b - 3)^2$ $4b^2-12b+9$

$16c^2-8cd+d^2$ $w^2-20wz+100z^2$ $x^2+18xy+81y^2$ $16+8s^2+s^4$

17. $(4c-d)^2$ **20.** $(w-10z)^2$ **23.** $(x+9y)^2$ **26.** $(4+s^2)^2$ x^2y

$4r^2-4rs+s^2$ **18.** $(2r-s)^2$ **21.** $(8r-1)^2$ **24.** $(x+11y)^2$ **27.** $(xy+2)^2$ $4xy$

$x^2-12xy+36y^2$ **19.** $(x-6y)^2$ **22.** $(10a-1)^2$ **25.** $(r^2-5)^2$ **28.** $(3-rs)^2$

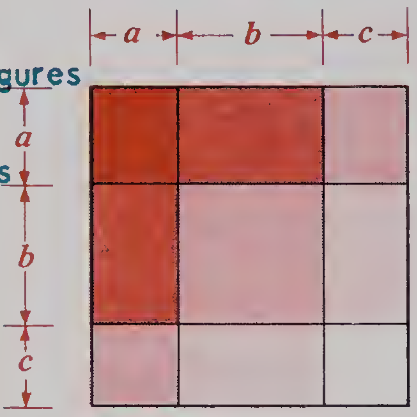
$100a^2-20a+1$ r^4-10r^2+25 $9-6rs+r^2s^2$

21. $64r^2-16r+1$ 24. $x^2+22xy+121y^2$

PROBLEMS

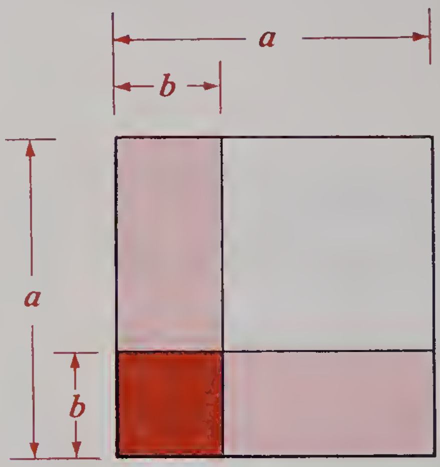
A **1.** Show that $(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$ by considering the figure at the left.

These problems and Figures 7-1 and 7-2 suggest some geometric methods used by the early Greeks to develop algebraic identities.



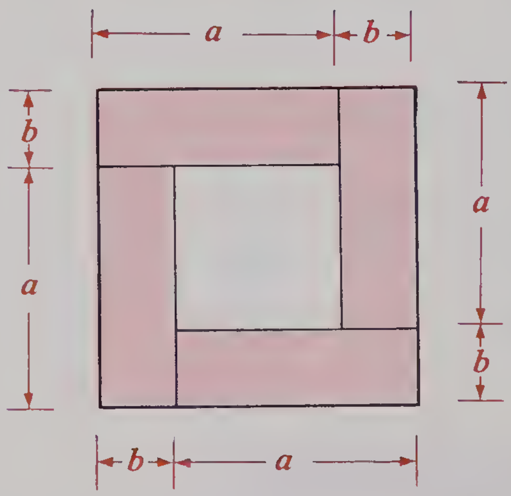
Ex. 1

2. Show that $(a-b)^2 = a^2-2ab+b^2$ by considering the adjoining figure.



Ex. 2

B **3.** Show that $(a+b)^2-(a-b)^2 = 4ab$ by considering the figure at the left.

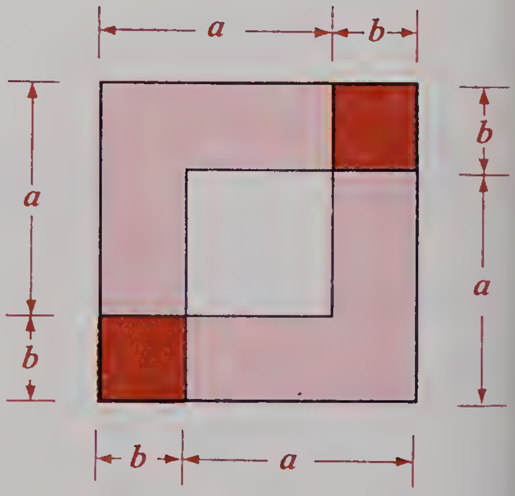


Ex. 3

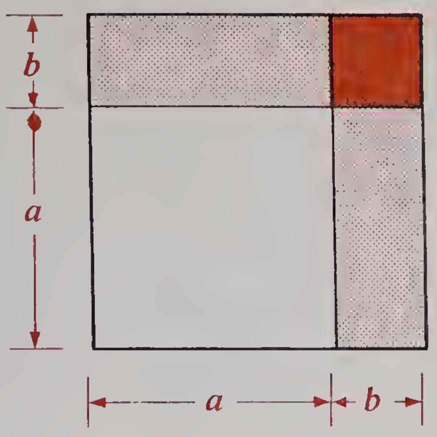
4. Show that

$$(a+b)^2+(a-b)^2 = 2(a^2+b^2)$$

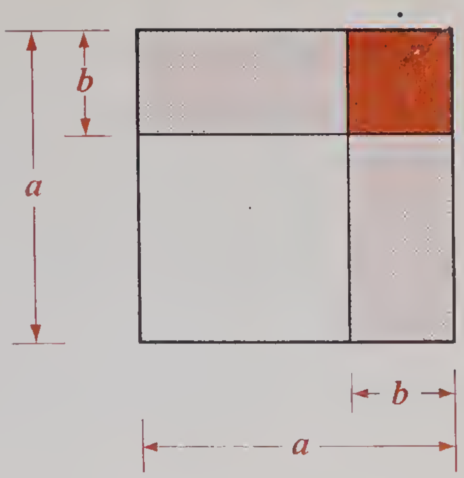
by considering the adjoining figure.



Ex. 4



Ex. 5



Ex. 6

5. Use the figure just above to show that

$$(a + b)^2 - b^2 = a^2 + 2ab.$$

6. Use the figure at right above to show that

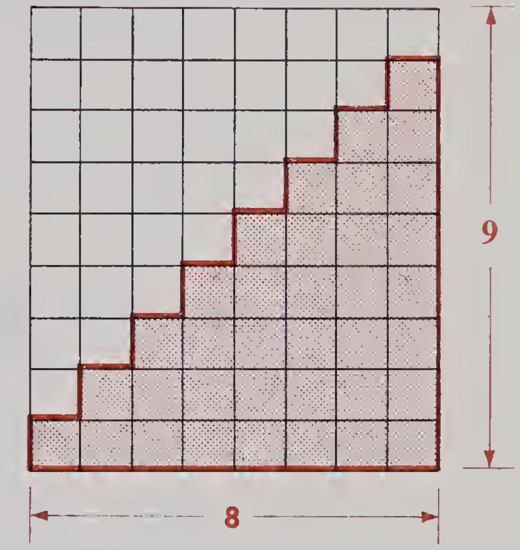
$$a^2 - (a - b)^2 = 2ab - b^2.$$

7. Use the adjoining figure to show that

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \frac{8(9)}{2}.$$

By the same method, show that for any positive integer n

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}.$$



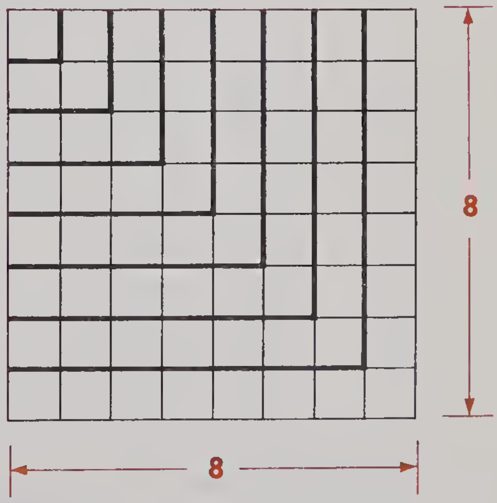
Ex. 7

8. Use the adjoining figure to show that

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = 8^2.$$

By the same method, show that for any positive integer n

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$



Ex. 8

7–6 Factoring a Trinomial Square

To factor a trinomial square, reverse the equations you use in squaring a binomial.

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$

Before you use one of these equations as a rule for factoring, be sure that the expression to be factored is a trinomial square. Examine each term of the trinomial to see how it may have been obtained. Consider $x^2 + 14x + 49$. Is the first term a square? Yes, x^2 is the square of x. Is the third term a square? Yes, 49 is 7^2. Is the middle term double the product of x and 7? Yes, $14x = 2(x)(7)$. Therefore, the trinomial is a square. Since all its terms are positive, it is the square of a sum;

$$x^2 + 14x + 49 = (x + 7)^2.$$

EXAMPLE. **Factor $196m^2 - 476mn + 289n^2$.**

Solution:

$$196m^2 \quad - \quad 476mn \quad + \quad 289n^2$$

$$14m \cdot 14m - 2 \cdot 14m \cdot 17n + 17n \cdot 17n$$

Thus, $196m^2 - 476mn + 289n^2 = (14m - 17n)^2$.

ORAL EXERCISES

Is each trinomial the square of a binomial? Justify your answer.

1. $x^2 + 2xy + y$ no
2. $a^2 + 2ab - b^2$ no
3. $r^2 + 4r + 4$ $(r+2)^2$
4. $w^2 + 2w + 4$ no
5. $z^2 + 4z + 2$ no
6. $x^2 + 2x + 1$ $(x+1)^2$
7. $4x^2 + 8x + 4$ $(2x+2)^2$
8. $9r^2 - 6x + 1$ no
9. $25x^2 - 10x + 1$ $(5x-1)^2$
10. $36x^2 + 12x + 1$ $(6x+1)^2$
11. $81t^2 + 18t + u^2$ no
12. $a^2b^2 + 6ab + 9$ $(ab+3)^2$
13. $x^2 - 2xy + y^2$ $(x-y$
14. $x^2 + 2xy - y^2$ no
15. $r^2 - 2rs + s$ no
16. $a^2 - 2a - 1$ no
17. $z^2 + 2z - 1$ no
18. $9r^2 - 18r + 1$ no
19. $25x^2 - 20x + 1$ no
20. $36w^2 - 24w + 1$ no
21. $a^2 - 20a + 100b^2$ no
22. $4a^2 - 24a + 9$ no
23. $x^2y^2 - 2xyz + z^2$ $(xy-z)^2$
24. $t^2u^2 + 2tuv + v^2$ $(tu+v)^2$

Caution pupils that some exercises may require rearrangement of terms and removal of common factors. Be sure that each final answer retains any common factor and is as complete as possi at this stage.

WRITTEN EXERCISES

Factor, and check.

A

1. $x^2 + 2bx + b^2$
2. $g^2 - 2gh + h^2$
3. $a^2 - 12a + 36$
4. $y^2 + 16y + 64$
5. $b^2 + 14b + 49$
6. $n^2 + 18n + 81$
7. $4a^2 - 4ab + b^2$
8. $25a^2 - 10ab + b^2$
9. $9x^2 + 6x + 1$
10. $16x^2 + 8x + 1$

11. $1 + 2n + n^2$
12. $1 + 4b + 4b^2$
13. $49x^2 - 28x + 4$
14. $25x^2 - 30x + 9$
15. $25a^2 + 60ab + 36b^2$
16. $144n^2 + 120nx + 25x^2$
17. $144x^2 - 24x + 1$
18. $16x^2 - 24xy + 9y^2$
19. $36a^2 + 60ab + 25b^2$
20. $16r^2 + 40rt + 25t^2$

21. $r^2 + 25 - 10r$
22. $4mn + n^2 + 4m^2$
23. $121a^2b^2 - 22ab + 1$
24. $243m^2n^2 + 54mn + 3$
25. $3k + 42k^2 + 147k^3$
26. $8n + 8n^2 + 2n^3$
27. $z^2 - 4a^2z + 4a^4$
28. $x^4 + 2x^2y + y^2$
29. $y^6 - 8y^3 + 16$
30. $n^4 - 2n^2 + 1$

B **31.** $x^4 - 18x^2 + 81$
32. $a^8 - 6a^4 + 9$
33. $x^2 - 4xy + 4y^2 - 9$

34. $a^2 + 6ab + 9b^2 - 1$
35. $m^2 - x^2 + 2x - 1$
36. $n^2 - y^2 - 6y - 9$

C Find k such that each trinomial will be a square.

37. $x^2 + kx + 9$ **38.** $x^2 - 2kx + 25$ **39.** $n^2 - 16n + k$

Exercises 37 and 38 have two solutions.

7–7 Multiplying Binomials at Sight

To learn to write the product of two binomials of the form $(ax + b)(cx + d)$ at sight, study these examples which apply the distributive property.

EXAMPLE 1. $(3x + 2)(5x - 7)$

Solution:

$$\begin{array}{rrr} 5x & -\ 7 & \\ 3x & +\ 2 & \\ \hline 15x^2 & -\ 21x & \\ & 10x & -\ 14 \\ \hline 15x^2 & -\ 11x & -\ 14 \end{array}$$

See 7-7 T.M. pg. 27 for teaching suggestion.

EXAMPLE 2. $(ax + b)(cx + d)$

Solution:

$$\begin{array}{rrr} & cx & +\ d \\ & ax & +\ b \\ \hline & acx^2 & +\ adx \\ & & bcx\ +\ bd \\ \hline acx^2 & +\ (ad + bc)x & +\ bd \end{array}$$

To write the terms in the trinomial product of two binomials at sight:

1. Multiply the first terms of the binomials.
2. Multiply the first term of each binomial by the last term of the other, and add these products.
3. Multiply the last terms of the binomials.

See 7-7 T.M. for comment on manipulation.

Each term of a trinomial like $15x^2 - 11x - 14$ has a special name. The first, $15x^2$, is the *quadratic term;* the second, $-11x$, is the *linear term;* and the third, -14, is the *constant term.* A **quadratic term** is a term of degree two. A **linear term** is a term of degree one. A **constant term** is a numerical term with no variable factor. The trinomial is itself called a *quadratic polynomial* because the term of the highest degree in it is a quadratic term.

ORAL EXERCISES

Give (a) the quadratic, (b) the linear, and (c) the constant terms, and (d) read the product as a quadratic trinomial.

a^2+6a+5 **1.** $(a + 1)(a + 5)$
b^2+5b+4 **2.** $(b + 4)(b + 1)$
$y^2+7y+10$ **3.** $(y + 2)(y + 5)$
$y^2+11y+28$ **4.** $(y + 4)(y + 7)$
$c^2+11c+28$ **5.** $(c + 4)(c + 7)$
$d^2+13d+40$ **6.** $(d + 8)(d + 5)$

7. $(s - 7)(s - 9)$ $s^2-16s+63$
8. $(t - 7)(t - 8)$ $t^2-15t+56$
9. $(u - 13)(u - 3)$ $u^2-16u+39$
10. $(w - 14)(w - 2)$ $w^2-16w+28$
11. $(a - 6)(a - 12)$ $a^2-18a+72$
12. $(b - 7)(b - 12)$ $b^2-19b+84$

13. $(x + 1)(x - 5)$ x^2-4
14. $(x + 4)(x - 5)$ x^2-x
15. $(x + 1)(x - 3)$ x^2-2
16. $(x - 3)(x + 2)$ x^2-x
17. $(y - 2a)(y + 5a)$ y^2+ -1
18. $(y - 3b)(y + 7b)$ $y^2+4by-2$

WRITTEN EXERCISES

Write each product.

A

1. $(4a + 3)(3a + 4)$
2. $(9z + 2)(3z + 1)$
3. $(5b - 9)(2b - 4)$
4. $(2y - 3)(7y - 5)$
5. $(3c - 2)(4c + 3)$
6. $(6x - 9)(7x + 5)$
7. $(2w + 11v)(3w - 5v)$
8. $(4h + 7k)(7h - 12k)$
9. $(2x + 5y)(8x - 3y)$
10. $(6x - 7y)(8x + 5y)$
11. $(x + 7)(-x + 3)$
12. $(-x + 5)(x + 6)$
13. $(-s - 3)(-s - 5)$
14. $(-t - 8)(-t - 2)$

B

15. $(-2y + 3)(y - 8)$
16. $(y + 5)(-3y - 4)$
17. $(-2n - 4)(-3n - 5)$
18. $(-5m - 2)(-2m - 3)$
19. $(-1.2m + 2.3)(.4m - .7)$
20. $(-3.1h - 3.2)(2.1h - 4.4)$

Solve, and check.

C

21. $(x + 5)(x - 5) = (x + 7)(x - 6)$
22. $(x - 4)(x + 4) = (x - 6)(x + 8)$

23. $(2r + 1)(8r - 3) = (4r - 1)^2$

24. $(9t - 1)(4t + 2) = (6t + 1)^2$

25. $x^2 - 1 = (x - 1)^2 + 2x$

26. $(x - 2)(x - 1) = (x + 1)^2 - (5x - 1)$

7–8 Factoring the Product of Binomial Sums or Differences

To factor the trinomial $x^2 + 5x + 6$, you recall that the product $(x + r)(x + s)$ is a similar trinomial. Compare the two.

$$\begin{array}{lccc} (x + r)(x + s) = & x^2 & + (r + s)x & + rs \\ & \updownarrow & \updownarrow & \updownarrow \\ & x^2 & + 5x & + 6 \end{array}$$

You see that were $(r + s) = 5$ and $rs = 6$, they would be exactly alike. With these clues, you can find two integers such that

$$rs = 6 \qquad \text{and} \qquad r + s = 5.$$

Stress the reasoning here.
See T. M. pg. 27 (2).

Observe that the product of the desired integers is positive, indicating that r and s are both positive or both negative. Observe next that their sum is positive, showing r and s both to be positive. There are two ways to express 6 as the product of two positive integers:

$$\begin{array}{ccc} 6 = \overset{\overset{\displaystyle\uparrow}{}}{1} \cdot 6 & \text{and} & 6 = 2 \cdot 3 \\ \begin{array}{cc} \uparrow & \uparrow \\ r & s \\ \downarrow & \downarrow \end{array} & & \begin{array}{cc} \uparrow & \uparrow \\ r & s \\ \downarrow & \downarrow \end{array} \\ r + s = 1 + 6 = 7 & & r + s = 2 + 3 = 5. \end{array}$$

The second set of factors satisfies both clues, so you conclude that

$$x^2 + 5x + 6 = (x + 2)(x + 3).$$

Of course, if you see the right pair of factors at the outset, there is no need to write the remaining possibilities. You check your conclusion by multiplying the factors, because the factored form is just another way of designating the original polynomial.

Factor: $x^2 - 11x + 24$

Solution: $x^2 - 11x + 24 = (x \quad)(x \quad)$
$= (x - \quad)(x - \quad)$
$= (x - 3)(x - 8)$, **Answer.**

Check: $(x - 3)(x - 8) = x^2 - 11x + 24$ ✓

Each of the quadratic polynomials you have factored has a positive constant term. Because $x^2 + 5x + 6$ has a positive linear term, the polynomial is expressed as a product of two binomial sums, $(x + 2)(x + 3)$. The second polynomial, $x^2 - 11x + 24$, having a negative linear term, factors into the product of two differences, $(x - 3)(x - 8)$.

Not every quadratic trinomial can be written as a product of binomials having integral coefficients. To factor $x^2 + 10x + 8$ you would have to find positive integers r and s such that

$$rs = 8 \qquad \text{and} \qquad r + s = 10.$$

The two ways of writing 8 as a product of positive integers are:

See T.M. pg. 27 (3).

$$8 = \underset{\substack{\uparrow \\ r}}{1} \cdot \underset{\substack{\uparrow \\ s}}{8} \qquad \text{and} \qquad 8 = \underset{\substack{\uparrow \\ r}}{2} \cdot \underset{\substack{\uparrow \\ s}}{4}$$

$$r + s = 9 \qquad\qquad\qquad r + s = 6.$$

In each case, $r + s \neq 10$; therefore, $x^2 + 10x + 8$ cannot be factored over the set of polynomials with integral coefficients. Such a polynomial is said to be *prime*, or *irreducible over this set of polynomials.* A polynomial which cannot be factored into polynomials of lower degree belonging to a designated set is said to be prime over that set of polynomials.

ORAL EXERCISES

For each trinomial, tell which two factors of the constant term have a sum equal to the coefficient of the linear term.

1. $x^2 + 6x + 8$ 2, 4
2. $x^2 + 5x + 6$ 2, 3
3. $s^2 + 8s + 15$ 3, 5
4. $t^2 + 12t + 35$ 5, 7
5. $w^2 - 7w + 10$ -2, -5
6. $w^2 - 13w + 22$ -2, -11
7. $r^2 - 18r + 77$ -7, -11
8. $z^2 - 10z + 21$ -3, -7
9. $v^2 + 16v + 55$ 5, 11
10. $m^2 + 19m + 34$ 2, 17
11. $y^2 - 15y + 26$ -2, -13
12. $x^2 - 7x + 12$ -3, -4
13. $x^2 + 5x + 4$ 1, 4
14. $x^2 - 9x + 8$ -1, -8
15. $y^2 - 9y + 18$ -3, -
16. $r^2 + 10r + 9$ 1, 9
17. $x^2 + 13x + 12$ 1, 12
18. $w^2 + 8w + 7$ 1, 7

WRITTEN EXERCISES

Factor each trinomial, and check by multiplication.

A

1. $n^2 + 14n + 33$
2. $z^2 + 12z + 27$
3. $x^2 + 11x + 18$
4. $x^2 + 10x + 16$
5. $h^2 - 13h + 36$
6. $y^2 - 15y + 56$

7. $x^2 - 19x + 90$
8. $a^2 - 47a + 90$
9. $m^2 + 21m + 90$
10. $r^2 + 33r + 90$
11. $33 - 34b + b^2$
12. $14 - 15k + k^2$
13. $42 + 17c + c^2$
14. $52 + 17s + s^2$
15. $x^2 + 20x + 51$
16. $y^2 + 52y + 51$
17. $x^2 + 14xy + 24y^2$
18. $y^2 + 26yz + 48z^2$
19. $m^2 - 22mn + 72n^2$
20. $s^2 - 21st + 20t^2$
21. $b^2 - 23bc + 76c^2$
22. $a^2 - 22ab + 57b^2$
23. $z^2 - 29zb + 120b^2$
24. $z^2 - 23zd + 120d^2$

Determine all integral values of b for which the trinomial can be factored over the set of binomials with integral coefficients.

SAMPLE. $x^2 + bx + 12$

Solution: 12 can be factored into a product of two integers as follows:

$1 \cdot 12,\ 2 \cdot 6,\ 3 \cdot 4,\ (-1)(-12),\ (-2)(-6),\ (-3)(-4).$

The corresponding values of b are

$13, 8, 7, -13, -8, -7$, **Answer.**

Ⓑ **25.** $y^2 + by + 20$
26. $z^2 + bz + 63$
27. $x^2 + bx + 1$
28. $y^2 + by + 4$
29. $m^2 + 2bm + 36$
30. $n^2 + 2bn + 44$

Find all positive integers c for which each trinomial can be factored over the set of binomials with integral coefficients.

31. $x^2 + 5x + c$
32. $x^2 + 7x + c$
33. $y^2 - 6y + c$

Show that each polynomial is prime over the set of polynomials with integral coefficients.

34. $x^2 + 3x + 7$
35. $x^2 + 15x + 9$
36. $y^2 - 4y + 1$
37. $y^2 - 3y + 3$
38. $x^2 + 4$
39. $x^2 + 9$

7–9 Factoring the Product of a Binomial Sum and a Binomial Difference

To factor $x^2 + 2x - 15$, proceed as before to look for r and s such that

$$x^2 + 2x - 15 = (x + r)(x + s) = x^2 + (r + s)x + rs.$$

Your clues are:

This is the reasoning in 7-8. Again, emphasize the reasoning.

$$rs = -15 \quad \text{and} \quad r + s = 2.$$

Here, the product rs is negative, indicating that one integer, say r, must be positive, while s must be negative. But, $r + s$ is 2, which means that r, the positive member of the pair, must have the greater absolute value. On the basis of these conclusions, consider:

$$\underset{\underset{r}{\uparrow}}{15}(\underset{\underset{s}{\uparrow}}{-1}) \qquad \text{and} \qquad \underset{\underset{r}{\uparrow}}{5}(\underset{\underset{s}{\uparrow}}{-3})$$

$$r + s = 14 \qquad\qquad r + s = 2$$

$$\therefore x^2 + 2x - 15 = (x + 5)(x - 3).$$

In factoring $x^2 - 2x - 15$, you would search for two integers of opposite sign, but with a negative sum. Therefore, the negative integer would have to have the larger absolute value. This trinomial is factored:

$$x^2 - 2x - 15 = (x - 5)(x + 3).$$

WRITTEN EXERCISES

Find the factors of each trinomial.

A

1. $x^2 + 3x - 10$
2. $y^2 + 5y - 14$
3. $y^2 - 4y - 21$
4. $x^2 - 2x - 15$
5. $h^2 - 2h - 63$
6. $n^2 + n - 56$
7. $k^2 + k - 110$
8. $w^2 - 4w - 96$
9. $r^2 - 2r - 99$
10. $x^2 - 13x - 30$
11. $u^2 - 6u - 55$
12. $b^2 + b - 132$
13. $x^2 - 8x - 33$
14. $n^2 - 8n - 48$
15. $x^2 - x - 90$
16. $b^2 - 5b - 24$
17. $z^2 - 3zt - 4t^2$
18. $t^2 - 3tu - 10u^2$
19. $-y^2 - yz + 2z^2$
20. $-x^2 - xy + 6y^2$
21. $a^2 + 3a - 40$
22. $c^2 + 9c - 36$
23. $d^2 - 6d - 16$
24. $m^2 - m - 30$

Determine all integral values of b for which the trinomial can be factored.

B

25. $x^2 + bx - 12$
26. $x^2 + bx - 14$
27. $y^2 + by - 4$
28. $y^2 + by - 16$
29. $t^2 + 2bt - 28$
30. $z^2 + 2bz - 20$

Find the two negative integers c of least absolute value for which each trinomial can be factored.

31. $x^2 + 5x + c$
32. $x^2 + 7x + c$
33. $y^2 - 6y + c$

Show that each of the following polynomials is prime over the set of polynomials with integral coefficients.

C
34. $x^2 + 8x - 7$
35. $x^2 + 5x - 4$
36. $y^2 - 4y - 3$
37. $y^2 - 6y - 5$
38. $y^2 - 3$
39. $y^2 - 6$

This section is optional in some courses. See 7-10 T.M. pg. 27.

7–10 General Method of Factoring Quadratic Trinomials

To factor a quadratic trinomial product whose quadratic term has a coefficient other than 1, you may use inspection and trial, as in this example:

Factor $6x^2 - 25x + 14$.

See T.M. pg. 27 (4) for alternative method.

First clue: The constant term is positive, and the linear term is negative.

$\therefore$ Both binomial factors are differences.

Second clue: The product of the linear terms of the binomials is $6x^2$, and the product of the constant terms of the binomials is 14.

$\therefore$ The possibilities to consider are as follows:

Possible Factors	*Corresponding Linear Terms*
$(x - 1)(6x - 14)$	$-14x - 6x = -20x$
$(x - 14)(6x - 1)$	$-x - 84x = -85x$
$(x - 2)(6x - 7)$	$-7x - 12x = -19x$
$(x - 7)(6x - 2)$	$-2x - 42x = -44x$
$(2x - 1)(3x - 14)$	$-28x - 3x = -31x$
$(2x - 14)(3x - 1)$	$-2x - 42x = -44x$
$(2x - 2)(3x - 7)$	$-14x - 6x = -20x$
$(2x - 7)(3x - 2)$	$-4x - 21x = -25x$

Third clue: The linear term of the trinomial is $-25x$. Only the last possibility satisfies all three clues.

$\therefore 6x^2 - 25x + 14 = (2x - 7)(3x - 2)$, **Answer.**

One other clue can help reduce the number of possible factors. If a trinomial has no common factor, none of its binomial factors can have a common factor. Thus, the above factor combinations containing $6x - 14$, $6x - 2$, $2x - 14$, or $2x - 2$ can be discarded at once, since each contains a common factor not in the given trinomial.

EXAMPLE. **Factor: $8x^2 + 2x - 3$**

Solution:

$$\begin{aligned} 8x^2 + 2x - 3 &= (\quad - \quad)(\quad + \quad) \\ &= (\quad - 1)(\quad + 3) \\ &= (2x - 1)(4x + 3), \text{ Answer.} \end{aligned}$$

Check: $(2x - 1)(4x + 3) = 8x^2 + 2x - 3$ ✓

WRITTEN EXERCISES

Find the factors of each trinomial.

Ⓐ **1.** $2y^2 + 7y + 3$
2. $3x^2 + 7x + 2$
3. $3n^2 - 4n + 1$
4. $3x^2 - 5x + 2$
5. $5x^2 - 2x - 7$
6. $2x^2 - 9x - 5$
7. $3a^2 + 2a - 1$
8. $5a^2 + 4a - 1$
9. $8x^2 - 14x + 3$
10. $3x^2 + 20x - 7$
11. $35y^2 - 22y + 3$
12. $13y^2 - 7y - 6$

Ⓑ **13.** $6x^2 + 25x + 21$
14. $14x^2 + 33x + 10$
15. $14x^2 - 15x - 11$
16. $12x^2 + 28xy - 5y^2$
17. $24x^2 - 14xy - 3y^2$
18. $6n^2 - 47ns - 63s^2$
19. $8m^2 + 14m - 15$
20. $10y^2 - 11y - 6$

Express each dimension as a binomial with integral coefficients.

Ⓒ **21.** The area of a rectangle is $6x^2 - x - 12$. What are the possible dimensions of the rectangle?

22. The area of a rectangle is $4n^2 + 3n - 10$. What are the possible dimensions of the rectangle?

23. What are the possible height and base of a triangle whose area is $15a^2 + 38a - 21$?

24. What are the possible height and base of a triangle whose area is $14b^2 - 25b - 25$?

EXTENSION OF FACTORING

7–11 Combining Several Types of Factoring

Sometimes a common factor conceals:

The difference of two squares	A trinomial square	A trinomial product
$4y^3 - 36y$	$-x^2 + 4x - 4$	$ax^2 - 5axy - 50ay^2$
$= 4y(y^2 - 9)$	$= -1(x^2 - 4x + 4)$	$= a(x^2 - 5xy - 50y^2)$
$= 4y(y + 3)(y - 3)$	$= -1(x - 2)^2$	$= a(x + 5y)(x - 10y)$

Here is a procedure to follow in factoring:

1. Is there a common factor? If so, factor the polynomial by finding the greatest common factor. Then consider each factor.
2. If a factor is a binomial, is it the difference of two squares? You can factor such a binomial.
3. If a factor is a trinomial, is it a square? You can factor a trinomial square.
4. If a factor is a trinomial which is not a square, assume that it is the product of two binomials, and search for them. Of course, you cannot factor a prime trinomial. But never decide that an expression is prime until you have tried all the ways you know of factoring.
5. If a factor is neither a binomial nor a trinomial, can you show a common polynomial factor by grouping? If you can, try to factor each of the resulting factors as in Steps **2**, **3**, and **4**.
6. After you factor a polynomial, write *all* the factors, including any monomial factor. The monomial factor may be a product, but all other factors should be prime; the *factoring should be complete.*
7. Always multiply the factors to see whether the product is the original expression; the *factoring should be correct.*

In general, to factor a polynomial product, first find any common factor; then find the binomial factors of the remaining product, and write all of the factors as an indicated product.

ORAL EXERCISES

In Written Exercises 1–27, state the common factor, if any.

Answers to Orals are written above Written Exercises 1–27.

If you have omitted Section 7-10, see 7-11 T.M. pg. 27.

WRITTEN EXERCISES

When factoring a polynomial, do not consider 1 as a common factor.

Factor each expression.

A

1. $3x^2 \overset{3}{-} 27$
2. $14y^2 \overset{14}{-} 56$
3. $2w^2 \overset{2}{+} 36w + 162$
4. $8z^2 \overset{8}{+} 112z + 392$
5. $6x^2 \overset{\text{None}}{-} 11x + 5$
6. $5r^2 \overset{\text{None}}{-} 13r + 6$
7. $49n^2 \overset{\text{None}}{-} 14n + 1$
8. $25x^2 \overset{\text{None}}{-} 20x + 4$
9. $-5ax^2 \overset{-5a}{-} 5ay^2$
10. $-rt^3 \overset{-rt}{-} r^3t$
11. $9x^2 \overset{\text{None}}{-} 24x + 16$
12. $4x^2 \overset{\text{None}}{-} 28x + 49$
13. $169a^2 \overset{\text{None}}{-} 49b^4$
14. $49y^4 \overset{\text{None}}{-} 144z^2$
15. $n^2 \overset{\text{None}}{-} 18n - 40$
16. $n^2 \overset{\text{None}}{-} 10n - 24$
17. $x^2 \overset{\text{None}}{+} 5x - 84$
18. $x^2 \overset{\text{None}}{+} 2x - 48$
19. $21 \overset{\text{None}}{-} 4m - m^2$
20. $161 \overset{\text{None}}{-} 160u - u^2$
21. $5s \overset{\text{None}}{+} 104 - s^2$
22. $3r \overset{\text{None}}{+} 180 - r^2$
23. $5 \overset{5}{-} 25y - 30y^2$
24. $3 \overset{3}{-} 21y - 24y^2$
25. $28n^2 \overset{\text{None}}{+} 87n + 54$
26. $6w^2 \overset{6}{-} 96w - 342$
27. $4t^2 \overset{4}{-} 28t - 480$

B

28. $az^2 - 12awz + 36aw^2$
29. $bx^2 - 14bxy + 49by^2$
30. $6y^3 + 3y^2 - 3y$
31. $6v^3 + 26v^2 + 8v$
32. $-x^2 + 2bx - b^2$
33. $-k^2 + 2kh - h^2$
34. $2n - n^2 - 1$
35. $4b - 4b^2 - 1$
36. $-14 - 9z - z^2$
37. $-16 - 10k - k^2$
38. $7a^2(a + 1) - 5a(a + 1) - 2(a + 1)$
39. $4b^2(b + 2) - 3b(b + 2) - (b + 2)$
40. $18x^2(x + 1) + 24x(x + 1) + 8(x + 1)$
41. $147y^2(y - 1) - 42y(y - 1) + 3(y - 1)$
42. $x^2 - 1 + 5x(x^2 - 1) + 4x^2(x^2 - 1)$
43. $y^2 - 4 + y(y^2 - 4) - 2y^2(y^2 - 4)$
44. $2w^4 - 162$
45. $3m^4 - 1875$
46. $-20y^2 + 43xy - 14x^2$
47. $-8m^2 + 50mn - 33n^2$

C

48. $x^4 - 7x^2 - 60$
49. $n^4 + 13n^2 - 30$
50. $y^4 - 5y^2 + 4$
51. $m^4 - 13m^2 + 36$
52. $5ax^2 - by^2 + bx^2 - 5ay^2$
53. $m^2r^2 - 9s^2 - 9r^2 + m^2s^2$
54. $y^3 - y^2 + y - 1$
55. $x^3 + x^2 - x - 1$

In Exercises 56–59, the given binomial is a factor of the trinomial over the set of polynomials with integral coefficients. Determine c in each case.

56. $2x - 3;\ 10x^2 - 3x + c$

57. $3y + 2;\ 21y^2 - y + c$

58. $5w + 1;\ cw^2 + 3w - 1$

59. $n - 3;\ cn^2 - 5n - 21$

7–12 Working with Factors Whose Product Is Zero

If you know that the product of two numbers is zero, what can you say about the numbers?

$$\text{Let } ab = 0 \quad \text{and} \quad a \neq 0.$$

See 7-12 T.M. pg. 28.

Since $a \neq 0$, the reciprocal of a, $\frac{1}{a}$, exists and is not zero. Using the multiplication property of equality, multiply each of the terms ab and 0 by $\frac{1}{a}$:

$$\frac{1}{a}(ab) = \frac{1}{a}\cdot 0.$$

$$\frac{1}{a}(ab) = \frac{1}{a}(0)$$

$$\left(\frac{1}{a}\cdot a\right)b = 0$$

$$1 \cdot b = 0$$

$$b = 0$$

On the left, use the associative property, the product of reciprocals, and the multiplicative property of 1.

On the right, use the multiplicative property of 0.

Emphasize this step. Here you use the fact that the product is zero. See T.M. pg. 28 (6).

Similarly, if $ab = 0$ and $b \neq 0$, you can show that a must equal 0. And, of course, if either $a = 0$ or $b = 0$, $ab = 0$.

A product is zero if, and only if, at least one of the factors is zero.

Point out the significance of "if, and only if." See T.M. pg. 28 (5).

WRITTEN EXERCISES

Solve each equation.

SAMPLE. $(x - 1)(x + 4) = 0$

Solution: If $(x - 1)(x + 4) = 0$, either $(x - 1)$ or $(x + 4)$ must be zero. Thus, the solution sets of $x - 1 = 0$ and $x + 4 = 0$ together form the solution set of $(x - 1)(x + 4) = 0$.

$\therefore$ The solution set is $\{1, -4\}$, **Answer.**

A

1. $13(a - 2) = 0$
2. $15(a - 8) = 0$
3. $-11(5 + c) = 0$
4. $37\left(16 - \frac{2}{y}\right) = 0$
5. $29\left(\frac{1}{b} + \frac{2}{7}\right) = 0$
6. $-31\left(\frac{1}{b} + \frac{3}{7}\right) = 0$
7. $24(12 + c) = 0$
8. $0(z + 9) = 0$
9. $0(z - 19) = 0$
10. $107\left(18 - \frac{3}{y}\right) = 0$
11. $0\left(\frac{2}{u} - \frac{4}{3}\right) = 0$
12. $0\left(\frac{9}{2} - \frac{3}{v}\right) = 0$

B

13. $(x - 3)(x - 5) = 0$
14. $(x + 2)(x + 7) = 0$
15. $0 = (y + 10)(y - 7)$
16. $0 = (y - 8)(y + 3)$
17. $0 = v(2v + 1)$
18. $0 = v(3v + 1)$
19. $(5r + 1)(2r - 6) = 0$
20. $(6r + 3)(7r + 14) = 0$

7-13 Solving Polynomial Equations by Factoring

A **polynomial equation** is an equation whose left and right members are polynomials. A polynomial equation is in *standard form* when one of its members is zero and the other is a polynomial in which all similar terms have been combined. Standard form for $x^2 - 20x = 300$ is $\mathbf{x^2 - 20x - 300 = 0}$. This equation is of degree two and is called a **quadratic equation**. The **degree of a polynomial equation** is the greatest of the degrees of the terms of the equation when written in standard form. An equation of degree one, like $3x + 5 = 2x - 1$, is called a **linear equation**, while an equation like $x^3 = 4x^2 + 5x$, whose degree is three, is a **cubic equation**. If you transform a polynomial equation into standard form, and if you can factor the left member, then you obtain its roots by finding the numbers for which at least one of those factors is zero.

EXAMPLE 1. Solve $x^2 - 20x = 300$.

Separating the factors with a vertical line reduces error.

Solution:

		$x^2 - 20x = 300$
1. Put the equation into standard form.		$x^2 - 20x - 300 = 0$
2. Factor the left member.		$(x - 30)(x + 10) = 0$
3. Set each factor equal to zero.	$x - 30 = 0$	$x + 10 = 0$
4. Solve the resulting linear equations.	$x = 30$	$x = -10$
5. Check each apparent root in the original equation.		$x^2 - 20x = 300.$

Stress application of the general principle.

$$\begin{aligned}(30)^2 - 20(30) &\overset{?}{=} 300 \\ 900 - 600 &\overset{?}{=} 300 \\ 300 &= 300 \checkmark\end{aligned} \qquad \begin{aligned}(-10)^2 - 20(-10) &\overset{?}{=} 300 \\ 100 + 200 &\overset{?}{=} 300 \\ 300 &= 300 \checkmark\end{aligned}$$

$\therefore$ The solution set is $\{30, -10\}$, Answer.

Several situations may arise when you try to solve a polynomial equation by factoring. First, the polynomial may have a common numerical factor. Since such a factor would be a nonzero number, you should eliminate it by applying the division property of equality. Second, two or more factors may be identical. Such factors will yield a *double* or *multiple root*, which should be written only once in the roster of the solution set. Of course, to use this method, you must be able to factor the polynomial in the equation.

EXAMPLE 2. Solve $3x^2 - 12x + 12 = 0$.

Solution:

$$3x^2 - 12x + 12 = 0$$

1. Divide each member by 3. $\quad x^2 - 4x + 4 = 0$
2. Factor the left member. $\quad (x - 2)(x - 2) = 0$
3. Set each factor equal to zero. $\quad x - 2 = 0 \mid x - 2 = 0$
4. Solve the resulting linear equations. $\quad x = 2 \mid x = 2$
5. Check in the original equation.

$$\begin{aligned}3x^2 - 12x + 12 &\overset{?}{=} 0 \\ 3(2)^2 - 12(2) + 12 &\overset{?}{=} 0 \\ 12 - 24 + 12 &\overset{?}{=} 0 \\ 0 &= 0 \checkmark\end{aligned}$$

$\therefore$ The solution set is $\{2\}$, Answer.

In other problems you may have a common monomial factor which contains a variable. Such a factor may be zero and give you a root. Therefore, you should not eliminate it by division. The following cubic equation illustrates this situation.

EXAMPLE 3. Solve $x^3 = 4x^2 + 5x$. ← Emphasize this example. Pupils often lose a root by dividing by x or other variable factors.

Solution:

$$x^3 = 4x^2 + 5x$$

1. Transform the equation into standard form. $\quad x^3 - 4x^2 - 5x = 0$
2. Factor the left member. $\quad x(x^2 - 4x - 5) = 0$

$$x(x - 5)(x + 1) = 0$$

(*cont. on p.* 266)

3. Set each factor equal to zero. $x = 0 \;\big|\; x - 5 = 0 \;\big|\; x + 1 = 0$
4. Solve the resulting equations. $x = 0 \;\big|\; x = 5 \;\big|\; x = -1$
5. **Check** each value in the original equation. $x^3 = 4x^2 + 5x$

$$\begin{array}{ccc|ccc|ccc}
(0)^3 & \overset{?}{=} & 4(0)^2 + 5(0) & (5)^3 & \overset{?}{=} & 4(5)^2 + 5(5) & (-1)^3 & \overset{?}{=} & 4(-1)^2 + 5(-1) \\
0 & \overset{?}{=} & 0 + 0 & 125 & \overset{?}{=} & 100 + 25 & -1 & \overset{?}{=} & 4 - 5 \\
0 & = & 0 \checkmark & 125 & = & 125 \checkmark & -1 & = & -1 \checkmark
\end{array}$$

$\therefore$ The solution set is $\{0, 5, -1\}$, Answer.

WRITTEN EXERCISES

Find the solution set of each equation.

Ⓐ

1. $w^2 + w - 90 = 0$
2. $w^2 + 5w - 14 = 0$
3. $t^2 - 2t = 15$
4. $s^2 - 3s = 18$
5. $x^2 + 8x = -15$
6. $y^2 + 11y = -18$
7. $y^2 + 3 = 4y$
8. $x^2 + 5 = 6x$
9. $x^2 = 108 + 3x$
10. $m^2 = 66 + 5m$
11. $25y^2 - 100 = 0$
12. $h^2 - 256 = 0$
13. $4z^2 = 25$
14. $9z^2 = 16$
15. $x^2 - 24x = 0$
16. $y^2 - 16y = 0$
17. $2r^2 + 9r + 10 = 0$
18. $3r^2 + 13r + 14 = 0$
19. $2n^2 - 11n = -5$
20. $3x^2 + 9x = 0$
21. $6n^2 - 15n = 0$
22. $2r^2 - 17r = -21$
23. $3k^2 + 10 = 17k$
24. $10t^2 + 3 = 11t$
25. $3x^2 + 6x = 144$
26. $4x^2 + 8x = 140$
27. $7t^2 - 35t = 168$
28. $8y^2 - 48y = 216$
29. $x^2 + 9 = 6x$
30. $16 + 8m + m^2 = 0$
31. $4m = m^2$
32. $6s^2 + s = 0$

Ⓑ

33. $n^3 - 6n^2 - 40n = 0$
34. $t^3 + 8t^2 - 84t = 0$
35. $4r^3 + 4r^2 + r = 0$
36. $9s^3 - 12s^2 + 4s = 0$
37. $x^4 - 17x^2 + 16 = 0$
38. $y^4 - 10y^2 + 9 = 0$

Ⓒ

39. $(2x - 1)^2 + 3x(x - 3) = 3(x - 2)(x - 1) - 5$
40. $(3y^2 - 2)^2 - (2 + 5y^2) = (5y^2 + 1)(y^2 - 2)$

41. $x^5 - 5x^3 + 4x = 0$ **42.** $y^5 - 8y^3 + 16y = 0$

Find an equation of the lowest degree having the given solution set.

SAMPLE. $x \in \{0, -1, 2\}$

Solution: Since $x = 0, x = -1$, or $x = 2$,

then $x = 0, x + 1 = 0$, or $x - 2 = 0$,

$$\begin{aligned} \therefore x(x + 1)(x - 2) &= 0. \\ x(x^2 - x - 2) &= 0, \\ x^3 - x^2 - 2x &= 0, \text{ Answer.} \end{aligned}$$

43. $y \in \{1, 3\}$ **45.** $w \in \{0, 1, -2\}$ **47.** $p \in \{-1, 1, 3\}$

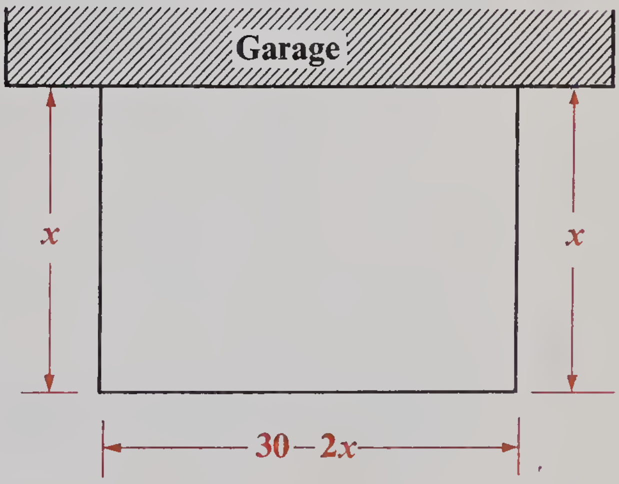
44. $z \in \{-1, -2\}$ **46.** $r \in \{0, 2, 4\}$ **48.** $y \in \{-3, -2, 2, 3\}$

7–14 Using Factoring in Problem Solving

With the ability to solve polynomial equations by factoring, you can solve problems which would have baffled you only a few weeks ago. You must exercise your judgment, rejecting answers which are not sensible in the light of the conditions of the problem.

EXAMPLE 1. **Mr. Gardner wishes to start a 100-square foot vegetable patch. Since he has only 30 feet of short chicken-wire fencing, he fences three sides of a rectangle, letting his garage wall act as the fourth side of the enclosure. How wide is the garden?**

Solution:



Let x **= number of feet in width;**

Then $(30 - 2x)$ **= length in feet,**

and $x(30 - 2x)$ **= area in square feet.**

$$\begin{aligned} x(30 - 2x) &= 100 \\ 30x - 2x^2 &= 100 \\ 2x^2 - 30x + 100 &= 0 \quad \text{(Since } 2 \neq 0\text{, divide by 2.)} \\ x^2 - 15x + 50 &= 0 \\ (x - 5)(x - 10) &= 0 \end{aligned}$$

$$\begin{array}{r|l} x - 5 = 0 & x - 10 = 0 \\ x = 5 & x = 10 \end{array}$$

4 *Check:* **Is the area of the vegetable patch 100 square feet?**

If $x = 5,\ 30 - 2x = 20.$ $\qquad 5 \times 20 = 100$ ✓

If $x = 10,\ 30 - 20 = 10.$ $\qquad 10 \times 10 = 100$ ✓

The width is 5 feet or 10 feet, Answer.

The next problem has only one solution, although the quadratic equation used in solving it has two roots. The problem is interesting also because it employs a most important rule: $d = rt + 16t^2$. This rule applies to any object falling freely to the ground.

EXAMPLE 2 **An object is ejected directly downward from an airplane 9600 feet above the ground. It starts to fall at 160 feet per second. How many seconds elapse before the object hits the ground?**

Solution:

1 **Let** t = **number of seconds before the object hits the ground.**

d = **number of feet of fall = 9600 feet.**

r = **rate at which fall starts = 160 feet per second.**

2

$$d = rt + 16t^2$$
$$9600 = 160t + 16t^2$$

3

$$16t^2 + 160t - 9600 = 0$$
$$t^2 + 10t - 600 = 0$$
$$(t - 20)(t + 30) = 0$$
$$t - 20 = 0 \quad \Big| \quad t + 30 = 0$$
$$t = 20 \quad \Big| \quad t = -30 \quad \text{(Rejected)}$$

4 **Does an object take 20 seconds to fall 9600 feet when it starts falling at the rate of 160 feet per second?**

$$d = rt + 16t^2$$
$$9600 \overset{?}{=} (160)(20) + (16)(20)^2$$
$$9600 \overset{?}{=} 3200 + 6400$$
$$9600 = 9600 \checkmark$$

The number of seconds elapsed is 20, Answer.

The root -30 is rejected because the object could not have hit the ground before it was thrown. Remember that you solve the problem by reasoning that if t satisfies the requirements stated in the problem, then t must satisfy the equation obtained in Step 3. On the other hand, just because a value of t satisfies the equation, you cannot conclude that the value will satisfy the problem. The solution set of the equation gives the *possible solutions* of the problem. By checking these possibilities in the words of the problem, you find the *actual solutions* of the problem. See 7-14 T.M. pg. 28.

PROBLEMS

Solve each problem, rejecting roots that do not fulfill its conditions.

A

1. A rectangle is 5 feet longer than it is wide. The area of the rectangle is 66 square feet. Find its dimensions.

2. A rectangle is 2 yards longer than it is wide. The area of the rectangle is 99 square yards. Find its dimensions.

3. An airplane can release objects with a downward speed of 64 feet per second. If an object is released when the airplane is at an altitude of 7680 feet, within how much time will the object hit the ground?

4. An object is thrown from an airplane at an altitude of 11,200 feet. It starts falling at 48 feet per second. Find how soon the object reaches the ground.

5. The number of telephone connections c that can be made through a switchboard to which n telephones are connected is given by the equation $c = \frac{1}{2}n(n - 1)$. If a switchboard operator can make 325 connections through her board, how many telephones are connected to it?

6. The number of straight lines n that can connect p points is given by the equation $n = \dfrac{p(p - 1)}{2}$. How many points has a figure if only 15 lines can be drawn connecting them?

7. Find two consecutive odd integers the sum of whose squares is 202.

8. Find two consecutive even integers the sum of whose squares is 100.

9. A builder decorates 36 square meters of a courtyard with a triangular garden whose base is 1 meter longer than its height. Find the base and height of the triangular garden.

10. Find the dimensions of a triangle whose area is 42 square centimeters and whose base and height together measure 19 centimeters.

The equation $h = rt - 16t^2$ is needed to solve the next four problems. It gives the height h, in feet, which an object will reach in t seconds when it is thrown upward with a starting speed of r feet per second.

11. A ball was thrown upward with a starting speed of 64 feet per second. In how many seconds did it reach a height of 64 feet?

12. A Fourth of July rocket was shot directly upward with a starting speed of 96 feet per second. In how many seconds did it reach a height of 144 feet?

13. A bullet left a gun at 1600 feet per second. In how many seconds did the bullet hit the balloon 4656 feet directly overhead?

14. A man shot at a balloon 2080 feet directly above him. The bullet left the gun with a muzzle speed of 2096 feet per second. How soon did the bullet reach the balloon?

B **15.** The plowed area of a field is a rectangle 80 feet by 120 feet. The owner plans to plow an extra strip of uniform width on each of the four sides of the field, in order to double the plowed area. How many feet should he add to each dimension of the field?

16. To make room for a barbecue pit, a man cuts the area of his garden in half by subtracting equal amounts from its length and width. If the garden originally is 30 feet by 40 feet, by how much should he reduce each dimension?

17. A ball is thrown directly upward with a starting speed of 48 feet per second. (a) When will the ball be 32 feet above the ground? Explain the two roots. (b) When will it return to the ground?

18. A bullet is fired directly upward with a muzzle velocity of 3216 feet per second. (a) When will the bullet be 3200 feet above the ground? Explain the two roots. (b) In how many minutes will it hit the ground?

19. What is the error in the following argument?

Be sure to assign this problem.

$$x^2 - 1 = 12$$

$$\therefore (x + 1)(x - 1) = 12 = 6 \cdot 2$$

$$\therefore x + 1 = 6 \quad \Big| \quad x - 1 = 2$$

$$x = 5 \quad \Big| \quad x = 3$$

$$\therefore \text{The solution set is } \{3, 5\}.$$

20. Jo is n years old. Her brother Al is n^2 years old. In 8 years, Al will be twice as old as Jo is then. How old is Al now?

21. In a right triangle, the hypotenuse exceeds a leg by 2 inches, and the perimeter is 60 inches. If the area is 120 square inches, find the hypotenuse.

22. Show that the sum of the squares of any two consecutive numbers is an odd number.

County Agents and Mathematics

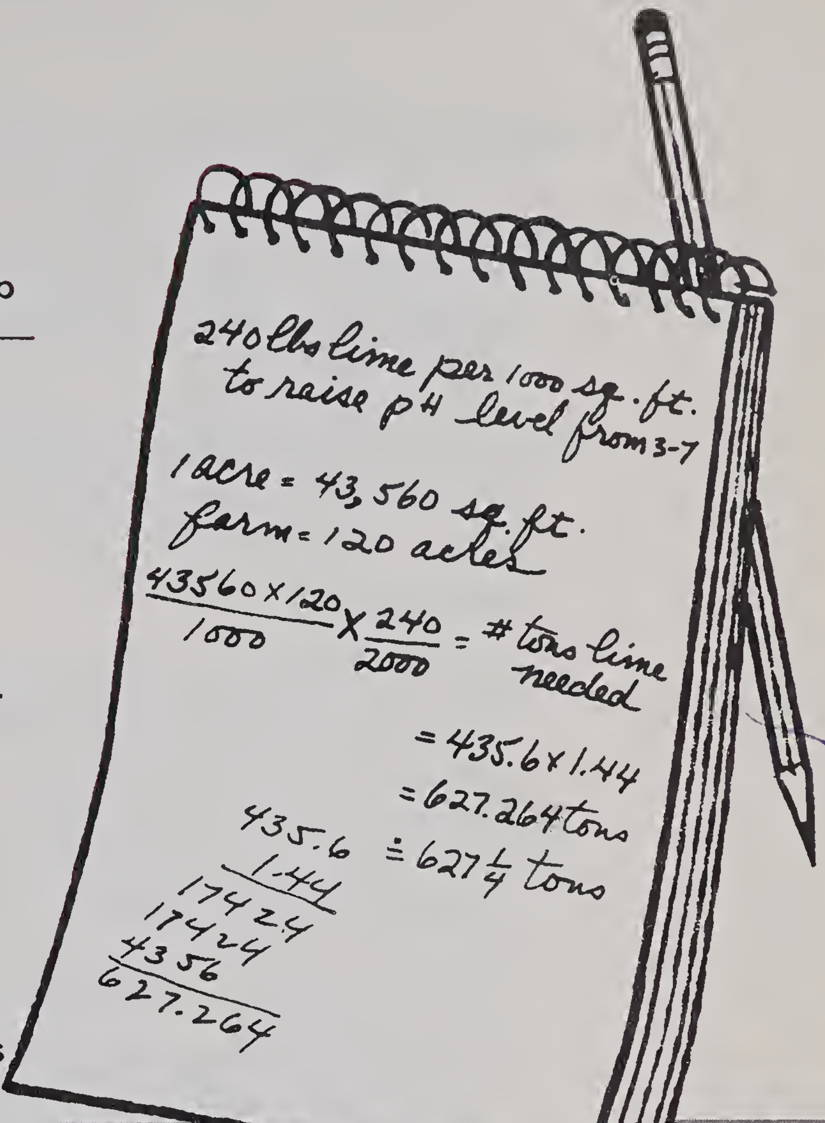
County agents are employed by federal, state, and county authorities to assist rural communities. Their chief tasks are to aid farmers with specific problems — such as crop failure, soil erosion, or marketing of produce — and to keep them informed of developments in agricultural research.

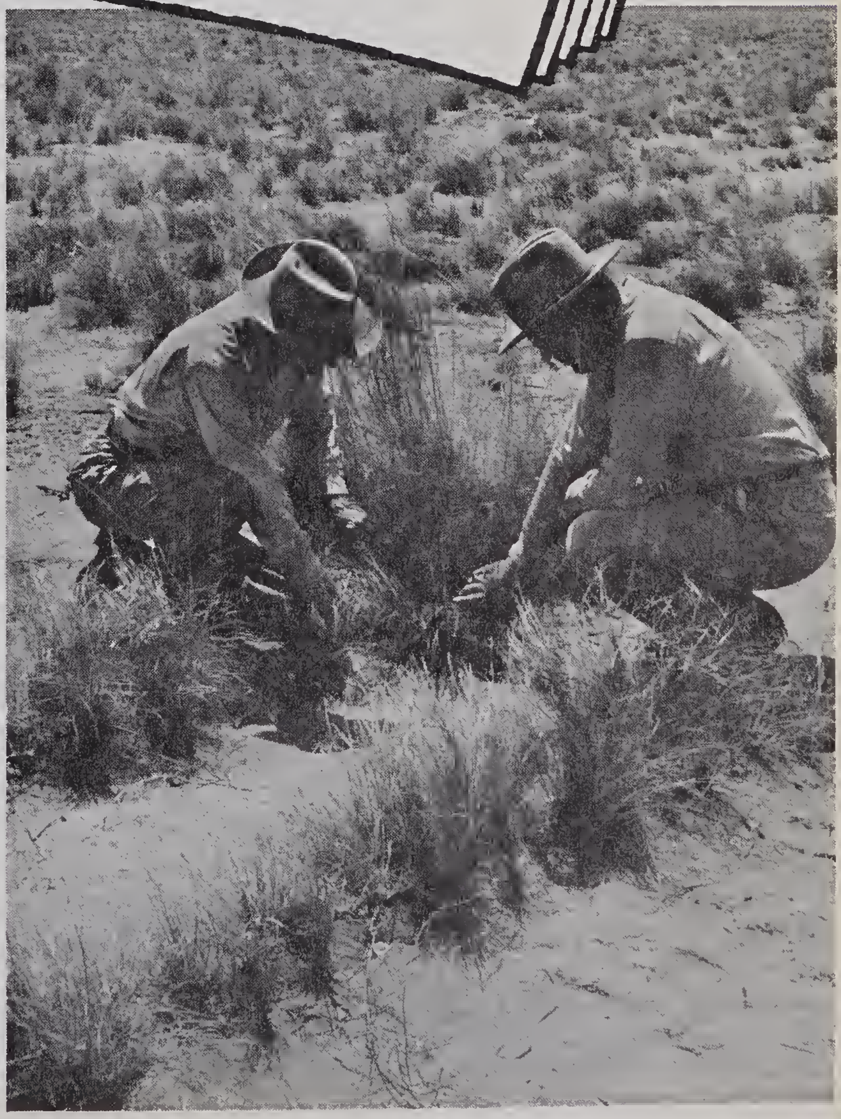
Agents often test soil to determine its pH, that is, whether the soil is acid, alkaline, or neutral. The degree of pH greatly affects the growth of many plants. Certain flowering shrubs — rhododendrons and azaleas, for example — grow best in somewhat acid soil, while cranberries require highly acid soils. Many vegetables, on the other hand, do best in neutral soils.

pH is measured in terms of a scale from 1 to 14; numbers from 1 to 7 indicate acidity in decreasing strength, 7 is neutral, and numbers from 7 to 14 indicate alkalinity in increasing strength. It is possible to alter the pH and thereby increase the productivity of a soil by adding lime, an alkaline substance, or sulphur, an acid-former.

The work pad shows a computation for modifying the composition of the topsoil on a particular farm. Soil samples taken over the 120-acre area showed a pH of 3. The county agent recommended that the farmer neutralize the soil by adding lime, and calculated that a total of 627$\frac{1}{4}$ tons would be needed for the 120 acres.

The county agent in the photograph is examining a tract of land that had gone to waste because poor vegetation and low rainfall subjected it to destructive dust storms. The area has been seeded with a hardy grass which retains moisture and binds the soil, providing good grazing land. In calculating the complex factors of rainfall, wind erosion, and plant growth, the agent again used mathematics to solve a difficult problem.





Chapter Summary

Inventory of Structure and Method

1. **To find the greatest common factor** of a number of integers, factor each as a product of prime numbers. The factors of a polynomial with integral coefficients usually are limited to positive integers and polynomials with integral coefficients.

2. **To factor a polynomial**, use the distributive property to form a product of the greatest common factor, if any, and the polynomial sum of the other factors of each term. Next, consider the possibilities of factoring this polynomial sum.

3. Certain **special products** should be read and factored at sight: The sum of two numbers times their difference: $(a + b)(a - b) = a^2 - b^2$; the square of a binomial sum: $(a + b)^2 = (a^2 + 2ab + b^2)$; and the square of a binomial difference: $(a - b)^2 = (a^2 - 2ab + b^2)$.

4. To **factor trinomial products** such as $ax^2 + bx + c$, $(a > 0)$: If b and c are positive, both binomial factors are sums; if b is negative and c is positive, both binomial factors are differences; if c is negative, the binomial factors are a sum and a difference. By inspection and trial find factors of the quadratic and constant terms which produce binomials whose product contains a linear term with coefficient b.

5. **Factoring** must be complete; each polynomial factor must be prime over the set of polynomials with appropriate coefficients. The correctness of factoring should be checked by multiplication.

6. To **solve a polynomial equation by factoring**: Transform the equation into standard form with the right member zero and the left member a polynomial in descending powers of the variable. Factor the left member. Set each factor equal to zero, applying the principle that if a product is zero at least one of its factors is zero. Solve the resulting linear equations. Check each possible root in the original equation. Write the solution set, listing multiple roots only once.

7. Problems leading to quadratic equations may have two answers. However, some problems have only one answer even though the equation has two roots. Therefore, all possible answers must be checked against the wording of the problem.

Vocabulary and Spelling

factoring a number (over a set of numbers) (*p. 237*)
prime number (*p. 238*)

greatest common factor (of two integers) (*p. 238*)
factoring a polynomial (*p. 239*)
common monomial factor (*p. 241*)
greatest common monomial factor (of a polynomial) (*p. 241*)
factoring by grouping (*p. 241*)
quadratic polynomial (*p. 254*)
quadratic term (*p. 254*)
linear term (*p. 254*)
constant term (*p. 254*)
prime polynomial (over a set of polynomials) (*p. 256*)
polynomial equation (*p. 264*)
standard form (*p. 264*)
degree of polynomial equation (*p. 264*)
linear equation (*p. 264*)
quadratic equation (*p. 264*)
cubic equation (*p. 264*)
multiple root (*p. 265*)

Chapter Test

7-1 **1.** Find the greatest common factor of 792 and 2520.
2. $42xy^2z^3 = 6xyz(?)$
7-2 **3.** Write $80a^2 - 16ab$ in factored form.
4. Write $x(x + 1) - 2(x + 1)$ in factored form.
5. Group the terms of $y^3 - 15 - 5y^2 + 3y$, and factor.
7-3 **6.** Find the square of $-9x^3$.
7. Write the product of $(rs + t^2)$ and $(rs - t^2)$.
7-4 **8.** Factor $8m^2 - 50$. **9.** Factor $-144 + x^2$.
7-5 **10.** $(3a + b)^2 = ?$ **11.** $(5k - 2m)^2 = ?$
7-6 **12.** Factor $25x^2 + 90xy + 81y^2$.
7-7 **13.** Solve and check: $(3x - 5)(4x + 1) = (6x - 7)(2x - 1)$.

Factor each trinomial.

7-8 **14.** $n^2 + 17n + 42$ **15.** $r^2 - 23rs + 90s^2$
7-9 **16.** $m^2 + 5m - 36$ **17.** $k^2 - 7k - 18$
7-10 **18.** $36n^2 + 95n + 56$ **19.** $3a^2 - 23ab - 36b^2$
7-11 **20.** $30t^2 + 10t - 100$

Find and check the solution set of each equation.

7-12 **21.** $(x - 2)(x + 5) = 0$ **22.** $y(2y - 1) = 0$
7-13 **23.** $z^2 - z = 90$ **24.** $49n^2 - 169 = 0$
7-14 **25.** A rectangle is 8 feet longer than it is wide. Its area is 105 square feet. Find its dimensions.

Chapter Review

7-1 Factoring in Algebra *Pages 237–240*

1. $360 = 2^{\underline{?}} \cdot 3^{\underline{?}} \cdot 5^{\underline{?}}$ **2.** $-75ab^2c^3 = 15abc(\underline{\ ?\ })$.

3. The greatest common factor of 360 and 400 is $\underline{\ ?\ }$.

4. The highest power of $2x^3y^3$ that is a factor of $6x^9y^9$ is $\underline{\ ?\ }$.

7-2 Identifying Common Factors *Pages 241–244*

5. Factor $5x^4 - 10x^3 + 15x^2$. **6.** Factor $x^2 - 2x^3 + x$.

7. By the distributive property,

$$n^2(n - 1) + 2(n - 1) = (n - 1)(\underline{\ ?\ } + \underline{\ ?\ }).$$

Factor each expression.

8. $y(y - 3) + 5(y - 3)$ **10.** $(2x - 1)(3x + 1) - (2x - 1)$

9. $r^2 + 3r + rs + 3s$ **11.** $2rt + 5st - 6ru - 15su$

7-3 Multiplying the Sum and Difference of Two Numbers *Pages 245–246*

Give the square of each number.

12. -16 **13.** $8x$ **14.** $-7a^2b^3$

Find each product.

15. $(5n - x)(5n + x)$ **16.** $(3y + \frac{1}{3})(3y - \frac{1}{3})$

7-4 Factoring the Difference of Two Squares *Pages 246–248*

Factor each expression.

17. $4x^2 - 25$ **18.** $8x^2 - 32y^2$ **19.** $2d^3 - d$ **20.** $x^4 - 81$

7-5 Squaring a Binomial: Plateau Section *Pages 248–251*

21. $(a + b)^2 = \underline{\ ?\ }$ **23.** $(2x - 1)^2 = \underline{\ ?\ }$

22. $(a - b)^2 = \underline{\ ?\ }$ **24.** $(5h + 3k)^2 = \underline{\ ?\ }$

7-6 Factoring a Trinomial Square *Pages 251–253*

25. Which of these trinomials are squares?

a. $x^2 - 20x - 100$ **b.** $4m^2 - 24m + 9$ **c.** $y^2 + 2y + 1$

Factor the trinomial squares.

26. $16x^2 - 8x + 1$ **27.** $9z^2 + 30az + 25a^2$

7-7 Multiplying Binomials at Sight *Pages 253–255*

Items 28–30 refer to $(2x + 3)(4x - 5) = 8x^2 + 2x - 15$.

28. A term of degree two, such as $8x^2$, is a $\underline{\ ?\ }$ term.

29. A term of degree one, such as $2x$, is a __?__ term.

30. A term with no variable factor, such as -15, is a __?__ term.

Give these products at sight.

31. $(8x + 3)(x + 7)$ **32.** $(5n - 4)(n - 12)$

33. Solve: $(5x - 8)(2x - 3) = (x + 2)(10x - 9)$

7-8 Factoring the Product of Binomial Sums or Differences

Pages 255–257

Items 34–36 refer to $x^2 + 8x + 12 = (x + r)(x + s)$.

34. To factor $x^2 + 8x + 12$, you must find two integers r and s whose product is __?__ and whose sum is __?__.

35. Since 12 is positive, r and s are both __?__ or both __?__.

36. Since $8x$ is positive, $r =$ __?__ and $s =$ __?__, if $r > s$.

Factor each expression.

37. $n^2 + 15n + 44$ **38.** $x^2 - 23x + 42$

7-9 Factoring the Product of a Binomial Sum and a Binomial Difference

Pages 257–259

Items 39–41 refer to the relationship $x^2 + x - 42 = (x + r)(x + s)$.

39. To factor $x^2 + x - 42$, you must find two integers whose product is __?__ and whose sum is __?__.

40. Since -42 is negative, r and s are __?__ in sign.

41. Since x is positive, if $r > s$, then $r =$ __?__ and $s =$ __?__.

Factor each expression.

42. $x^2 - x - 42$ **43.** $x^2 - 2x - 35$ **44.** $x^2 + 2x - 35$

7-10 General Method of Factoring Quadratic Trinomials

Pages 259–261

Determine all the integral values of b for which the trinomial can be factored over the set of binomials with integral coefficients.

45. $3x^2 + bx + 2$ **46.** $14y^2 - by + 3$ **47.** $10n^2 - bn - 6$

Write in factored form.

48. $20y^2 - 19y + 3$ **49.** $26z^2 + z - 6$

7-11 Combining Several Types of Factoring

Pages 261–263

50. If a factor is a binomial, determine whether it is the __?__ of two __?__.

51. If a factor is a trinomial, determine whether it is a __?__ or __?__.

7-12 Working with Factors Whose Product Is Zero

Pages 263–264

Solve each equation.

52. $3(x + 2) = 0$ **53.** $(y + 2)(2y - 5) = 0$

7-13 Solving Polynomial Equations by Factoring *Pages 264–267*

54. Transform $x^2 - 5x = 50$ into standard form.

Solve by factoring.

55. $x^2 + 9x = 70$

56. $x^2 = 19x - 84$

57. $x^3 - 7x^2 - 30x = 0$

7-14 Using Factoring in Problem Solving *Pages 267–270*

58. Twice the square of a certain integer is 3 less than 7 times the integer. Find the integer.

Extra for Experts

Scientific Notation

Scientists use very large and very small numbers.

The speed of light is 29,900,000,000 centimeters per second;
the mass of a proton is .000,000,000,000,000,000,000,001,65 grams.

It is customary to express such numbers in a shorter way in scientific notation (or standard notation), as the product of a whole number or decimal between 1 and 10 and an integral power of 10. Thus, $29{,}900{,}000{,}000 = 2.99 \times 10^{10}$ and $.00000000000000000000000165 = 1.65 \times 10^{-24}$.

To determine a method of transforming a number from ordinary decimal form to scientific notation, study the change in the exponent of ten in the following:

$$\begin{aligned}
.00194 &= 1.94 \times .001 &&= 1.94 \times 10^{-3}\\
.0194 &= 1.94 \times .01 &&= 1.94 \times 10^{-2}\\
.194 &= 1.94 \times .1 &&= 1.94 \times 10^{-1}\\
1.94 &= 1.94 \times 1 &&= 1.94 \times 10^{0}\\
19.4 &= 1.94 \times 10 &&= 1.94 \times 10^{1}\\
194 &= 1.94 \times 100 &&= 1.94 \times 10^{2}\\
1940 &= 1.94 \times 1000 &&= 1.94 \times 10^{3}
\end{aligned}$$

The effect of multiplying or dividing a number in the decimal system by 10 is to shift the position of the decimal point. Therefore, changing from one form to the other becomes a matter of counting the number of places you must shift the decimal point.

These examples illustrate a procedure for changing a number from one form to the other.

EXAMPLE 1. A starfish lays an average of 2,520,000 eggs annually. Express this number in scientific notation.

Solution: $2{,}520{,}000 = 2.52 \times 10^n$. To find n, place a caret ($\wedge$) to correspond to the position of the decimal point in the answer: $2_\wedge520{,}000$. Count the number of places from the caret to the decimal point. You count 6 to the right.

$\therefore 2{,}520{,}000 = 2.52 \times 10^6$, Answer.

EXAMPLE 2. The charge of an electron is .00000000048 electrostatic units. Express this number in scientific notation.

Solution: $.00000000048 = 4.8 \times 10^n$
Place the caret sign as above: $.0000000004_\wedge8$. Count the number of places from the caret sign to the decimal point. You count 10 to the left.

$\therefore .00000000048 = 4.8 \times 10^{-10}$, Answer.

EXAMPLE 3. Express in decimal form the average white blood cell count of a human male, 5.43×10^6 per cubic millimeter.

Solution: Here $n = 6$, so count 6 places to the right of the caret to locate the decimal point.

$5._\wedge43 \times 10^6 = 5{,}430{,}000$, Answer.

EXAMPLE 4. Express 8.0×10^{-4} without exponents.

Solution: Since $n = -4$, count 4 places to the left of the caret to locate the decimal point.

$8._\wedge0 \times 10^{-4} = .0008$, Answer.

Scientific notation shows the degree of accuracy of a measured or computed number. Thus, the 5.43 in 5.43×10^6 is an example of three-digit accuracy, but the 8.0 in 8.0×10^{-4} shows two-digit accuracy.

Two numbers expressed in scientific notation can be multiplied or divided readily. Thus,

$$\begin{aligned}(4.3 \times 10^{-12})(1.92 \times 10^8) &= (4.3)(1.92) \times (10^{-12})(10^8)\\ &= 8.3 \times 10^{-4}\end{aligned}$$

Questions

Express each number in scientific notation.

1. The diameter of the earth's orbit: 186,000,000 miles
2. The diameter of a large molecule: .00000017 centimeter
3. The length of the largest virus (parrot fever): .000025 centimeter
4. The number of molecules in 22.4 liters of a gas (Avogadro's number): 602,000,000,000,000,000,000,000
5. The diameter of the sun: 130,000,000,000 centimeters
6. The electronic charge: .00000000000000000016 coulomb

Express each number without using exponents.

7. 9.3×10^7 miles
8. 1.86×10^5 miles per second
9. 9.6×10^{22} centimeters
10. 9.5×10^{-6} centimeter
11. 6.67×10^{-8} cgs unit
12. 1.6×10^{-19} coulomb

Express the result of each operation in standard notation.

13. $(6.9 \times 10^{-6}) \div (2.3 \times 10^{-2})$
14. $(5.83 \times 10^{-9}) \times (1.39 \times 10^{-8})$
15. $\dfrac{1.82 \times 10^5}{9.1 \times 10^{17}}$
16. $(9.72 \times 10^8) \times (4.8 \times 10^{-20})$
17. $8.3 \times 10^3 + 2.7 \times 10^4 + 3.1 \times 10^4 - 6.7 \times 10^3 - 8.1 \times 10^4$

Just for Fun

Perfect and Amicable Numbers

In ancient and medieval times, learned men considered 6 a "perfect" number, whereas 8 was "deficient" and 12, "excessive."

This sort of classification of numbers depends on the sum of their *aliquot parts*. The aliquot parts of a number are all its factors except itself. Notice the results when you add the aliquot parts of each of the three numbers mentioned above:

⑥ ⟶ $1 + 2 + 3 = 6$ → The sum of the aliquot parts equals the number, so the number is perfect.

⑧ $\longrightarrow 1 + 2 + 4 = 7 \rightarrow$ The sum of the aliquot parts is less than the number, so the number is deficient.

⑫ $\longrightarrow 1 + 2 + 3 + 4 + 6 = 16 \rightarrow$ The sum of the aliquot parts is greater than the number, so the number is excessive.

Now identify each of the following four consecutive numbers as perfect, deficient, excessive, or prime: 27, 28, 29, 30.

The Greek mathematician Euclid developed a rule for finding perfect numbers: $N = 2^{p-1}(2^p - 1)$, in which p is a prime number. The rule doesn't give all the perfect numbers, but it works with these values of p: 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127. Try it with some of the smaller values. Check the perfection of your result.

The ancients called 220 and 284 "friends" or "amicable" because the aliquot parts of 220 total 284, and the aliquot parts of 284 total 220.

$$\begin{array}{lrr} (220) \rightarrow & 1 & \\ & 2 & 110 \\ & 4 & 55 \\ & 5 & 44 \\ & 10 & 22 \\ & \underline{11} & \underline{20} \\ & 33 & +\ 251 = 284 \end{array} \qquad \begin{array}{lrr} (284) \rightarrow & 1 & \\ & 2 & 142 \\ & \underline{4} & \underline{71} \\ & 7 & +\ 213 = 220 \end{array}$$

As time went on, mathematicians tried to find other numbers whose aliquot parts equaled each other. In 1636, Fermat told a friend that he had found another pair: 17,296 and 18,416. Below are the factors of these numbers; find their aliquot parts, and check Fermat's statement that they are amicable.

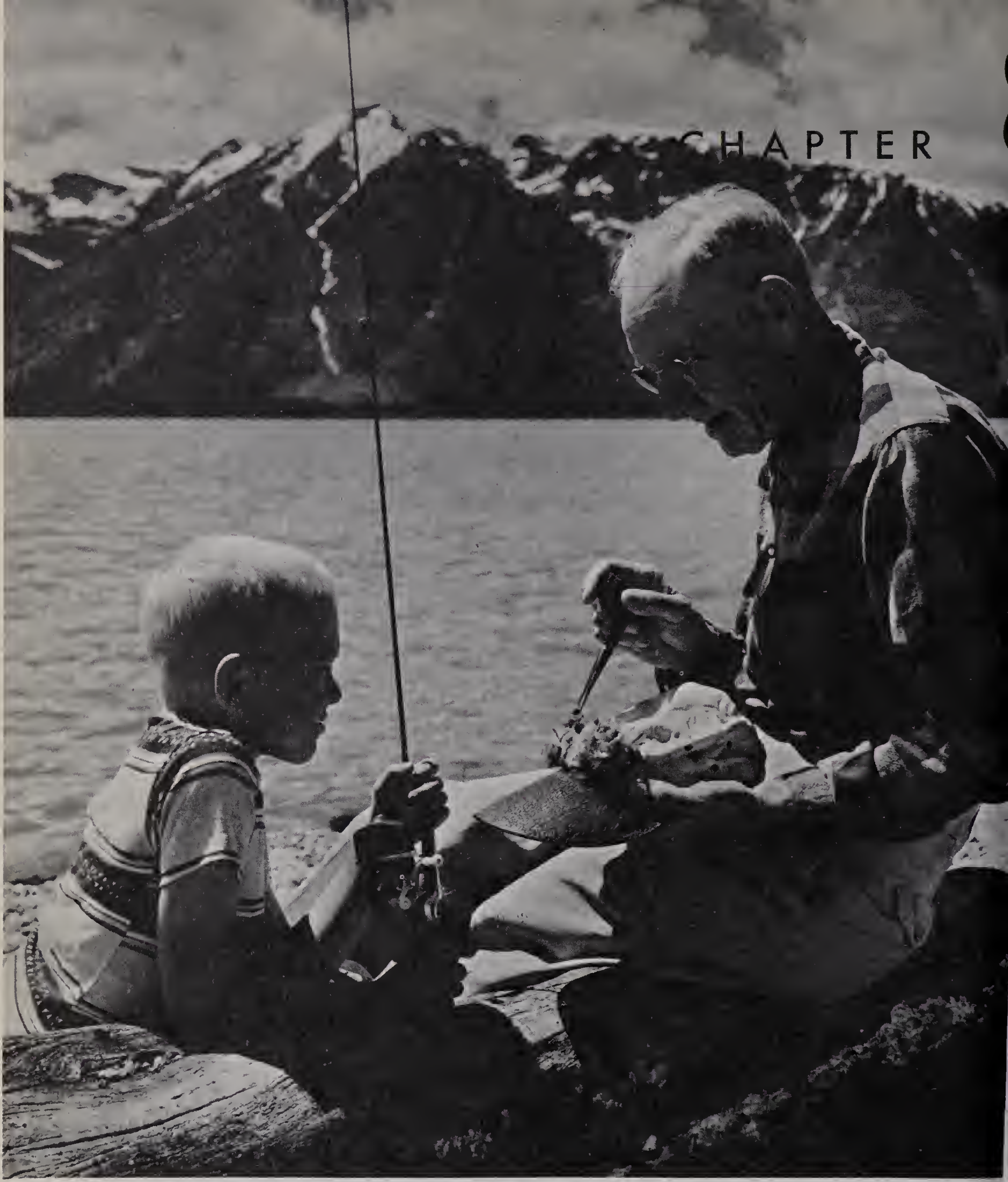
$$17{,}296 \rightarrow 2 \cdot 2 \cdot 2 \cdot 2 \cdot 23 \cdot 47$$

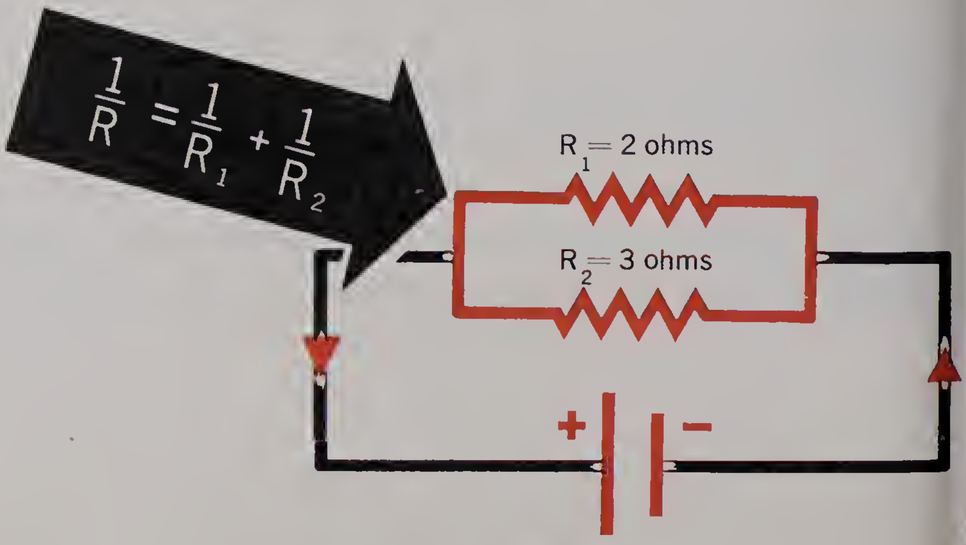
$$18{,}416 \rightarrow 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1151$$

A little over a hundred years later, Euler published a list of 64 pairs of amicable numbers, including the pair that Fermat had found and the pair that had been known since antiquity. All the other 62 are larger than Fermat's pair. But there is a pair of amicable numbers smaller than Fermat's, although larger than 220 and 284 which Euler completely missed.

The missing pair was found in 1866 by the Italian mathematician Paganini, who was then 16 years old. The numbers are 1184 and 1210. If he could find them, surely you can show that they are amicable!

CHAPTER





$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$$

R_1 = 2 ohms
R_2 = 3 ohms
+
−

Working with Fractions

Tempus fugit. To a child, it may seem that next Christmas will never come. An adult may feel that one Christmas follows too closely upon another. Why does time seem to pass more quickly as you get older?

Psychologists explain this phenomenon in terms of the relationship of one year to the length of your life. When you are six years old, one year represents one-sixth of your whole life. When you are sixty, it is only one-sixtieth. Unconsciously, you compare numbers in terms of quotients; although you may be aware only of comparing numbers in terms of differences.

You will find comparing numbers by division to be a useful tool in solving problems in science (lower illustration). When you learn to think of two numbers in terms of their *ratio*, you have new insight into the properties of numbers.

FRACTIONS AND RATIOS

8–1 Defining Algebraic Fractions

Any indicated quotient of two algebraic expressions like $\frac{2}{1}$, $\frac{5}{9}$, $-\frac{7}{2}$, $\frac{x}{3}$, $\frac{1}{y}$, and $\frac{r^3 + 3r + 2}{5r - 10}$ is called an *algebraic fraction*. Since division by zero is not permitted, *a fraction is defined only when its denominator is not zero.* Do you see why the indicated numbers must be excluded from the replacement set of x in the fractions below? To find such excluded numbers, set the denominator of each fraction equal to zero, and solve the resulting equation. See T.M. pg. 29 (1) for a discussion of word usage.

Fraction	$\frac{1}{x}$	$\frac{1}{x - 2}$	$\frac{5x + 4}{x + 2}$	$\frac{7}{x^2 - 4}$	$\frac{3x + 6}{4}$
Excluded Numbers	0	2	-2	2 and -2	no exclusions

ORAL EXERCISES

Give the value of the variable for which the fraction is not defined.

SAMPLE. $\frac{x}{3x - 6}$ *What you say:* When $x = 2$, $3x - 6 = 0$.

1. $\frac{1}{5x}$ $x = 0$
2. $\frac{1}{2y}$ $y = 0$
3. $\frac{3a + 4}{a - 6}$ $a = 6$
4. $\frac{b^2 - 1}{b + 3}$ $b = -3$
5. $\frac{x}{5x + 10}$ $x = -2$
6. $\frac{t}{6t - 18}$ $t = 3$
7. $\frac{a - 7}{a - 7}$ $a = 7$
8. $\frac{c + 2}{c + 2}$ $c = -2$
9. $\frac{d + 4}{d^2}$ $d = 0$
10. $\frac{z - 3}{z^2}$ $z = 0$
11. $\frac{3t - 15}{15 - 3t}$ $t = 5$
12. $\frac{7k - 35}{35 - 7k}$ $k = 5$

WRITTEN EXERCISES

Express as fractions. Find any values of the variables for which the fraction is not defined.

A

1. $5 \div y$
2. $-7 \div z$
3. $.17$
4. $.9$
5. $.2x$
6. $3y$
7. $(b - 1) \div b$
8. $a \div (a - 1)$
9. $k \div (6k - 12)$
10. $f \div (3f + 21)$
11. $(12x^2 + 4) \div 3$
12. $(15y^2 - 10) \div 5$
13. $(g - 2) \div (14g + 7)$
14. $(h - 5) \div (33h - 11)$
15. $1 \div x(x - 1)$
16. $1 \div y(y + 4)$

Give the set of excluded values of the variables.

B

17. $\frac{4c + 16}{c^2 - 8c + 12}$
18. $\frac{9d - 18}{d^2 + 9d + 14}$
19. $\frac{1}{5p^2 + 14p - 3}$
20. $\frac{1}{7g^2 - 5g - 2}$
21. $\frac{a - 2}{a^2 - 4}$
22. $\frac{b - 3}{b^2 - 9}$
23. $\frac{3y - 5}{y^2 + 9}$
24. $\frac{2x + 1}{x^2 + 4}$
25. $\frac{x + 4}{x^2 - 16}$

State the restrictions on the values of the variables.

C

26. $\frac{1}{a(a - b)}$
27. $\frac{cd}{c^2 + 2cd + d^2}$
28. $\frac{rs}{r^2 - 2rs + s^2}$

8–2 Reducing Fractions

Why do the fractions $\frac{3}{2}$ and $\frac{15}{10}$ name the same number? Using the property of quotients, you have $\frac{15}{10} = \frac{3 \cdot 5}{2 \cdot 5} = \frac{3}{2} \cdot \frac{5}{5} = \frac{3}{2} \cdot 1 = \frac{3}{2}$. Similarly, any fraction of the form $\frac{3c}{2c}$ equals $\frac{3}{2}$ because $\frac{c}{c}$ equals 1 if $c \neq 0$.

Have pupils state this property (see page 216).

This example illustrates the ***multiplication property of fractions*:**

> Dividing or multiplying the numerator and denominator of a fraction by the same nonzero number produces a fraction equal to the given one.
>
> $$\frac{ac}{bc} = \frac{a}{b}, \quad \text{provided} \quad c \neq 0$$

See T.M. pg. 29 (2) for a general proof.

Thus, $-\frac{20}{64} = -\frac{5 \cdot 4}{16 \cdot 4} = -\frac{5}{16}$ and $\frac{24x}{12x^2} = \frac{2 \cdot 12x}{x \cdot 12x} = \frac{2}{x}$, if $x \neq 0$.

A *fraction* is said to be *in lowest terms* when its numerator and denominator have no common factor other than 1 and −1. **Reducing a fraction to lowest terms** is the process of dividing the numerator and denominator by their *greatest common factor*.

EXAMPLE 1. **Reduce $\frac{15b + 5}{6b^2 - b - 1}$ to lowest terms.**

Solution:

Factor the numerator and denominator. →

$$\frac{15b + 5}{6b^2 - b - 1} = \frac{5(3b + 1)}{(2b - 1)(3b + 1)}$$

Notice the common factors of the numerator and denominator.

Divide numerator and denominator by the greatest common factor, $3b + 1$. →

$$= \frac{5(3b + 1) \div (3b + 1)}{(2b - 1)(3b + 1) \div (3b + 1)}$$

Simplify the result. →

$$= \frac{5}{2b - 1}, \text{ if } b \notin \{\tfrac{1}{2}, -\tfrac{1}{3}\},$$

Answer.

Recall: If $b = \frac{1}{2}$ or $b = -\frac{1}{3}$, the original fraction is meaningless.

This division is explicitly shown for teaching emphasis. Pupils, however, need not write this step out. See T.M. pg. 29 (3).

EXAMPLE 2. Simplify $\frac{6-t}{t^2-36}$.

Solution:

$$\frac{6-t}{t^2-36} = \frac{6-t}{(t+6)(t-6)}$$

To show the common factor, express the numerator as a product having -1 as a factor. →

$$= \frac{-1(t-6)}{(t+6)(t-6)}$$

Red indicates commo
factors of the numera
and denominator.

This is true because $\frac{a}{b} = a \cdot \frac{1}{b}$.

$$= \frac{-1}{t+6}, \text{ if } t \notin \{6, -6\}, \textbf{ Answer.}$$

The fraction $\frac{-1}{t+6}$ also can be written in the form $-\frac{1}{t+6}$, because

$$\frac{-1}{t+6} = (-1) \cdot \frac{1}{t+6} = -\frac{1}{t+6}.$$

This is true because of the multiplicative property of -1.

You can reduce a fraction only when the numerator and denominator have a common *factor*. Compare the fractions below:

$\frac{2 \cdot 3}{2} = \frac{3}{1}$ $\frac{ab}{a} = \frac{b}{1} = b$, if $a \neq 0$. a and b are factors of the numerator. This fraction can be reduced because a is a common factor of the numerator and denominator.	$\frac{2+3}{2}$ $\frac{a+b}{a}$, if $a \neq 0$. a and b are *not* factors of the numerator. This fraction can *not* be reduced, for no factor (other than 1 or -1) is common to both numerator and denominator.

Hereafter, it will be assumed that the replacement sets of the variables include no value for which the denominator is zero. Emphasize this assumption.

WRITTEN EXERCISES

Write each fraction in lowest terms, noting all restrictions on the values of the variables.

Ⓐ **1.** $\frac{30h^2k}{30hk}$ **2.** $\frac{5st^2}{30st}$ **3.** $\frac{-21m^2n^2}{28m^3n^3}$

4. $\dfrac{-35x^2z^2}{63x^3y^3}$

5. $\dfrac{3a + 3b}{4a + 4b}$

6. $\dfrac{5c - 5d}{5c + 5d}$

7. $\dfrac{4x + 4y}{x^2 - y^2}$

8. $\dfrac{r^2 - 25}{3r + 15}$

9. $\dfrac{2a + 3b}{2ab}$

10. $\dfrac{5r + 3b}{5rs}$

11. $\dfrac{x^2 - 16}{4 - x}$

12. $\dfrac{1 - a^2}{a - 1}$

13. $\dfrac{m^2 - 4}{m^2 - 4m + 4}$

14. $\dfrac{x^2 - 16}{x^2 - 8x + 16}$

15. $\dfrac{3a + 3}{a^2 + 2a + 1}$

16. $\dfrac{5x + 10}{x^2 + 4x + 4}$

17. $\dfrac{r^2s - s}{r - 1}$

18. $\dfrac{2d^2 - 2}{d + 1}$

19. $\dfrac{x^2 + 1}{x + 1}$

20. $\dfrac{x - 4}{x^2 - 4}$

21. $\dfrac{6a^2 - 54}{2a^2 + 8a + 6}$

22. $\dfrac{8a^2 + 40a + 32}{32 - 2a^2}$

23. $\dfrac{5c^2 - 45c + 90}{180 - 5c^2}$

24. $\dfrac{3d^2 - 27}{24 - 11d + d^2}$

B

25. $\dfrac{c^2 + 4cd + 4d^2}{c^2 - 4d^2}$

26. $\dfrac{9a^2 + 6ab + b^2}{9a^2 - b^2}$

27. $\dfrac{r^2 - 3r}{r^2 - 4r + 3}$

28. $\dfrac{t^2 - 7t}{t^2 - 8t + 7}$

29. $\dfrac{a^2 - 3a - 4}{a^2 + 2a + 1}$

30. $\dfrac{b^2 + 5b - 6}{b^2 - 2b + 1}$

31. $\dfrac{z^2 - 8z + 15}{z^2 + z - 12}$

32. $\dfrac{m^2 - 2m - 8}{m^2 - m - 6}$

33. $\dfrac{x^2 - 4x + 4}{4 - x^2}$

Explain why each reduction is incorrect.

Assign these exercises. Stress again that a fraction can be reduced only when numerator and denominator have a common factor.

34. $\dfrac{x + 1}{2x} \overset{?}{=} \dfrac{1 + 1}{2} = 1$

35. $\dfrac{r - 5}{r + 5} \overset{?}{=} \dfrac{1 - 1}{1 + 1} = \dfrac{0}{2} = 0$

36. $\dfrac{ay + y}{y} \overset{?}{=} \dfrac{a}{1} = a$

37. $\dfrac{6 + b}{2 + b} \overset{?}{=} \dfrac{6}{2} = 3$

38. $\dfrac{3 + t^2}{3 + t} \overset{?}{=} \dfrac{1 + t}{1 + 1} = \dfrac{1 + t}{2}$

39. $\dfrac{10 + 2x}{x^2} \overset{?}{=} \dfrac{10 + 2}{x} = \dfrac{12}{x}$

Write each fraction in lowest terms, noting all restrictions on the values of the variables.

40. $\dfrac{6n^2 - 13n - 5}{6n^2 - 17n + 5}$

41. $\dfrac{8n^2 + 10n - 3}{8n^2 - 14n + 3}$

42. $\dfrac{6m + 3m^2 - 3m^3}{6m^3 + 18m^2 + 12m}$

43. $\dfrac{10t^3 - 15t^2 - 10t}{4t - 16t^3}$

44. $\dfrac{2x^3 + 20x^2 + 50x}{4x^3 + 100x}$

45. $\dfrac{5p^3 + 20p^2 + 20p}{25p^3 + 100p}$

46. $\dfrac{6z^4 + 54xz^3 + 120x^2z^2}{3z^2 + 3xz - 36x^2}$

47. $\dfrac{4m^3n - 8m^2n^2 - 32mn^3}{6m^4 - 6m^3n - 36m^2n^2}$

8–3 Ratio

"Ratio" is just a special name for a familiar idea. Proportion is discussed in Chapter 12.

To compare the daily output of two oil wells, one producing 750 barrels and the other, 250 barrels daily, you can say that the first yields three times as many barrels as the second. This comparison is made by computing the quotient of the numbers: $\frac{750}{250} = 3$. You also can say that the daily yields are in the *ratio* of 3 to 1 ($\frac{3}{1}$ or 3:1).

A **ratio** of one number to another is the quotient of the first number divided by the second. You can express the ratio 5 to 4 by:

1. **An indicated quotient using the division sign ÷** → **$5 \div 4$**
2. **An indicated quotient using the ratio sign :** → **5:4**
3. **A fraction** → $\frac{5}{4}$
4. **A fraction in decimal notation** → **1.25**

By the multiplication property of fractions, the ratio 5:4 compares not only the numbers 5 and 4, but also 10 and 8, 15 and 12, -25 and -20, and $5n$ and $4n$, where n is not zero. However, if you compare a 2-foot line to a 4-inch line, you must change the 2 feet to 24 inches, and then find the ratio, $\frac{24}{4}$ or 6:1. Emphasize that knowing the ratio of two numbers i not enough to identify the number. See the Example below.

To compare the measures of two quantities of the same kind, express the measures in the same unit; then compute their quotient.

EXAMPLE **In ferric oxide the ratio of iron to oxygen, by weight, is 7 to 3. How many pounds of each element are in 500 pounds of ferric oxide?**

Solution:

1	Choose a variable to use in representing the desired weights.	**Let $7n$ = pounds of iron.** **Then $3n$ = pounds of oxygen.**
2	Form an equation.	$7n + 3n = 500$
3	Solve the equation. Stress that the value of n is not the answer.	$10n = 500$ $n = 50$ $\therefore 7n = 350, 3n = 150$

Check: **Are the weights in the ratio of 7 to 3?**

$$\frac{350}{150} \stackrel{?}{=} \frac{7}{3}, \quad \frac{7}{3} = \frac{7}{3} \checkmark$$

Do the weights of iron and oxygen total 500 pounds?

$$350 + 150 \stackrel{?}{=} 500$$

$$500 = 500 \checkmark$$

350 pounds of iron,
150 pounds of oxygen, } **Answer.**

ORAL EXERCISES

Give each ratio in its lowest terms.

1. 4:8 1:2
2. 5:15 1:3
3. $\frac{ab}{ad}$ b:d
4. $\frac{adx}{bdx}$ a:b
5. $\frac{200}{15}$ 40:3
6. $\frac{369}{6}$ 123:2
7. $\frac{3a}{3b}$ a:b
8. $\frac{10x}{10y}$ x:y
9. $6x$ to $7x$ 6:7
10. $2y$ to $5y$ 2:5
11. $\frac{.9}{9}$ 1:10
12. $\frac{2.5}{5}$ 1:2
13. x^2 to $4x^2$ 1:4
14. $2y^3$ to y^3 2:1
15. $\frac{12g}{60g^2}$ 1:5g
16. $\frac{6g^3}{9g^2}$ 2g:3
17. $(x^2 + 1):(x^2 + 1)^3$ $1:(x^2+1)^2$
18. $(y^2 + 4)^2:(y^2 + 4)$ $(y^2+4):1$
19. 3 feet to 15 feet 1:5
20. 4 meters to 16 meters 1:4
21. 1 quart to 1 gallon 1:4
22. 1 lb. to 1 oz. 16:1
23. 3 m. to 5 cm. 60:1
24. 1200 g. to 2 kg. 3:5
25. 35 cents to 1 dollar 7:20
26. 12 months to 2 years 1:2
27. 1 lb. 6 oz. to 2 lb. 1 oz. 2:3
28. 1 kg. 20 g. to 3 kg. 10 g. 102:301

WRITTEN EXERCISES

Give each ratio in its lowest terms.

1. The area of an 8- by 12-inch rectangle to that of one 4 by 36 inches
2. The area of a 6- by 9-foot rectangle to that of one 9 by 12 feet
3. The area of a 1- by 2-foot rectangle to that of one 8 by 20 inches
4. The area of a 9-inch square to that of a 1.5-foot square
5. A baseball player's 25 hits to his 100 times at bat
6. The cost per pound for screws selling at $15 for 25 pounds

7. \$7,250,000 in assets to \$2,500,000 in liabilities

8. A profit of \$360 to a cost of \$1200

9. Men to women in a college with 3500 women in 10,500 students

10. 175 pounds of sodium to chlorine in a compound of the two weighing 290 pounds

In Exercises 11–14, use the rule that in a triangle, Area $= \dfrac{\text{Base} \times \text{Altitude}}{2}$.

11. The area of a triangle with a 12-yard base and a 6-yard height to that of one with an 18-yard base and a 5-yard height

12. The area of a triangle with a 14-meter base and a 20-meter height to that of one with a 28-meter base and a 5-meter height

13. The area of a triangle 9 inches high with a 1-foot base to that of one 4 inches high with a $1\frac{1}{2}$-foot base

14. The area of a triangle 2 feet high with a 4-yard base to that of one 4 feet high with a 3-yard base

Find the ratio $x{:}y$ in each case.

15. $4x = 3y$

16. $5y = 2x$

17. $x = y$

18. $x = 7y$

19. $3x - 2y = 0$

20. $6y - 9x = 0$

Ⓑ **21.** $\dfrac{3x + 2y}{2y} = \dfrac{4}{3}$

22. $\dfrac{2x - y}{2x} = -\dfrac{3}{4}$

23. $\dfrac{9x - 4y}{6y} = 5\dfrac{1}{3}$

24. $\dfrac{4x + 6y}{3y} = 3\dfrac{1}{3}$

25. $\dfrac{x^2 - 2y^2}{y^2} = \dfrac{2x - 3y}{y}$

26. $\dfrac{x^2 + 7y^2}{y^2} = \dfrac{3y - 4x}{y}$

PROBLEMS

Ⓐ **1.** Find the larger of two numbers in the ratio of 2 to 5, whose difference is 56.

2. Find the smaller of two numbers in the ratio of 5 to 3, whose sum is −24.

3. How many of 35 delegates should each of two cities, whose populations are in the ratio 2:5, send to a convention?

4. The ratio of tin to silver in 78 pounds of an alloy is 12:1. How much silver has the alloy?

5. An artist plans a landscape in which the land's height has a ratio of 8:5 to the sky's height. How high in the 104-inch mural is his horizon?

Here a ratio appears as a rate.

6. Sandy makes bread in which the ratio of whole-wheat to white flour is $\frac{3}{1}$. How much white flour is in the total 8 cups?
7. A dime contains copper and silver in the ratio 1:9. How much of each is in 45 pounds of dimes?
8. Ross and Morgan need \$24,200 in capital. Ross invests \$500 for every \$600 of Morgan. How much does each invest?
9. Mr. A charges \$1.05 for 21 pounds of nails, and Mr. B charges 88 cents for 16 pounds of nails. Which is the better buy?
10. One worker assembles 34 units in 90 minutes. A second assembles 40 units in 1 hour 44 minutes. Which man works faster?
11. Divide a 30-cm. line into two parts with a ratio of 2:3.
12. Divide a line 21 cm. long into two parts whose ratio is $\frac{3}{4}$.

B

13. Only 2 of every 7 of a city's 1,343,790 dwelling units were built after 1950. How many of its dwelling units were over 10 years old in 1960?
14. In a 41,140-mile highway system, 3 of every 17 miles are primary, and the rest, secondary roadway. How many miles are secondary roadway?
15. In sulfuric acid, the ratio of sulfur to hydrogen is 16:1, and of oxygen to sulfur is 2:1. How much of each is in 490 lb. of sulfuric acid?
16. If the ratio of silt to clay is 1:1 and of sand to clay is 2:9, how much of each does 500 pounds of soil contain?
17. Find the largest of the three parts of \$1200, having a ratio of 3:5:7.
18. Jack, Walt, and Randy receive a total of \$6.75 for delivering newspapers. If they deliver in the ratio 2:3:4, what is Randy's share?

8–4 Per Cent and Percentage Problems

The ratio of one number to another is often expressed as a *per cent*. The words **per cent** (%) stand for *divided by 100* or *hundredths*. Hence, 4% is another way of writing $\frac{4}{100}$ or .04; and 125% is another name for $\frac{125}{100}$ or 1.25; and 1% = $\frac{1}{100}$. Most important, 100% = $\frac{100}{100}$ = 1. Pupils have trouble with per cents exceeding 100.

> To write a ratio as a per cent, write the ratio as a fraction with denominator 100; then write the numerator followed by a per cent sign.

EXAMPLE 1. **Express each number as a per cent: $\frac{5}{6}$, 2.3.**

Solution:

$$\frac{5}{6} = \left(\frac{5}{6}\cdot 100\right)\frac{1}{100} = \frac{83\frac{1}{3}}{100} = 83\tfrac{1}{3}\%$$

$$2.3 = (2.3)(100)\frac{1}{100} = \frac{230}{100} = 230\%$$

A **percentage** is a number equal to a per cent of another number called the **base**. Since per cent is the ratio of the percentage to the base, it is often called the **rate**, to avoid confusion with *percentage*. The key to per cent and percentage problems is this basic relationship:

$$\frac{\text{percentage}}{\text{base}} = \text{rate} \quad \text{or} \quad p = rb, \quad \text{if} \quad b \neq 0.$$

EXAMPLE 2. **How much is a 20% markup on an item whose cost is $35?**

Solution:

1 **Let p = markup (percentage)**

$r = 20\%$

$b = 35$

2 $\frac{p}{b} = r$ or $p = rb$

$\frac{p}{35} = \frac{20}{100}$

$p = (.20)(35)$

Show that the markup (percentage) is $7.

WRITTEN EXERCISES

Determine the indicated percentage.

A
1. 7% of 250
2. 2% of 2.5
3. 32% of 12.5
4. 90% of 1000
5. 75% of 6
6. 100% of 72
7. 200% of 12
8. 150% of 38
9. 300% of 1
10. 2.25% of 16
11. $\frac{1}{5}$% of 635
12. .03% of 1000

Find each number.

13. 24 is 30% of the number.
14. 17 is 25% of the number.
15. 5% of the number is 2.1.
16. 80% of the number is 5.2.
17. 100% of the number is 195.
18. 150% of the number is 63.
19. $\frac{1}{3}$% of the number is .72.
20. $1\frac{1}{5}$% of the number is 20.4.

Determine each rate.

21. What % of 52 is 39?
22. What % of 35 is 21?
23. What % of 3 is 12?
24. What % of 6 is 15?

25. 1 is what % of 200?

26. 3 is what % of 240?

27. 60 is what % of 5?

28. 500 is what % of 25?

PROBLEMS

1. If 2% of its output is defective, how many of a machine's 1500-bolt output are good?

2. How much copper is in 25 pounds of an alloy containing 5% copper?

3. How much hydrogen chloride is in one quart of acid containing 40% hydrogen chloride?

4. The average weight of men in World War II was 8% more than in World War I. If the average in the first war was 140 pounds, what was it in the second?

5. A family spends $116 a month for rent. Find the total monthly income, if 20% is spent for rent.

6. An agent received $47.50 as his 5 per cent commission on a car sale. What was the car's price?

7. Find the cost of a suit if 80 per cent of it is $42.40.

8. A camera's sale price of $7.50 is 60% of the list price. What was the list price?

9. Ben sold a car for $1120, which was 160 per cent of its cost to him. What had the car cost him?

10. What is the cost of a coat, if 175 per cent of it is $105?

11. The salesman's commission on a $1250 car is $62.50. What is his rate of commission?

12. On $8000 Mr. Todd pays a tax of $1960. What per cent is this?

13. Mr. Jones paid $2500 for a car. In a year, its value was $2000. By what per cent had the car depreciated in value?

14. Mr. Williams bought a house for $12,000 and sold it for $13,800. By what per cent had the house increased in value?

15. A camera and four rolls of film cost $11.55, 30 per cent below the original price. What was the original price?

16. Rice contains only 5 per cent fat. How much rice would you have to eat to consume a quarter of a pound of fat?

17. If a dealer's expenses are 15%, and his profit, 30% of the selling price, for what does he sell a painting which cost $330?

18. An article is marked $13 and a 15% discount is given. What profit does a dealer make if the article cost him $8?

MULTIPLYING AND DIVIDING FRACTIONS

8–5 Multiplying Fractions

When you read the equality (see page 216)

$$\frac{xy}{cd} = \frac{x}{c} \cdot \frac{y}{d}$$

from right to left, you see the ***rule for multiplying fractions:*** if $c \neq 0$, $d \neq 0$,

$$\frac{x}{c} \cdot \frac{y}{d} = \frac{xy}{cd}$$

When fractions are multiplied, the product is a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators of the given fractions.

$$\frac{m+1}{m-2} \cdot \frac{m-3}{m+2} = \frac{(m+1)(m-3)}{(m-2)(m+2)} = \frac{m^2-2m-3}{m^2-4}$$

$$5r \cdot \frac{2r}{s} = \frac{5r}{1} \cdot \frac{2r}{s} = \frac{10r^2}{s}$$

A product of fractions, not in lowest terms, should be reduced.

$$\frac{n}{d} \cdot \frac{d}{n} = \frac{nd}{dn} = \frac{nd}{nd} = 1; \qquad \frac{a-b}{a+b} \cdot \frac{a+b}{7} = \frac{(a-b)(a+b)}{7\ \ (a+b)} = \frac{a-b}{7}$$

You can simplify the multiplication of fractions by first factoring where possible.

$$\frac{3x-3}{5x+2} \cdot \frac{25x^2-4}{x^2-2x+1} = \frac{3(x-1)}{(5x+2)} \cdot \frac{(5x-2)(5x+2)}{(x-1)(x-1)}$$

$$= \frac{3(5x-2)(x-1)(5x+2)}{(x-1)(x-1)(5x+2)}$$

$$= \frac{3(5x-2)}{x-1} \quad \textit{or} \quad \frac{15x-6}{x-1}$$

Remind pupils that the equations are true for every value of x, except $\frac{-2}{5}$ and 1.

See 8-5 T.M. pg. 29 for note on cancellation.

ORAL EXERCISES

Determine each product.

1. $\frac{1}{5} \cdot \frac{1}{3}$ $\frac{1}{15}$
2. $\frac{1}{2} \cdot \frac{1}{7}$ $\frac{1}{14}$
3. $\frac{3}{5} \cdot \frac{11}{4}$ $\frac{33}{20}$
4. $\frac{9}{4} \cdot \frac{3}{2}$ $\frac{27}{8}$
5. $\left(-\frac{1}{2}\right)\left(\frac{13}{7}\right)$ $-\frac{13}{14}$
6. $\left(\frac{1}{4}\right)\left(-\frac{17}{5}\right)$ $-\frac{17}{20}$
7. $\frac{x}{2} \cdot \frac{y}{3}$ $\frac{xy}{6}$
8. $\frac{b}{4} \cdot \frac{c}{2}$ $\frac{bc}{8}$
9. $7 \cdot \frac{5}{3}$ $\frac{35}{3}$
10. $8 \cdot \frac{6}{5}$ $\frac{48}{5}$
11. $m \cdot \frac{m}{4}$ $\frac{m^2}{4}$
12. $t \cdot \frac{t}{8}$ $\frac{t^2}{8}$
13. $\left(-\frac{2}{a}\right)\left(-\frac{7}{b}\right)$ $\frac{14}{ab}$
14. $\left(-\frac{5}{r}\right)\left(-\frac{3}{s}\right)$ $\frac{15}{rs}$
15. $\frac{2x}{y} \cdot \frac{x}{3}$ $\frac{2x^2}{3y}$
16. $\frac{3c}{d} \cdot \frac{c}{2}$ $\frac{3c^2}{2d}$
17. $\left(-\frac{1}{2c}\right)\left(\frac{1}{5c}\right)$ $-\frac{1}{10c^2}$
18. $\left(\frac{1}{4a}\right)\left(-\frac{1}{4a}\right)$ $-\frac{1}{16a^2}$
19. $\frac{5x^2}{3y} \cdot \frac{2x}{3y}$ $\frac{10x^3}{9y^2}$
20. $\frac{4k^2}{3m} \cdot \frac{2k}{5m}$ $\frac{8k^3}{15m^2}$
21. $\frac{a+b}{b} \cdot (2a-b)$ $\frac{2a^2+ab-b^2}{b}$
22. $\frac{3x-y}{3} \cdot (3x+y)$ $\frac{9x^2-y^2}{3}$
23. $\frac{2}{t+2} \cdot \frac{3}{t+2}$ $\frac{6}{t^2+4t+4}$
24. $\frac{k+4}{5} \cdot \frac{k+4}{5}$ $\frac{k^2+8k+16}{25}$
25. $\frac{1}{v^2+1} \cdot \frac{1}{v^2-1}$ $\frac{1}{v^4-1}$
26. $\frac{1}{m^2-1} \cdot \frac{1}{m^2-1}$ $\frac{1}{m^4-2m^2+1}$
27. $\frac{x+y}{3} \cdot \frac{6}{y+x}$ 2
28. $\frac{u+v}{4} \cdot \frac{2}{v+u}$ $\frac{1}{2}$
29. $\frac{s}{s-r} \cdot \frac{r-s}{s}$ -1
30. $\frac{k-t}{t} \cdot \frac{t}{t-k}$ -1

WRITTEN EXERCISES

Determine each product in lowest terms.

A

1. $\frac{3}{7} \cdot \frac{5}{4} \cdot \frac{1}{7}$
2. $\frac{1}{8} \cdot \frac{3}{2} \cdot \left(-\frac{7}{5}\right)$
3. $\frac{3}{8} \cdot \frac{4}{9}$
4. $\frac{5}{7} \cdot \frac{14}{25}$
5. $\frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6}$
6. $\frac{2}{9} \cdot \frac{18}{35} \cdot \frac{7}{4}$
7. $5c^2 \cdot \frac{c}{3} \cdot \frac{c^2}{4}$
8. $7x \cdot \frac{x}{2} \cdot \frac{x^2}{8}$
9. $\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a^2}$
10. $\frac{r}{s} \cdot \frac{s}{t} \cdot \frac{t}{r^2}$
11. $\frac{6x}{11y} \cdot \frac{77y}{15x}$
12. $\frac{9b}{7c} \cdot \frac{28c}{81b}$

13. $\frac{7ab}{5cd} \cdot \frac{15c^2d^2}{28a^2b^2}$

14. $\frac{2r^2s^2}{9t^2u^2} \cdot \frac{45tu}{14rs}$

15. $\frac{t - v}{t + 2v} \cdot \frac{t + v}{t - 2v}$

16. $\frac{3r + w}{r - w} \cdot \frac{3r - w}{r + w}$

17. $\frac{a^2 - b^2}{4} \cdot \frac{12}{a + b}$

18. $\frac{2}{1 - x^2} \cdot \frac{1 - x}{2}$

19. $\frac{4 - y^2}{14} \cdot \frac{7}{2 - y}$

20. $\frac{m^2 - n^2}{6} \cdot \frac{3}{m + n}$

21. $\frac{2u + 2v}{5} \cdot \frac{10}{u + v}$

Give each indicated product in its lowest terms.

Ⓑ **22.** $\frac{r^2 - s^2}{m^2 - n^2} \cdot \frac{m + n}{r - s}$

23. $\frac{a^2 - 49}{b^2 - 25} \cdot \frac{b - 5}{a + 7}$

24. $\frac{y^2 - 2y + 1}{3x^2} \cdot \frac{x}{y - 1}$

25. $\frac{b^2 + 2b + 1}{ab^2} \cdot \frac{a}{b + 1}$

26. $\frac{k^2 + k}{k^2} \cdot \frac{3k - 3}{k^2 - 1}$

27. $\frac{6m + 6n}{m^2 - n^2} \cdot \frac{m^2 - mn}{2m}$

28. $\frac{9x^2 - 1}{6x + 2} \cdot \frac{2}{3x - 1}$

29. $\frac{6a - 3b}{4a^2 - b^2} \cdot \frac{2a + b}{3}$

30. $\frac{s - t}{2} \cdot \frac{4s + 4t}{s^2 + 2st + t^2}$

31. $\frac{x^2 + 2x + 1}{5x - 5} \cdot \frac{15}{x + 1}$

32. $\frac{64r^2 - 1}{64r^2 + 16r + 1} \cdot \frac{8r + 1}{16r - 2}$

33. $\frac{x^2 + 3x + 2}{x + 1} \cdot \frac{x + 3}{x^2 + 5x + 6}$

34. $\frac{a^2 + 5a + 4}{a + 4} \cdot \frac{a + 5}{a^2 + 6a + 5}$

35. $\frac{c^2 - 5c - 6}{c^2 + 3c} \cdot \frac{c + 3}{6 - c}$

Ⓒ **36.** $\frac{2 - x}{2 + x} \cdot \frac{x + 2}{x - 2}$

37. $\frac{4 - b^2}{5b - 10} \cdot \frac{4}{b + 2}$

38. $\frac{2x^2 - 50}{5x^2 - 80} \cdot \frac{4x + 16}{3x - 15}$

39. $\frac{a^2 + ab}{b - a} \cdot \frac{a^2 - ab}{b + a}$

40. $\frac{3n^2 - 27}{6 - n - n^2} \cdot \frac{4 - 2n}{3 - n} \cdot \frac{3n + n^2}{6}$

41. $\frac{4k^2 - 9}{6 + k - 2k^2} \cdot \frac{2 - k}{15k - 10k^2} \cdot \frac{k^2}{2k + 3}$

42. $\frac{2x^2 + 7x + 3}{x^2 - 9} \cdot \frac{x^2 - 3x}{2x^2 + 11x + 5}$

43. $\frac{3n^2 + 7n + 2}{1 - 9n^2} \cdot \frac{3n^2 - 7n + 2}{4 - n^2}$

44. $\frac{4 - 16x^2}{16x^2 - 4x - 6}(4x^2 - 11x + 6)\frac{x^6}{4x^2 - 8x^3}$

45. $\frac{90a^2 - 6ab - 12b^2}{15a - 6b} \times \frac{3a^4 - a^2b^2}{81a^4 - b^4}(9a^2 + b^2)$

8–6 Dividing Fractions

A quotient can be expressed as the product of the dividend and the reciprocal of the divisor (page 139).

$$6 \div 3 = 6 \times \tfrac{1}{3}; \quad 12 \div \tfrac{1}{3} = 12 \times 3; \quad \tfrac{4}{5} \div \tfrac{2}{15} = \tfrac{4}{5} \times \tfrac{15}{2}$$

Since the reciprocal of $\frac{c}{d}$ is $\frac{d}{c}$, if $c \neq 0$, $d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

EXAMPLE. $\dfrac{t^2 - 9}{t + 3} \div \dfrac{t^2 - 6t + 9}{3 - t}$

Solution:

$$\frac{t^2 - 9}{t + 3} \div \frac{t^2 - 6t + 9}{3 - t} = \frac{t^2 - 9}{t + 3} \cdot \frac{3 - t}{t^2 - 6t + 9}$$

$$= \frac{(t + 3)(t - 3)}{(t + 3)} \cdot \frac{(3 - t)}{(t - 3)(t - 3)}$$

$$= \frac{(t + 3)(t - 3)(-1)(t - 3)}{(t + 3)(t - 3)(t - 3)} = -1$$

Have your class explain why $3 - t = (-1)(t - 3)$. See 8-6 T.M. pg. 30.

WRITTEN EXERCISES

Find each quotient.

A

1. $\dfrac{3}{7} \div \dfrac{11}{14}$

2. $\dfrac{12}{11} \div \dfrac{6}{7}$

3. $\dfrac{a}{b} \div \dfrac{a}{b^2}$

4. $\dfrac{c^2}{d} \div \dfrac{c}{d}$

5. $\dfrac{3b}{4c^2} \div \dfrac{15b^2}{16c^3}$

6. $\dfrac{13r^2}{20a^2} \div \dfrac{39r^3}{5a}$

7. $\dfrac{3g^2}{10} \div \dfrac{9g^2}{25}$

8. $\dfrac{12x^3}{35} \div \dfrac{48x^3}{77}$

9. $\dfrac{a - b}{4} \div \dfrac{a - b}{2}$

10. $\dfrac{2x + 1}{2} \div \dfrac{2x + 1}{6}$

11. $\dfrac{9}{11x} \div 18$

12. $\dfrac{4a^2}{7} \div 8a$

13. $\dfrac{m^2 - n^2}{mn} \div (m - n)$

14. $\dfrac{a^2 + 2ab + b^2}{ab} \div (a + b)$

15. $\dfrac{x}{x^2 - 4x + 4} \div \dfrac{1}{x - 2}$

16. $\dfrac{m}{m^2 + 6m + 9} \div \dfrac{1}{m + 3}$

17. $\dfrac{1}{x^2 - 16} \div \dfrac{x - 4}{x + 4}$

18. $\dfrac{2x + 4}{3x + 9} \div \dfrac{4x + 8}{5x + 15}$

19. $\dfrac{ax - 2a}{bx - 3b} \div \dfrac{cx - 2c}{dx - 3d}$

20. $\dfrac{m^2 - n^2}{9m^2n^2} \div \dfrac{3m - 3n}{mn}$

21. $\dfrac{5a^2 - 5b^2}{a^2b^2} \div \dfrac{a + b}{10ab}$

22. $\dfrac{x^2 - y^2}{ax - ay} \div \dfrac{x^2 + 2xy + y^2}{x^2 + xy}$

23. $\dfrac{x^2 + 4x + 3}{x^2 - 4x - 5} \div \dfrac{x + 3}{x - 5}$

B **24.** $\dfrac{a + 2a^2}{9 - a^2} \div \dfrac{1 - 4a^2}{3a + a^2}$

25. $\dfrac{r^3 - rs^2}{r^2s} \div \dfrac{r^2 + rs}{3rs + 3s^2}$

26. $\dfrac{x^4 - y^4}{3x + 3y} \div \dfrac{2x^2 + 2y^2}{6}$

27. $\dfrac{x^2 - 3x + 2}{x^2 - 1} \div \dfrac{2 - x}{1 - x}$

28. $\dfrac{5c^2 - 5cd}{cd + d^2} \div \dfrac{d^3 - dc^2}{cd^2}$

29. $\dfrac{14}{7t^2 + 7u^2} \div \dfrac{2u + 2t}{u^4 - t^4}$

8–7 Fractions Involving Multiplication and Division

In the absence of parentheses the rule for order of performing multiplications and divisions is applied to an expression containing fractions. You replace only the fraction immediately following a division sign by its reciprocal. See Section 1-9 page 23.

EXAMPLE. $\dfrac{(y + 1)^2}{15} \div \dfrac{y + 1}{5} \cdot \dfrac{3}{y + 1}$

Solution:
$$\frac{(y + 1)^2}{15} \div \frac{y + 1}{5} \cdot \frac{3}{y + 1} = \frac{(y + 1)^2}{15} \cdot \frac{5}{y + 1} \cdot \frac{3}{y + 1}$$
$$= \frac{(y + 1)(y + 1)(5)(3)}{(y + 1)(y + 1)(15)}$$
$$= 1, \text{ Answer.}$$

Check: Let $y = 3$. The check is left to you.

Use these exercises to review work with fractions.

WRITTEN EXERCISES

Multiply or divide, and check by substitution.

A **1.** $\dfrac{3}{x^3} \cdot \dfrac{x^2}{5} \div \dfrac{6}{xy}$

2. $\dfrac{a^4}{4} \cdot \dfrac{3}{a^3} \div \dfrac{6a}{b}$

3. $\frac{a^2b}{c} \cdot \frac{c^3d}{a^3} \div b^2d$

4. $\frac{6r^2s^3}{t} \div 9r^2s^4 \cdot \frac{3t^4}{s^2}$

5. $\frac{x^2 - 1}{x^2} \cdot \frac{x}{x - 1} \div \frac{x + 1}{5}$

6. $\frac{3a}{3a + 2} \cdot \frac{6a + 4}{4a} \div \frac{a + 2}{2}$

7. $\frac{9m + n}{9mn} \div \frac{3m - n}{3m} \cdot \frac{12}{3m + n}$

8. $\frac{2c - d}{2c} \div \frac{8c + d}{8cd} \cdot \frac{16c + 2d}{4}$

9. $\frac{a^2 - b^2}{a^2} \cdot \frac{a + b}{a - b} \div \frac{a^2 + 2ab + b^2}{ab}$

10. $\frac{r^2 - 2rs + s^2}{rs} \cdot \frac{r + s}{r - s} \div \frac{r^2 - s^2}{r^2}$

11. $\frac{m}{m + 1} \div \frac{2m^2}{2m + 6} \cdot \frac{m^2 - 2m - 3}{m^2 - 9}$

12. $\frac{3x - 1}{3x^2} \div \frac{3x + 1}{3x} \cdot \frac{9x^2 + 6x + 1}{9x^2 - 1}$

13. $\frac{a^3}{3a} \div \frac{a^2b^2 - 9}{b} \cdot \frac{3ab - 9}{ab}$

14. $\frac{4t}{s^2t^2 - 4} \div \frac{16}{st} \cdot \frac{4st - 8}{st^2}$

B 15. $\frac{x^2 + 10x + 25}{x^2 + 10x} \cdot \frac{10x}{x^2 + 15x + 50} \div \frac{x + 5}{x + 10}$

16. $\frac{a^2 - 12a + 32}{8a} \cdot \frac{a^2 - 8a}{a^2 - 8a + 16} \div \frac{a - 8}{a - 4}$

17. $\frac{3a^3}{3a + c} \div \frac{9a^2}{9a^2 - c^2} \cdot \frac{3ab}{3a^2 - ac}$

18. $\frac{x^3y^3}{x^3 - xy^2} \div \frac{xyz}{x^2 - xy} \cdot \frac{xy + yz}{xy}$

19. $\frac{ab + ac - ad}{abc} \cdot \frac{bcd^2}{bd + cd - d^2} \div d^2$

20. $\frac{v^2w^2x^3}{vx - wx - x^2} \cdot \frac{uv - uw - ux}{uvw} \div x^2$

ADDING AND SUBTRACTING FRACTIONS

8–8 Combining Fractions with Equal Denominators

Consider the sum $\frac{a}{b} + \frac{c}{b}$. Since $\frac{a}{b} = a\left(\frac{1}{b}\right)$ and $\frac{c}{b} = c\left(\frac{1}{b}\right)$, by

Emphasize the use of the distributive property. See 8-8 T.M. pg. 30.

the distributive principle, you know that

$$\frac{a}{b} + \frac{c}{b} = a\left(\frac{1}{b}\right) + c\left(\frac{1}{b}\right) = (a + c)\left(\frac{1}{b}\right) = \frac{a + c}{b}.$$

This chain of equalities gives the ***rule for adding fractions:***

> The sum of fractions with equal denominators is a fraction whose numerator is the sum of the numerators and whose denominator is the common denominator of the given fractions.

EXAMPLES.

$$\frac{4}{9} + \frac{7}{9} - \frac{y}{9} = \frac{4 + 7 - y}{9} = \frac{11 - y}{9}$$

$$\frac{5}{p + q} + \frac{q}{p + q} - \frac{q - p}{p + q} = \frac{5 + q - (q - p)}{p + q} = \frac{5 + p}{p + q}$$

Sometimes you get a sum which can be reduced.

$$\begin{aligned} \frac{7a}{a^2 - b^2} - \frac{5a + 4b}{a^2 - b^2} + \frac{2b}{a^2 - b^2} &= \frac{7a - (5a + 4b) + 2b}{a^2 - b^2} \\ &= \frac{2a - 2b}{a^2 - b^2} \\ &= \frac{2\cancel{(a - b)}}{(a + b)\cancel{(a - b)}} \\ &= \frac{2}{a + b} \end{aligned}$$

WRITTEN EXERCISES

Give each sum in lowest terms.

Ⓐ

1. $\frac{4}{15} - \frac{8}{15} - \frac{7}{15}$
2. $\frac{3}{50} - \frac{29}{50} + \frac{1}{50}$
3. $\frac{8}{3z} - \frac{1}{3z} + \frac{2}{3z}$
4. $\frac{10}{7a} + \frac{15}{7a} - \frac{4}{7a}$

5. $\frac{12a}{5r} - \frac{2a + 5}{5r}$

6. $\frac{2b}{9t} - \frac{3 - 7b}{9t}$

7. $\frac{2a + 1}{5} + \frac{4 - 2a}{5}$

8. $\frac{3b + 5}{7} + \frac{2 - 3b}{7}$

9. $\frac{3r}{r + s} + \frac{3s}{r + s}$

10. $\frac{x}{x - 3} - \frac{3}{x - 3}$

11. $\frac{a^2}{a + b} - \frac{b^2}{a + b}$

12. $\frac{z^2}{z - 4} - \frac{16}{z - 4}$

13. $\frac{3x - 3y}{2xy} - \frac{2x + 5y}{2xy} + \frac{8y - x}{2xy}$

14. $\frac{5}{3ab} + \frac{1 - 2a}{3ab} - \frac{6 - 2a}{3ab}$

B 15. $\frac{2xy}{x + y} + \frac{x^2 + y^2}{x + y}$

16. $\frac{m^2 + n^2}{m - n} - \frac{2mn}{m - n}$

17. $\frac{3}{x^2 - 3x - 4} + \frac{x - 2}{x^2 - 3x - 4}$

18. $\frac{2x}{x^2 - x - 2} - \frac{x + 2}{x^2 - x - 2}$

19. $\frac{1 + a}{a^2 + 3a - 4} - \frac{2a + a^2}{a^2 + 3a - 4}$

20. $\frac{5 - b}{b^2 - 16} - \frac{b^2 - 3}{b^2 - 16}$

8–9 Adding Fractions with Unequal Denominators

See T.M. pg. 30 (4).

To add $\frac{5}{12}$ and $\frac{41}{90}$, express them as fractions with equal denominators. Any positive integer having both 12 and 90 as factors will do, but for convenience you usually want the *lowest common denominator* (**L.C.D.**). To find the L.C.D. systematically, write 12 and 90 as the products of primes:

This also is called the "least" common denominator. The L.C.D. is the least common multiple of the denominators. See T.M. pg. 30 (5).

$$12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3 \quad \textit{and} \quad 90 = 2 \cdot 3 \cdot 3 \cdot 5 = 2 \cdot 3^2 \cdot 5$$

$$\therefore \textit{the } \text{L.C.D.} = 2^2 \cdot 3^2 \cdot 5 = 180.$$

To convert these fractions to the denominator 180, you do this:

$$\frac{5}{12} = \frac{5 \cdot 3 \cdot 5}{12 \cdot 3 \cdot 5} = \frac{75}{180} \quad \textit{and} \quad \frac{41}{90} = \frac{41 \cdot 2}{90 \cdot 2} = \frac{82}{180}.$$

Point out the use of the multiplicative property of fractions, on page 283.

To complete the problem, you add.

$$\frac{5}{12} + \frac{41}{90} = \frac{75}{180} + \frac{82}{180} = \frac{75 + 82}{180} = \frac{157}{180}$$

EXAMPLE $\dfrac{a+1}{a^2-2a-3} - \dfrac{1}{a^2+a} - \dfrac{3}{a^2-3a}$

Solution:

1. Find the L.C.D.

Factor each denominator. →

$$a^2 - 2a - 3 = (a-3)(a+1)$$
$$a^2 + a = a(a+1)$$
$$a^2 - 3a = a(a-3)$$

Take each factor the greatest number of times it appears in any denominator → $a(a+1)(a-3)$

2. Write with factored denominators.

$$\frac{a+1}{(a-3)(a+1)} - \frac{1}{a(a+1)} - \frac{3}{a(a-3)}$$

3. Replace each fraction by an equivalent fraction.

$$\frac{(a+1)a}{(a-3)(a+1)a} - \frac{1(a-3)}{a(a+1)(a-3)} - \frac{3(a+1)}{a(a-3)(a+1)}$$

4. Combine the fractions, and simplify.

$$\frac{a(a+1) - 1(a-3) - 3(a+1)}{a(a+1)(a-3)} = \frac{a^2 + a - a + 3 - 3a - 3}{a(a+1)(a-3)}$$
$$= \frac{a^2 - 3a}{a(a+1)(a-3)}$$
$$= \frac{a(a-3)}{a(a+1)(a-3)}$$
$$= \frac{1}{a+1}, \text{ Answer.}$$

Help students to see that this procedure is essentially the same as that used in changing common fractions to have a common denominator.

Check: **Let $a = 2$. The check is left to you.**

ORAL EXERCISES

Give the L.C.D. of these denominators.

1. 6, 9 18
2. 10, 4 20
3. 2, a $2a$
4. 3, b $3b$
5. 15, 6 30
6. 14, 4 28
7. ab, bc abc
8. m^2n^2, mn m^2n^2

9. x, xy, y xy **10.** a, b, ab ab **11.** $n - m, m - n$ $n-m$ or $m-n$ **12.** $c - b, b - c$ $c-b$ or $b-c$

Give the lowest common denominator.

13. $\frac{a}{a+b} - \frac{b}{a^2-b^2}$ a^2-b^2

14. $\frac{1}{r^2-s^2} + \frac{1}{r-s}$ r^2-s^2

15. $\frac{2}{x^2+2xy+y^2} - \frac{3}{x+y}$ $(x+y)^2$

16. $\frac{5}{a-b} + \frac{1}{a^2-2ab+b^2}$ $(a-b)^2$

17. $\frac{4x}{x+4} + \frac{2x}{x+3}$ $x^2+7x+12$

18. $\frac{3t}{t+2} + \frac{5t}{t+3}$ t^2+5t+6

19. $\frac{x}{x^2-y^2} + \frac{y}{x+y} + \frac{xy}{y-x}$ x^2-y^2 or y^2-x^2

20. $\frac{x}{x^2-9} + \frac{3}{x+3} + \frac{3x}{3-x}$ x^2-9 or $9-x^2$

WRITTEN EXERCISES

Combine the fractions.

A

1. $\frac{4b}{a} - \frac{1}{2}$

2. $\frac{1}{3} + \frac{2a}{x}$

3. $\frac{7}{6c} + \frac{1-3c}{3c}$

4. $\frac{1}{10a} - \frac{2a+5}{2a}$

5. $\frac{2a+1}{5} - \frac{a-3}{2}$

6. $\frac{7n-4}{4} - \frac{3n+2}{3}$

7. $\frac{2h+1}{4} + \frac{h-3}{6}$

8. $\frac{3h-5}{9} - \frac{2h+1}{6}$

9. $\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x^3}$

10. $\frac{3}{a^3} - \frac{2}{a^2} + \frac{1}{a}$

11. $\frac{1}{ab} + \frac{a-2}{bc}$

12. $\frac{4}{rs} - \frac{2-t}{rt}$

13. $\frac{x}{bc} + \frac{x+c}{ac} - \frac{x+b}{ab}$

14. $\frac{a}{yz} - \frac{a+z}{xz} + \frac{a+y}{xy}$

15. $\frac{3}{xy^2} + \frac{2-x}{x^2y}$

16. $\frac{4}{r^2s} - \frac{2-r}{rs^2}$

17. $\frac{1}{a+b} + \frac{a}{a^2-b^2}$

18. $\frac{d}{c^2-d^2} - \frac{1}{d-c}$

19. $\frac{5-4k}{8} - \frac{2-3k}{6}$

20. $\frac{4+3k}{8} + \frac{k-6}{12}$

21. $\frac{1}{6} + \frac{3x-1}{4} - \frac{x+2}{2}$

22. $\frac{2}{9} - \frac{2x-3}{6} + \frac{x-1}{2}$

23. $\frac{2x-5}{2-x} + \frac{x}{2x-4}$

24. $\frac{3a}{2a+6} - \frac{a-1}{a+3}$

B **25.** $\dfrac{4x-7}{x^2-3x+2} - \dfrac{3}{x-1}$

26. $\dfrac{6m-13}{m^2-5m+6} - \dfrac{5}{m-3}$

27. $\dfrac{x}{x^2-25} - \dfrac{1}{2x+10}$

28. $\dfrac{27}{x^2-81} + \dfrac{3}{2x+18}$

29. $\dfrac{2a+3b}{3a^2b} - \dfrac{a+2b}{4ab^2} - \dfrac{1}{6ab}$

30. $\dfrac{3}{2m^2} + \dfrac{2m}{8m^2n} - \dfrac{3m+n}{4mn^2}$

31. $\dfrac{2}{x-1} - \dfrac{3}{1+x} - \dfrac{x-5}{1-x^2}$

32. $\dfrac{3}{2+n} + \dfrac{2}{n-2} + \dfrac{5n-2}{4-n^2}$

33. $\dfrac{2}{3+y} + \dfrac{5}{y^2-9} + \dfrac{2y-1}{3-y}$

34. $\dfrac{3}{z^2-25} - \dfrac{1}{5+z} - \dfrac{z+1}{5-z}$

C **35.** $\dfrac{a-2}{(a+2)^2} + \dfrac{a}{5(a+2)} - \dfrac{1}{25}$

36. $\dfrac{x+1}{(x-1)^2} - \dfrac{x}{x^2-1} + \dfrac{1}{x-1}$

37. $\dfrac{a-6b}{2a^2+5ab+2b^2} + \dfrac{3}{2a+b} - \dfrac{7}{a+2b}$

38. $\dfrac{x-y}{(x+y)^2} - \dfrac{x}{x^2-y^2} - \dfrac{1}{x-y}$

8–10 Mixed Expressions

A mixed numeral like $2\frac{3}{8}$ denotes the sum of an integer and a fraction. When you transform it into an improper fraction, you write the integer as a fraction with denominator 1 and add two fractions with unequal denominators.

Emphasize the analogy between mixed numbers and mixed expressions.

$$2\tfrac{3}{8} = \frac{2}{1} + \frac{3}{8} = \frac{16}{8} + \frac{3}{8} = \frac{19}{8}$$

The sum or difference of a polynomial and a fraction is called a **mixed expression**. A mixed expression can be written as a single fraction:

$$x + \frac{3}{x} = \frac{x}{1} + \frac{3}{x} = \frac{x^2}{x} + \frac{3}{x} = \frac{x^2+3}{x}$$

$$2 - \frac{a-b}{a+b} = \frac{2}{1} - \frac{a-b}{a+b} = \frac{2(a+b)}{a+b} - \frac{a-b}{a+b} = \frac{a+3b}{a+b}$$

If the numerator is a polynomial, you can change a fraction to a mixed expression by doing the division:

$$\frac{3y^2 - 2}{3y} = y - \frac{2}{3y}$$

$$\frac{2c^2 + 5c - 6}{c - 1} = 2c + 7 + \frac{1}{c - 1}$$

ORAL EXERCISES

Read each expression as a single fraction.

1. $2\frac{2}{3}$ $\frac{8}{3}$

2. $5\frac{1}{6}$ $\frac{31}{6}$

3. $-3\frac{3}{4}$ $-\frac{15}{4}$

4. $-7\frac{2}{5}$ $-\frac{37}{5}$

5. $a + \frac{3}{b}$ $\frac{ab+3}{b}$

6. $4 + \frac{x}{y}$ $\frac{4y+x}{y}$

7. $x - \frac{3}{2x}$ $\frac{2x^2-3}{2x}$

8. $y + \frac{6}{7y}$ $\frac{7y^2+6}{7y}$

9. $3 - \frac{s}{t + 2}$ $\frac{3t+6-s}{t+2}$

10. $1 + \frac{n}{m - 3}$ $\frac{m-3+n}{m-3}$

11. $8 + \frac{a}{b - c}$ $\frac{8b-8c+a}{b-c}$

12. $4 - \frac{xy}{x - y}$ $\frac{4x-4y-xy}{x-y}$

Read each fraction as a mixed expression.

13. $\frac{x^2 + 2}{x}$ $x+\frac{2}{x}$

14. $\frac{a^2 + 5}{a}$ $a+\frac{5}{a}$

15. $\frac{b^2 + 2b + 2}{b}$ $b+2+\frac{2}{b}$

16. $\frac{z^2 + 3z + 7}{z}$ $z+3+\frac{7}{z}$

17. $\frac{ab + b + c}{b}$ $a+1+\frac{c}{b}$

18. $\frac{2y^2 + 4y + 7}{y}$ $2y+4+\frac{7}{y}$

WRITTEN EXERCISES

Express each mixed expression as a single fraction, in simplest form.

A **1.** $y + \frac{1}{y + 1}$

2. $b + \frac{a}{a + b}$

3. $1 + \frac{x - y}{x + y}$

4. $3 + \frac{a + b}{a - b}$

5. $a + 1 + \frac{1}{a - 1}$

6. $x - 2 + \frac{4}{x + 2}$

7. $\frac{2}{x - 1} + 1$

8. $\frac{4}{x - 2} + 1$

9. $\frac{x}{y} + 2 + \frac{y}{x}$

10. $\frac{a}{b} + 2 + \frac{b}{a}$

11. $s - 3 + \frac{5s - 6}{s - 2}$

12. $x + 5 - \frac{x - 20}{x - 4}$

Change each fraction to a mixed expression.

13. $\frac{1324}{15}$

14. $\frac{2591}{13}$

15. $\frac{12x^3 + 6}{3x}$

16. $\dfrac{8 - 2y^2}{2y}$

17. $\dfrac{6a^3 - 1}{3a^3}$

18. $\dfrac{6 - 8c^8}{4c^4}$

19. $\dfrac{3a^2b^2 + 9ab}{3a^2b^2}$

20. $\dfrac{8rs - 4r^2s^2}{4r^2s^2}$

21. $\dfrac{x^2 + 5x + 6}{x + 1}$

22. $\dfrac{a^2 + 7a + 8}{a + 1}$

23. $\dfrac{12x^2 - 8x - 17}{2x - 3}$

24. $\dfrac{12a^2 + 14a - 17}{4a + 2}$

8–11 Complex Fractions (Optional)

A **complex fraction** is a fraction whose numerator or denominator contains one or more fractions. Complex fractions may be changed to simple ones by two methods.

Method I: Multiply the numerator and denominator by the L.C.D. of all the fractions within them.

Method II: Express the fraction as a quotient, using the sign ÷, and divide.

See 8-11 T.M. pg. 30 for choice of method.

EXAMPLE 1. $\dfrac{\frac{5}{6}}{\frac{7}{4}}$ *Solution:*

Method I

$$\frac{\frac{5}{6}}{\frac{7}{4}} = \frac{\frac{5}{6}(12)}{\frac{7}{4}(12)} = \frac{10}{21}$$

Method II

$$\frac{\frac{5}{6}}{\frac{7}{4}} = \frac{5}{6} \div \frac{7}{4} = \frac{5}{6} \cdot \frac{4}{7} = \frac{10}{21}$$

EXAMPLE 2. $\dfrac{\dfrac{y - 3}{2y^2}}{\dfrac{3y + 1}{4y}}$

Solution:

Method I

$$\frac{\dfrac{y - 3}{2y^2}}{\dfrac{3y + 1}{4y}} = \frac{\dfrac{y - 3}{2y^2}(4y^2)}{\dfrac{3y + 1}{4y}(4y^2)} = \frac{2(y - 3)}{y(3y + 1)}$$

Method II

$$\frac{\dfrac{y - 3}{2y^2}}{\dfrac{3y + 1}{4y}} = \frac{y - 3}{2y^2} \div \frac{3y + 1}{4y}$$

$$= \frac{y - 3}{2y^2} \cdot \frac{4y}{3y + 1}$$

$$= \frac{4y(y - 3)}{2y^2(3y + 1)}$$

$$= \frac{2 \cdot 2y(y - 3)}{y \cdot 2y(3y + 1)}$$

$$= \frac{2(y - 3)}{y(3y + 1)}$$

WRITTEN EXERCISES

Simplify each fraction.

A

1. $\dfrac{\frac{7}{8}}{\frac{21}{16}}$

2. $\dfrac{\frac{15}{16}}{\frac{45}{64}}$

3. $\dfrac{\frac{a}{b}}{\frac{a^2}{b^2}}$

4. $\dfrac{\frac{x^2}{y^2}}{\frac{x}{y}}$

5. $\dfrac{\frac{8a^2b}{9c}}{\frac{4ac}{3b}}$

6. $\dfrac{\frac{22r}{7s^2t}}{\frac{11s}{21rt}}$

7. $\dfrac{\frac{k-m}{m}}{\frac{k+m}{3m}}$

8. $\dfrac{\frac{x+2}{2x}}{\frac{x-2}{x}}$

9. $\dfrac{\frac{x^2-16}{x}}{x-4}$

10. $\dfrac{a^2-25}{\frac{a+5}{a}}$

11. $\dfrac{\frac{5}{6}+\frac{1}{2}}{\frac{1}{5}-4}$

12. $\dfrac{\frac{3}{4}+\frac{2}{5}}{\frac{7}{10}-2}$

13. $\dfrac{\frac{m+2}{m-2}}{\frac{2m+4}{m^2-4}}$

14. $\dfrac{\frac{ax+ab}{x^2-b^2}}{\frac{x+b}{x-b}}$

15. $\dfrac{\frac{m}{n}-1}{\frac{m}{n}+1}$

16. $\dfrac{\frac{t}{p}+1}{\frac{t}{p}-1}$

B

17. $\dfrac{\frac{a^2+b^2}{ab}+2}{\frac{a^2-b^2}{2ab}}$

18. $\dfrac{\frac{x^2-9y^2}{6xy}}{\frac{x^2+3y^2}{2xy}-2}$

19. $\dfrac{\frac{p+q}{p^2+q^2}-\frac{1}{q}}{\frac{p+q}{p^2+q^2}-\frac{1}{p}}$

20. $\dfrac{\frac{r-s}{r^2+s^2}-\frac{1}{r}}{\frac{r-s}{r^2+s^2}-\frac{1}{s}}$

21. $\dfrac{\frac{2}{a-1}+\frac{a-1}{a+1}}{1+\frac{a+1}{a-1}}$

22. $\dfrac{\frac{b-3}{b+3}-1}{\frac{b+3}{b-3}+\frac{2}{b+3}}$

C

23. $1+\dfrac{1}{2+\frac{1}{3}}$

24. $1+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a}}}$

25. $1+\cfrac{1}{1+\cfrac{1}{b+\cfrac{1}{b}}}$

Determine each solution set.

26. $1+\dfrac{2+\frac{6}{x}}{x+3}=\dfrac{1}{x}$

27. $1+\dfrac{4-\frac{8}{y}}{y-2}=\dfrac{1}{y}$

FRACTIONS IN OPEN SENTENCES AND PROBLEMS

8–12 Open Sentences with Fractional Coefficients

Methods previously used to solve open sentences may be used also when the numerical coefficients are fractions.

EXAMPLE. $\frac{5x}{4} + \frac{x+3}{10} = 3$ See 8-12 T.M. pg. 30.

Solution 1:

Point out the use of the distributive property.

$$\frac{5x}{4} + \frac{x+3}{10} = 3$$

$$20\left(\frac{5x}{4} + \frac{x+3}{10}\right) = 20(3)$$

$$20\left(\frac{5x}{4}\right) + 20\left(\frac{x+3}{10}\right) = 20(3)$$

$$25x + 2x + 6 = 60$$

Solution 2:

$$\frac{5x}{4} + \frac{x+3}{10} = 3$$

$$\frac{25x}{20} + \frac{2(x+3)}{20} = 3$$

$$\frac{25x + 2x + 6}{20} = 3$$

$$\frac{27x + 6}{20} = 3$$

$$27x + 6 = 60$$

$$27x = 54$$

$$x = 2$$

The check is left to you.

ORAL EXERCISES

State the L.C.D., and read each open sentence after multiplying all its terms by the L.C.D.

SAMPLE. $\frac{4p}{21} + \frac{p}{6} = 2$ *What you say:* The L.C.D. is 42; $8p + 7p = 84$.

1. $\frac{2x}{3} + \frac{x}{5} = 3$ 15; $10x + 3x = 45$
2. $\frac{h}{2} + \frac{5h}{7} = 10$ 14; $7h + 10h = 140$
3. $\frac{s}{3} - s > \frac{7}{5}$ 15; $5s - 15s > 21$
4. $\frac{1}{2}m < \frac{5}{6}m - 6$ 6; $3m < 5m -$

5. $\frac{1}{3}n + \frac{1}{2}n = n - 2$ 6; $2n+3n = 6n-12$

6. $\frac{1}{4}g + \frac{1}{3}g = g + 3$ 12; $3g+4g = 12g+36$

7. $\frac{t}{4} - t \leq \frac{11}{12}$ 12; $3t-12t \leq 11$

8. $\frac{x}{3} \geq \frac{7x}{9} - 9$ 9; $3x \geq 7x-81$

9. $\frac{x^2}{2} + \frac{x}{6} - \frac{1}{3} = 0$ 6; $3x^2+x-2=0$

10. $\frac{u^2}{5} - \frac{u}{2} + \frac{7}{10} = 0$ 10; $2u^2-5u+7=0$

11. $b^2 - \frac{2b}{39} + \frac{3}{13} = 0$ 39; $39b^2-2b+9=0$

12. $q^2 + \frac{5q}{22} - \frac{7}{2} = 0$ 22; $22q^2+5q-77=0$

WRITTEN EXERCISES

Solve. If the sentence is an inequality, graph its solution set.

A

1. $\frac{b}{5} - \frac{b}{10} = \frac{1}{10}$

2. $\frac{k}{4} - \frac{k}{8} = \frac{3}{8}$

3. $\frac{1}{3}y + \frac{1}{4}y > \frac{7}{2}$

4. $\frac{1}{4}x + \frac{1}{5}x < \frac{9}{2}$

5. $\frac{2}{3}g - \frac{3}{5}g = 1$

6. $\frac{3}{7}g - \frac{1}{3}g = 4$

7. $.02c \geq .01c - .1$

8. $\frac{2c}{3} + 1 \leq \frac{c}{2}$

9. $3z - \frac{7}{3}z = 1$

10. $.5s - 1.4s = .9$

11. $\frac{n}{5} - \frac{n}{3} < \frac{6}{5}$

12. $\frac{x}{7} - \frac{x}{4} > \frac{6}{7}$

13. $\frac{4}{3}h = \frac{1}{2}h + 45$

14. $\frac{3}{4}n = 26 - \frac{1}{3}n$

15. $\frac{3m - 5}{2} - \frac{m}{3} = 8$

16. $\frac{3m}{2} + \frac{8 - 4m}{7} = 3$

17. $\frac{n + 3}{2} \geq \frac{n - 8}{5} + 1$

18. $\frac{n - 3}{4} \leq \frac{2n - 5}{5} + 1$

19. $\frac{3}{4}(2r + 5) - \frac{5}{8}(3r - 1) = 1$

20. $\frac{2}{9}(s - 3) - 3 = \frac{1}{3}(2s + 5)$

21. $\frac{n + 3}{8} - \frac{n - 2}{6} = 1$

22. $\frac{n + 5}{12} - \frac{n + 3}{8} = 1$

23. $.04x + .06(20{,}000 - x) = 960$

24. $.03k + .05(1000 - k) = 34$

25. $\frac{2w + 5}{4} - \frac{10w + 13}{8} = 2w + 1$

26. $\frac{3w + 5}{3} = 3w + 2 - \frac{3w - 4}{6}$

27. $.15(y - 5) - .02(4y - 3) + .9 = 0$

28. $.08(4y + 5) - .03(2y - 3) = .36$

29. $\dfrac{2t + 3}{9} - \dfrac{4t - 1}{6} - \dfrac{9 - 8t}{18} = 0$

30. $\dfrac{1 - 6t}{10} - \dfrac{2t + 3}{6} - \dfrac{13 + 6t}{4} = 0$

8–13 Investment Problems

See 8-13 T.M. pg. 31.

The simple interest i on p dollars at the interest rate r per cent per year for t years is given by the equation $i = prt$. Thus, the interest on \$100 invested at 6% for 2 months is

$$i = prt = 100(\tfrac{6}{100})\tfrac{2}{12} = \$1.$$

Investment problems may concern money invested at different rates of interest.

EXAMPLE **An alumni association sets up a fund to grant annual scholarships totaling at least \$1800. They invest 25% of the fund in stocks yielding 3%, the rest in bonds at 5%. How large is the fund?**

Solution:

1

Let x = the number of dollars in the fund

25% of x or $.25x$ = the number of dollars invested at 3%

75% of x or $.75x$ = the number of dollars invested at 5%

$(.03)(.25x)$ = yearly income from stocks

$(.05)(.75x)$ = yearly income from bonds

2

$$\underbrace{\text{Stock income}}_{(.03)(.25x)} + \underbrace{\text{Bond income}}_{(.05)(.75x)} \geq \underbrace{\text{least amount for scholarships}}_{1800}$$

3

$$\begin{aligned} 75x + 375x &\geq 1800(10{,}000) \\ 450x &\geq 18{,}000{,}000 \\ x &\geq 40{,}000 \end{aligned}$$

4 The check is left to you.

Multiply each member of the open sentence by 10^4, since $(.03)(.25) = \frac{3}{10^2} \cdot \frac{25}{10^2} = \frac{75}{10^4}$ and $(.05)(.75) = \frac{5}{10^2} \cdot \frac{75}{10^2} = \frac{375}{10^4}$.

PROBLEMS

A

1. How much interest is earned in 2 years on $500 invested at 4% per year?
2. A man invests $600 at 5% interest per year. How much interest does he receive after 3 years?
3. John invests a sum at 4% per year. After half a year, he receives $20 interest. Find the investment.
4. An investment at 6% simple annual interest earned $360 in 5 years. Find the investment.
5. A man borrowed $1200 for 2 years at 6 per cent per year. What amount did he pay back at the end of the time? $[a = p(1 + rt)]$
6. A man invested $750 at simple interest of 3% per year. What amount of money did he have after 4 years?
7. Mr. Collins invests a sum at 4%, and an equal amount at 6%. His return totals $40 a year. How much money is invested at each rate?
8. Mr. Samson invested half a sum at 3%, half at 4%. His total annual income was $189. How much did he invest at each rate?
9. Mr. Turner invested a sum of money at 7 per cent, and twice that sum at 3 per cent. His yearly return was $390. How much did Mr. Turner invest at each rate?
10. Mr. Martin risked a little money in an investment at 9%. He invested seven times as much money at 3%. After one year he received $750. What did he invest at each rate?
11. Mr. Paxton invested $2000 more than his wife. The income from both investments at 5% was $250 a year. How much did each invest?
12. Tom and Sally Avery each have money in postal savings, at 2% per year. Each year they use their $12 interest to buy a joint birthday present for their father. Tom's savings total $100 more than Sally's. How much has each?
13. A man invests $10,000, part at 4 per cent per year, and the remainder at 6 per cent. From this he receives $500 annually. How much does he invest at each rate?
14. A man invested some money at 5 per cent and $800 less at $3\frac{1}{2}$ per cent. He received $210 a year from these investments. How much was each?

15. The Best Mortgage Company invests $35,000 at 8% per year. How much must it invest at 10% per year for an annual income of $5000 from both investments?
16. Mrs. Able invests $7000 at 4%. Mr. Able wishes to invest enough at 6% for their combined annual income to total $1000. How much should he invest?

17. One sum is invested at 4%, and another, \$500 larger than the first is invested at 5%. The interest on the second amount exceeds that on the first by \$33 a year. How much is invested at each rate?

18. Part of \$6000 is invested at $3\frac{1}{2}\%$, and the rest, at 6%. The income at $3\frac{1}{2}\%$ exceeds the income at 6% by \$67.50 per year. Find the investment at each rate.

19. Thirty per cent of a fund is invested at 5% per year. The rest is invested at 4%. How much is invested at each rate, if the total income is \$860?

20. A man leaves 60% of his estate to his wife and the rest to his son. The wife invests at 5.5% per year and the son, at 4.5% per year. Find the wife's yearly income, if the son's is \$5400.

8–14 Per Cent Mixture Problems

Another type of problem involving per cent is the per cent mixture problem.

EXAMPLE **A solution of antifreeze is said to be 60% antifreeze, if 60% of the solution is pure antifreeze and 40% is water. How much water must be added to 10 quarts of a 60% antifreeze solution to obtain a 25% antifreeze mixture?**

Solution:

1 **Let y = number of quarts of water to be added.**

Amount of original solution: **10 quarts**

Amount of final solution: **$(10 + y)$ quarts**

Per cent pure antifreeze in original solution: **60%**

Per cent pure antifreeze in final solution: **25%**

Amount of pure antifreeze in original solution: **60% of 10 quarts**

Amount of pure antifreeze in final solution: **25% of $(10 + y)$ quarts**

2

$$\underbrace{\text{Amount of pure antifreeze in final solution}}_{.25(10 + y)} \underset{\downarrow}{=} \underbrace{\text{Amount of pure antifreeze in original solution}}_{.60(10)}$$

Show that 25% of the new solution equals 60% of the original.

Point out this application of the basic percentage relationship: P (percentage) = B (base) x R (rate).

PROBLEMS

A

1. How much water must be added to 2 quarts of a disinfectant containing 30% active ingredient to form a solution containing 20% active ingredient?

2. To reduce 16 ounces of a 25% solution of antiseptic to a 10% solution, how much distilled water should a nurse add?

3. How much water must be added to a barrel containing 48 pounds of a 10% brine (salt and water) to obtain a 6% brine?

4. How much water must be evaporated from 100 pounds of a 4% brine to get a 5% brine?

5. If a dairyman has 200 pounds of milk testing 3% butterfat, how many pounds of skimmed milk must he remove to have 3.6% butterfat?

6. In certain localities, "approved milk" must contain 3.2% butterfat. How much skimmed milk may a farmer add to 200 pounds of milk testing 4.8% butterfat to obtain a milk testing 3.2% butterfat?

7. How much alcohol must be added to a pint of tincture of arnica, containing 20% arnica and 80% alcohol to reduce it to a 10% arnica solution?

8. One quart of a 10% iodine solution (90% is alcohol) can be reduced to a 2% solution by adding how much alcohol?

B

9. A solution contains 40 grams of sugar in 200 grams of water. How much sugar must be added to make a 50% sugar solution?

10. A brine contains 45 pounds of water and 3 pounds of salt. How much salt must be added to obtain a 10% salt solution?

11. In 14-carat gold are 14 parts, by weight, of gold and 10 parts of other metal (usually copper). Coin gold is 90% gold and 10% copper. How many ounces of pure gold must be added to 24 ounces of 14-carat gold to make an alloy of coin gold?

12. How many ounces of copper must be alloyed with 185 ounces of pure silver to produce sterling silver, which contains 92.5% pure silver?

13. A chemist has a solution of 50% pure acid and another of 80% pure acid. How many ounces of each will make 300 ounces of a solution which is 72% pure acid?

14. The capacity of an automobile cooling system is 16 quarts. If it is full of a 15% antifreeze solution, how many quarts must be replaced by a 90% solution to give 16 quarts of a 65% solution?

8–15 Fractional Equations

An equation like $\frac{60}{z^2 - 36} + 1 = \frac{5}{z - 6}$ which has a variable in the denominator of one or more terms, is called a **fractional equation**.

See 8-15 T.M. pg. 31.

EXAMPLE Solve $\frac{60}{z^2 - 36} + 1 = \frac{5}{z - 6}$

Solution:

$$\frac{60}{z^2 - 36} + 1 = \frac{5}{z - 6}$$

$$(z^2 - 36) \cdot \frac{60}{z^2 - 36} + (z^2 - 36) \cdot 1 = (z^2 - 36) \cdot \frac{5}{(z - 6)}$$

$$60 + z^2 - 36 = 5z + 30$$

$$z^2 - 5z - 6 = 0$$

$$(z - 6)(z + 1) = 0$$

$$z = 6 \quad | \quad z = -1$$

When you test 6 and −1 in the original equation, notice what happens:

$$\frac{60}{z^2 - 36} + 1 = \frac{5}{z - 6}$$

Let $z = 6$.

$$\frac{60}{6^2 - 36} + 1 \stackrel{?}{=} \frac{5}{6 - 6}$$

$$\frac{60}{0} + 1 \stackrel{?}{=} \frac{5}{0}$$

The fractions $\frac{60}{0}$ and $\frac{5}{0}$ are meaningless, so 6 is not a root of the given equation.

Let $z = -1$.

$$\frac{60}{(-1)^2 - 36} + 1 \stackrel{?}{=} \frac{5}{-1 - 6}$$

$$\frac{60}{-35} + 1 \stackrel{?}{=} \frac{5}{-7}$$

$$-\frac{12}{7} + 1 \stackrel{?}{=} -\frac{5}{7}$$

$$-\frac{5}{7} = -\frac{5}{7} \checkmark$$

$\therefore$ The solution set is $\{-1\}$.

Multiplying the given equation by $z^2 - 36$ leads to an equation that is *not equivalent* to the given one. This new equation has the extra root 6 because the multiplier $z^2 - 36$ represents zero when z is replaced by 6.

> Multiplying an equation by a variable expression which can represent zero may produce an equation having roots not satisfying the original equation. Therefore, test each root found by substituting it in the original equation. Only values producing true statements belong to the solution set.

Because the original and the transformed equations are not equivalent, the check is an *essential* part of the solution not just a guard against numerical errors.

WRITTEN EXERCISES

Solve each equation.

A

1. $\dfrac{12}{s+2} = \dfrac{4}{s-2}$

2. $\dfrac{7}{t-3} = \dfrac{2}{t+2}$

3. $\dfrac{5}{x} = \dfrac{x-3}{2}$

4. $\dfrac{6}{z-1} = \dfrac{z}{2}$

5. $\dfrac{t}{t-4} = \dfrac{t+4}{6}$

6. $\dfrac{5}{v+6} = \dfrac{v-6}{v}$

7. $\dfrac{3}{4x} + \dfrac{1}{x} = \dfrac{7}{8}$

8. $\dfrac{2}{3x} + \dfrac{1}{x} = \dfrac{5}{9}$

9. $\dfrac{n+5}{2n} - \dfrac{7}{3n} = \dfrac{5}{12}$

10. $\dfrac{4}{3n} - \dfrac{n+4}{6n} = 2$

11. $a - \dfrac{2}{a-3} = \dfrac{a-1}{3-a}$

12. $c - \dfrac{c}{1-c} = \dfrac{2-c}{c-1}$

13. $\dfrac{4}{3t-2} + \dfrac{7}{3t} - \dfrac{1}{t} = 0$

14. $\dfrac{3}{4k} + \dfrac{4}{3k-1} - \dfrac{2}{k} = 0$

15. $\dfrac{2x}{x+2} - 2 = \dfrac{x-8}{x-2}$

16. $\dfrac{4y}{y-3} - 3 = \dfrac{3y-1}{y+3}$

17. $5c - \dfrac{c+1}{c+1} + 6 = 0$

18. $4k - \dfrac{k-1}{k-1} = 3$

19. $\dfrac{p+2}{3p-6} - \dfrac{2}{3p+6} + \dfrac{7}{9} = 0$

20. $\dfrac{3n-2}{15} - \dfrac{16-3n}{n+6} = \dfrac{n+3}{5}$

21. $\dfrac{3}{2m-6} - \dfrac{2m-3}{2m^2-5m-3} + \dfrac{1}{4m+2} = 0$

22. $\dfrac{2}{3m+12} - \dfrac{1}{9m-3} - \dfrac{m-2}{3m^2+11m-4} = 0$

23. $\dfrac{2t+1}{2t-3} - 1 = \dfrac{t-4}{2t+3} - \dfrac{7t}{9-4t^2}$

24. $\dfrac{2t+3}{t-1} - 4 - \dfrac{2}{t+3} = \dfrac{5-6t}{t^2+2t-3}$

25. $\dfrac{1}{3s-2} + \dfrac{1}{3s+4} = \dfrac{6}{9s^2+6s-8}$

26. $\dfrac{1}{2d+3} + \dfrac{1}{2d+1} = \dfrac{2}{4d^2+8d+3}$

8–16 Work Problems

An equation for work is $w = rt$, where w is the amount of work done, r is the rate of doing work, and t is the time worked. If several persons work together, the amount of work done is assumed to be the sum of the individual amounts.

EXAMPLE 1

A deck of punched cards can be read by one electronic reader in 20 minutes. A second can read the same deck in 12 minutes. In how much time could both readers together process the cards?

Solution:

1 **Let x = number of minutes needed for the readers processing the cards together.**

$\frac{1}{20}$ = rate of first reader (one-twentieth of the job in one minute)

$\frac{1}{12}$ = rate of second reader (one-twelfth of the job in one minute)

1 = work done (one whole job)

2

$$\underbrace{\textbf{Work done by both}}_{1} = \underbrace{\textbf{work done by first}}_{\frac{1}{20}x} \underset{+}{\underset{\downarrow}{\textbf{plus}}} \underbrace{\textbf{work done by second.}}_{\frac{1}{12}x}$$

3

$$\begin{aligned} (60)1 &= (60)\left(\frac{x}{20} + \frac{x}{12}\right) \\ 60 &= 3x + 5x \\ 60 &= 8x \\ \tfrac{15}{2} &= x \end{aligned}$$

4 *Check:* In $\frac{15}{2}$ minutes, the first reader processes $\frac{15}{2}(\frac{1}{20})$ of the deck.

In $\frac{15}{2}$ minutes, the second reader processes $\frac{15}{2}(\frac{1}{12})$ of the deck.

Does $\frac{15}{2}(\frac{1}{20}) + \frac{15}{2}(\frac{1}{12}) = 1$ (the job)? $\frac{3}{8} + \frac{5}{8} = 1$ ✓

It takes $7\frac{1}{2}$ minutes when both readers are used, Answer.

Sometimes those doing a job may not work for the same time. In such cases, the values substituted for time in the equation may differ.

EXAMPLE 2 **How long would it take to process the deck of punched cards of Example 1 if the first reader stopped after working 5 minutes?**

Solution:

1 **Let t = number of minutes required to process deck (time second reader operated).**

5 = number of minutes first reader operated

2 $$1 = \frac{1}{20}(5) + \frac{1}{12}t$$

Steps 3 and 4 are left to you.

PROBLEMS

A
1. One pipe can fill a tank in 5 hours. A second can fill it in 3 hours. How long will it take both pipes together to fill the tank?
2. A press can print one day's newspapers in 4 hours. A high-speed press can do the job in half that time. How fast can both presses together do the job?
3. A payroll is prepared by two computers in 6 hours. The faster computer can do the job itself in 10 hours. In what time can the slower computer do the job?
4. An air conditioner lowers the temperature 10 degrees in 12 minutes. With a second air conditioner also working, this change takes 4 minutes. How long would the second device need to produce that change?

5. One lathe can ream holes in a shipment of metal parts in 3 hours; a second also can do it in 3 hours, but a third needs 4 hours. How long will the job take, if all three lathes are used?

6. Each of two incinerators can process a day's refuse in 20 hours. Together with a third incinerator, they process the refuse in 6 hours. In what time can the third incinerator do the job?

7. One bulldozer clears land twice as fast as another. Together they clear a large tract in $1\frac{1}{2}$ hours. How long would the larger bulldozer take?

8. One pump fills a tank twice as fast as another. If together they fill the tank in 16 minutes, how long does the larger pump take?

Ⓑ 9. A swimming pool has two inlet pipes. One fills the pool in 3 hours, the other, in 6 hours. The outlet pipe empties the pool in 4 hours. Once the outlet pipe was left open when the pool was being filled. In how many hours was the pool full?

10. The hot water faucet fills a tub in 40 minutes, and the cold water faucet, in 30 minutes. The tub can be drained in 20 minutes. If both faucets are open while the drain is open, how soon will the tub be full?

11. It takes Mr. Shea 8 hours to paint his barn. If he works 1 hour and then asks a painter to help him, they finish in 3 more hours. In what time can the painter do the whole job?

12. One machine labels 1200 cans in one hour, and a second labels 900 cans in an hour. If the faster machine starts 1 hour before the slower, how long will it take to label 8200 cans?

Ⓒ 13. A job can be done by 8 men in 3 hours, or by 15 boys in 5 hours. How long would it take 3 men and 25 boys together?

14. A man contracts to build a road in 72 days, a job requiring 60 men. The man hires 50 men who work for a while until he realizes that he must hire 30 more to finish on time. How many days do these 30 men work?

15. Together, a man and his two sons assemble an electric train in 12 minutes. The job would take the man alone 10 minutes less than it would take either son. How soon could both sons, together, set up the train?

16. Together, three men paint a barn in 6 hours. Alone, the first man takes twice as long as the second, and the second takes 6 hours longer than the third. In how many hours can the slowest man paint the barn?

8–17 Motion Problems

See 8-17 T.M. pg. 31.

You can solve certain motion problems by using fractional equations.

EXAMPLE An airplane that is flying 600 miles per hour in calm air can cover 2520 miles with the wind in the same time that it can cover 2280 miles against the wind. Find the speed of the wind.

Solution:

1 Let s = the speed, or rate, of the wind in miles per hour.

Since the plane's speed in calm air is 600 miles per hour,

$600 + s$ = speed with the wind. Distance with wind = 2520 miles.

$600 - s$ = speed against the wind. Distance against wind = 2280 miles.

2

	r ·	t =	d
With wind	$600 + s$	$\dfrac{2520}{600 + s}$	2520
Against wind	$600 - s$	$\dfrac{2280}{600 - s}$	2280

$$\underbrace{\text{time with the wind}} = \underbrace{\text{time against the wind}}$$

$$\frac{2520}{600 + s} = \frac{2280}{600 - s}$$

3

$$(600 + s)(600 - s) \cdot \frac{2520}{600 + s} = (600 + s)(600 - s) \cdot \frac{(2280)}{600 - s}$$

$$2520(600 - s) = 2280(600 + s)$$

$$\therefore\ 63(600 - s) = 57(600 + s)$$

$$63(600) - 57(600) = 63s + 57s$$

$$6 \cdot 600 = 120s$$

$$30 = s$$

4 Is the rate of the wind 30 miles per hour?

The speed of the plane with the wind: $600 + 30 = 630$ miles per hour.

∴ with the wind it travels 2520 miles in $\dfrac{2520}{630}$, or 4 hours.

The speed of the plane against the wind: $600 - 30 = 570$ miles per hour.

Against the wind it travels 2280 miles in $\dfrac{2280}{570}$, or 4 hours. ✓

∴ the speed of the wind is 30 miles per hour, Answer.

PROBLEMS

A

1. Jim Black rows 9 miles downstream in the same time that he rows 3 miles upstream. The current flows at 6 miles per hour. How fast does Jim row in still water?
2. A motorboat goes 25 miles downstream in the time it goes 15 miles upstream. The current flows at 5 miles per hour. What is the boat's rate in still water?
3. A man rows 4 miles an hour in still water. He takes as long rowing 4 miles upstream as 12 miles downstream. How fast is the current?
4. A plane can fly 180 miles per hour in calm air. It can travel 800 miles with the wind in the same time as it can travel 640 miles against the wind. Find the speed of the wind.
5. A river steamer travels 36 miles downstream in the same time that it travels 24 miles upstream. Its engines drive it in still water at 12 miles an hour more than the rate of the current. Find the rate of the current.
6. In still water Jim's outboard motor drives his boat 4 times as fast as the current in Pony River. He takes a 15-mile trip up the river and returns in 4 hours. Find the rate of the current.
7. Chris can paddle his canoe at 5 miles per hour on Loon Lake. In Fallow Run, he paddles 4 miles upstream and returns. The upstream trip takes 4 times as long as the downstream trip. How fast is Fallow Run?
8. Leo swims at 2 miles per hour in still water. After he swims down a river for a quarter of a mile, returning takes three times as long as swimming downstream. Find the rate of the current.

B

9. A sound made at one end of a 8250-foot railroad rail reaches an observer at the other end 7 seconds before he hears it in air. Find the speed of sound in the rail, if sound travels 15 times faster in the metal than in air.
10. The total resistance R in an electrical circuit consisting of two resistances of a ohms and b ohms connected in parallel is given by the equation $\frac{1}{R} = \frac{1}{a} + \frac{1}{b}$. Find the larger of the two parallel resistances if it is 2 ohms more than the smaller, and if the total resistance is two-thirds of the smaller resistance.
11. A boat takes 1 hour longer to sail 36 miles up a river than to return. If the river flows at 3 m.p.h., find the speed of the boat in still water.
12. A man rows 1 mile up a river in order to board a motorboat which takes him 10 miles down the river. The man rows at 4 m.p.h., and the

motorboat makes 12 m.p.h. If the trip takes 1 hour and 40 minutes, find the speed of the current.

13. On a 6400-kilometer rocket test range, one rocket takes 8 minutes longer than a second, which travels 40 kilometers a minute faster. Find the speed of the second rocket.

14. A machine folds and closes 5500 cartons. A second machine does the same job in 10 minutes less, processing 5 cartons more each minute. How many cartons does the second machine handle in a minute?

Just for Fun

Fractured Fractions

With certain fractions, you can obtain interesting number patterns by breaking the numerator into two equal factors and at the same time breaking the denominator into an indicated sum. One of these fractions has the value 121, or $(11)^2$:

$$121 = \frac{484}{4} = \frac{22 \cdot 22}{1 + 2 + 1}.$$

Notice the numerator of the "fractured" fraction. Each factor is made up of two *twos*. Two is the middle digit in 121, the value of the fraction. Now look at the denominator. It contains the same digits as the number 121, but they are separated by plus signs.

Another fractured fraction is shown below. Its value is 12,321, or $(111)^2$:

$$12{,}321 = \frac{333 \cdot 333}{1 + 2 + 3 + 2 + 1}.$$

Each factor in its numerator is made up of three threes: 3 is the middle digit in the number 12,321. The denominator is the indicated sum of the digits in 12,321.

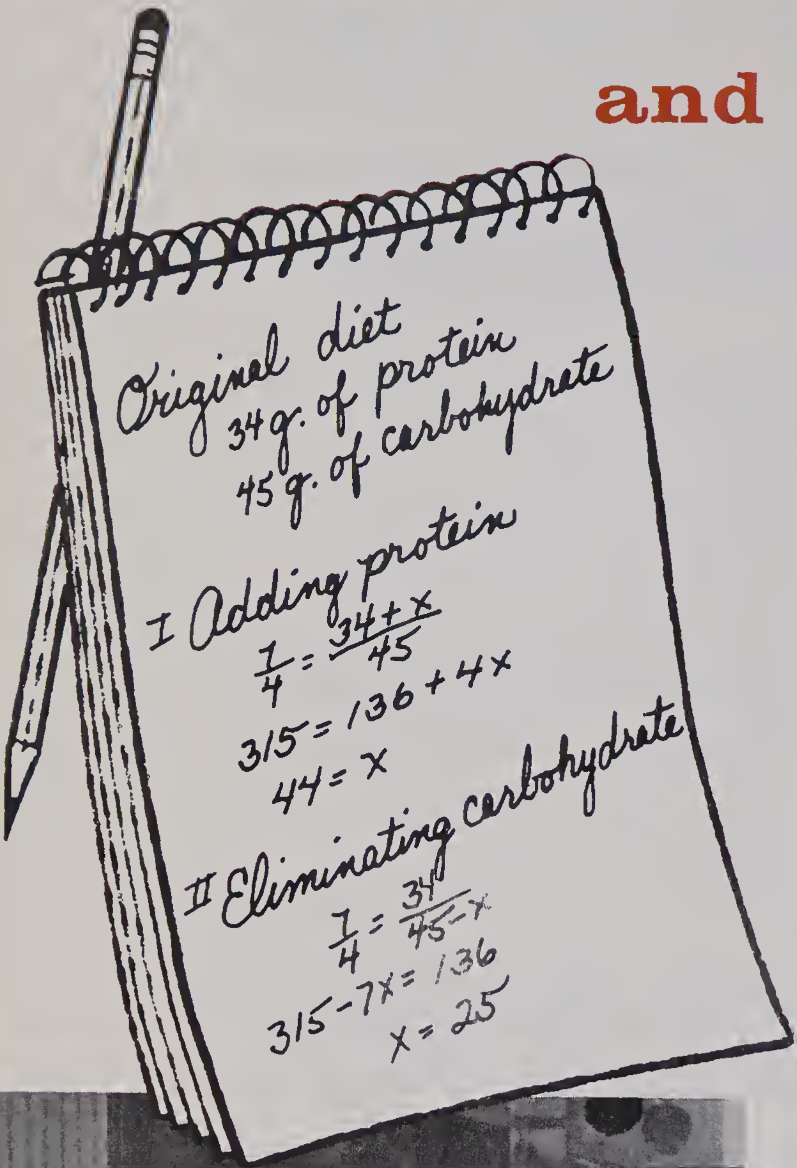
What is the value of this fraction?

$$\frac{4444 \cdot 4444}{1 + 2 + 3 + 4 + 3 + 2 + 1}.$$

Can you write fractured fractions having the following values?

123,454,321
12,345,654,321
1,234,567,654,321
123,456,787,654,321
12,345,678,987,654,321

Home Economists and Mathematics



Original diet
34 g. of protein
45 g. of carbohydrate

I Adding protein

$$\frac{7}{4} = \frac{34+x}{45}$$

$$315 = 136 + 4x$$

$$44 = x$$

II Eliminating carbohydrate

$$\frac{7}{4} = \frac{34}{45-x}$$

$$315 - 7x = 136$$

$$x = 25$$

Home economics is a deceptively simple title for a field including subjects as varied as nutrition, fashion, interior design, and child psychology. Planning a budget, calculating the mortgage on a home, determining how much can be saved by buying a product in a "king-size" container as opposed to the "regular" size, calculating the amount of carpet needed to cover a particular floor area — these are some of the mathematical problems frequently encountered by homemakers as well as professional home economists.

The technician in the photograph is applying her training in home economics to the testing of new food products at the experimental kitchens of a leading food company. She is using a paddle mechanism to determine the gel strength, after different setting intervals, of fruit preserves made with a new jelling agent. The meter in the foreground is used for determining the relative acidity of fruit products. In these experiments as in other aspects of home economics, a knowledge of mathematics is a definite asset.

The work pad illustrates one application of mathematics to the field of dietetics. A hospital dietician must plan a meal for a patient whose doctor has prescribed a diet with a 7:4 protein-to-carbohydrate ratio. The dietician's computations on a tentative menu show that she can arrive at the desired ratio by (1) adding about 44 grams of protein to the meal planned or (2) omitting the potato. Omitting the potato would satisfy the 7:4 ratio without adding an excess amount of food to the meal total and is the preferred alternative.

Chapter Summary

Inventory of Structure and Method

1. The **multiplication property of fractions**: For each a, each b, and each c, other than zero, $\frac{ac}{bc} = \frac{a}{b}$. This property permits you to **reduce a fraction to lowest terms** by dividing its numerator and denominator by the same **common nonzero factor.**
2. The property of quotients $\frac{xy}{ab} = \frac{x}{a} \cdot \frac{y}{b}$ gives the rule for **multiplying fractions**: The product is the fraction whose **numerator** is the **product** of the **numerators** and whose **denominator** is the **product** of the **denominators** of the fractions. The product should be expressed in lowest terms. By **first factoring** numerators and denominators, you may be able to **reduce** the **product** of fractions. To **divide fractions, multiply** the dividend by the **reciprocal** of the **divisor**; thus, $\frac{ab}{c} \div \frac{ac}{b} = \frac{ab}{c} \cdot \frac{b}{ac} = \frac{b^2}{c^2}$.
3. To find the **lowest common denominator** of several fractions, factor each denominator, and find the product of these different factors, each taken the greatest number of times it appears in any one denominator. To **add and subtract fractions**, use the multiplication property to replace each fraction by one equal to it and having as its denominator the lowest common denominator of the given fractions. The **sum** of **fractions** with equal denominators is the fraction whose **numerator** is the **sum** of the **numerators** and whose **denominator** is the **common denominator** of the fractions.
4. To change an algebraic fraction to a mixed expression, divide the numerator by the denominator. To change a **mixed expression** to a fraction, write the polynomial as a fraction whose denominator is 1, and add this fraction to the fractional part of the expression.
5. Equations whose numerical coefficients are fractions, and **fractional equations** having the variable in the denominator of a fraction often occur in the solution of **investment problems**, **per cent mixture problems**, **work problems**, and **motion problems**. You simplify equations by multiplying each member of the lowest common denominator of the terms of the equation. Whenever the L.C.D. may represent zero, the roots of the resulting equation may not satisfy the original equation.
6. A **ratio** is used to compare like measurements or to express a **rate**.

Vocabulary and Spelling

algebraic fraction (*p. 281*)
multiplication property of fractions (*p. 283*)
reducing a fraction (*p. 283*)
fraction in lowest terms (*p. 283*)

ratio (*p. 286*)
per cent (*p. 289*)
percentage (*p. 290*)
base (*p. 290*)
rate (*p. 290*)
lowest common denominator (L.C.D.) (*p. 299*)
mixed expression (*p. 302*)
complex fraction (*p. 304*)
fractional equation (*p. 312*)

Chapter Test

8–1 **1.** For what values of the variable is $\frac{v^2}{v^2 - 16}$ not defined?

8–2 Express in lowest terms.

2. $\frac{21r - 35s}{7r + 14s}$ **3.** $\frac{z^2 - 9}{z^2 + 2z - 15}$ **4.** $\frac{2 - t}{t^2 - 8t + 12}$

8–3 Express the ratio in lowest terms. **5.** 54 volts to 81 volts

6. How long are two parts of a 16-meter line in the ratio 3 to 5?

8–4 **7.** At what discount was a \$3.75 item bought for \$3.30?

Express in lowest terms.

8–5 **8.** $\frac{5p}{6q} \cdot \frac{2p}{qt^2}$ **9.** $\frac{a^2 - 25}{8ab} \cdot \frac{2b}{a + 5}$

10. $\frac{m^2 - 2mn + n^2}{m + n} \cdot \frac{m + n}{m^2 - mn}$

8–6 **11.** $\frac{1}{2y^2 - 50} \div \frac{y + 5}{2y - 10}$ **12.** $\frac{6a^2b^2}{a^2 - b^2} \div \frac{3a^2b^2}{a^2 - 2ab + b^2}$

8–7 **13.** $\frac{c + d}{c - d} \cdot \frac{c - 3d}{c + 3d} \div \frac{c^2 - 2cd - 3d^2}{c^2 + 2cd - 3d^2}$

8–8 **14.** $\frac{4y}{y + 3} + \frac{4}{y + 3} - \frac{y - 5}{y + 3}$

8–9 **15.** $\frac{m + p}{3mp} - \frac{2m + 3}{6m^2}$ **17.** $\frac{x + 1}{x - 1} - \frac{3x^2 - 1}{1 - x^2}$

16. $\frac{h + 2}{h + 3} + \frac{h + 3}{h + 2}$

8–10 **18.** Write as a single fraction: $w + v - \frac{wv + v^2}{2w + v}$

19. Change to a mixed expression: $\frac{4x^2 + 8x + 5}{4x}$

8-11 **20.** (Optional) Simplify.

a. $\dfrac{\frac{y}{t^2}}{\frac{y^2}{t}}$ **b.** $\dfrac{\frac{2x - 2a}{x^2 + 2ax + a^2}}{\frac{x - a}{x + a}}$ **c.** $\dfrac{5 - \frac{n}{k}}{25 - \frac{n^2}{k^2}}$

8-12 **21.** Graph the solution set of the inequality $\frac{x - 1}{3} \leq \frac{x}{2} + 1$.

8-13 **22.** Investments at 3% and 5% total \$1000. What sum at 3% yields \$6 more income than the other?

8-14 **23.** How many pounds of salt would change 75 pounds of a 4% salt solution to a 10% solution?

8-15 **24.** Solve: $\dfrac{x^2 - 7}{x^2 - 16} - \dfrac{3 - x}{4 - x} = 0$

8-16 **25.** Henry can chop a bin of wood in 24 minutes. How fast can George do the job if the two boys together do it in 15 minutes?

8-17 **26.** Tom's boat goes 48 miles with a current of 2 m.p.h. in the time it goes 32 miles upstream. How fast is the boat in still water?

Chapter Review

8-1 Defining Algebraic Fractions *Pages 281–282*

1. A fraction may not have a denominator whose value is $\underline{\ ?\ }$.

For which values of the variables are Exercises 2–5 not defined?

2. $\dfrac{1}{y^2 - 9}$ **3.** $\dfrac{x + 3}{x - 2}$ **4.** $\dfrac{ut}{x}$ **5.** $\dfrac{a^2 - b^2 + c^2}{a^2 - 2ab + b^2}$

8-2 Reducing Fractions *Pages 283–285*

Reduce Exercises 6–8 to lowest terms.

6. $\dfrac{4x^2y}{6xy^2}$ **7.** $\dfrac{a - b}{a^2 - b^2}$ **8.** $\dfrac{4x + 3y}{4xy}$

9. The term $(2 - a)$ is the product of -1 and $\underline{\ ?\ }$.

10. $\dfrac{2 - a}{a - 2} = \underline{\ ?\ }$ **11.** $\dfrac{x^2 - 2x + 1}{1 - x} = \underline{\ ?\ }$

12. $\dfrac{4 - y^2}{2y^2 - y - 6} = \dfrac{?}{2y + 3}$, if $y \notin \{\underline{\ ?\ }, \underline{\ ?\ }\}$

8-3 Ratio *Pages 286–289*

13. Similar quantities to be compared by ratios are expressed in the ___?___ ___?___.

Express each ratio in lowest terms.

14. 18 liters to 6 liters **15.** 50 centimeters to 3 meters

16. The area of an eight-inch square to that of a rectangle 1 foot by 6 inches

8-4 Per Cent and Percentage Problems *Pages 289–291*

17. $17\% = \frac{?}{100}$ **18.** $\frac{3}{4} = \underline{\ ?\ }\%$ **19.** $\frac{3}{2} = \underline{\ ?\ }\%$

20. How much is a tax which equals 28% of an $8000 income?

21. What per cent of a 500-mile trip did a boy drive if his father drove 259 miles?

8-5 Multiplying Fractions *Pages 292–294*

Find the products.

22. $\frac{x}{t} \cdot \frac{u}{v}$

23. $\frac{2atx}{3bz} \cdot \frac{5a^2xz}{7bt^3}$

24. $\frac{5}{m+n} \cdot \frac{5}{m-n}$

25. $\frac{r+t}{r-t} \cdot \frac{r-t}{t}$

26. $\frac{3z-2}{3z+2} \cdot \frac{2z+3}{2z-3}$

27. $\frac{a^2+2a+1}{a^2-2a+1} \cdot \frac{2a-2}{5a+5}$

8-6 Dividing Fractions *Pages 295–296*

28. $\frac{r^2t^2}{4} \div \frac{r^2s^2t^2}{12}$

29. $\frac{a^2-4}{2a+2} \div \frac{a-2}{a+1}$

30. $\frac{x^2+10x+25}{x^2-12x+36} \div \frac{x+5}{x-6}$

8-7 Fractions Involving Multiplication and Division *Pages 296–297*

31. *Simplify:* $\frac{r+3k}{3r} \cdot \frac{r-3k}{3} \div \frac{3r+k}{k}$

8-8 Combining Fractions with Equal Denominators *Pages 297–299*

Combine in Exercises 32–34.

32. $\frac{9s}{4t} + \frac{3s}{4t}$ **33.** $\frac{6g}{2g+1} + \frac{3}{2g+1}$ **34.** $\frac{b^2+1}{b^2-1} - \frac{2b}{b^2-1}$

8-9 Adding Fractions with Unequal Denominators *Pages 299–302*

Combine the fractions.

35. $\frac{4}{2a+b} - \frac{2}{a+2b}$

36. $\frac{s^2+rs}{r^2-s^2} + \frac{r}{r-s}$

37. $\frac{x^2 - 1}{3x + 3} - \frac{x^2 - 1}{4x + 4}$ **39.** $\frac{a}{a + 2} + \frac{6}{a - 2} - \frac{8}{a^2 - 4}$

38. $\frac{2n - 3}{n - 5} + \frac{2}{5 - n}$ **40.** $\frac{y + 1}{y + 2} - \frac{2}{y + 4} + \frac{4}{y^2 + 6y + 8}$

8–10 Mixed Expressions *Pages 302–304*

Change to a single fraction.

41. $a^2 - \frac{a^3 - 1}{a}$ **42.** $x + y - \frac{2xy}{x + y}$

Change to mixed expressions:

43. $\frac{8m^2 - 16m + 2}{8m}$ **44.** $\frac{w^2 + 5w + 7}{w + 2}$

8–11 Complex Fractions (Optional) *Pages 304–305*

Simplify each fraction.

45. $\frac{\frac{n}{x}}{\frac{n^2}{x^2}}$ **46.** $\frac{9 - \frac{a^2}{h^2}}{3 + \frac{a}{h}}$

8–12 Open Sentences with Fractional Coefficients *Pages 306–308*

Find the solution sets.

47. $\frac{z}{3} + \frac{z}{5} = 45 - \frac{z}{15}$ **48.** $\frac{3}{2}n \geq \frac{4}{3}n + 9$

8–13 Investment Problems *Pages 308–310*

49. Mr. Keith invested half his money at 3% and half at 5% for an annual income of \$120. How much is each investment?

50. Mrs. Willis invested \$2500 at 4%. How much must she invest at 5% to make her income from both investments \$325?

51. A man invested some money at 5% and twice as much at 6%. If his annual income from both was \$289, find each amount.

52. Mr. Clety invests part of \$7000 at $3\frac{1}{2}$% and the remainder at 7% for a total annual yield of \$350. Find each investment.

8–14 Per Cent Mixture Problems *Pages 310–311*

Items 53–57 refer to this problem: How many pounds of water evaporate from 100 pounds of 3% salt brine to make a 4% salt solution?

53. The number of pounds of brine is originally ___?___.

54. Originally, _?_ per cent of the brine is salt.

55. There are _?_ pounds of salt in the original solution, and _?_ pounds of salt in the final solution.

56. The per cent of salt in the final solution is _?_.

57. If x pounds of water evaporate, _?_ pounds of brine remain.

58. A science teacher diluted 100 milliliters of pure acid to make a 20% acid solution. How much water did he add?

8–15 Fractional Equations *Pages 312–314*

Solve each equation.

59. $\frac{3}{m} + \frac{2m - 3}{6m} - \frac{m + 1}{2m} = 0.$

60. $\frac{n + 5}{4 - n^2} + \frac{3}{n - 2} = \frac{1}{n + 2}$

8–16 Work Problems *Pages 314–316*

Items 61–63 refer to this situation: In 24 hours two machines do a job which the faster machine can do alone in 40 hours.

61. The faster machine can produce _?_ of the order in 1 hour, and its rate is _?_.

62. The second machine completes the order in x hours at the rate of _?_.

63. When the machines work together, the first machine produces _?_, and the second, _?_ of the order.

64. Joe can wash his car in 30 minutes. If Arno helps, they do the job in 18 minutes. How fast can Arno wash the car alone?

8–17 Motion Problems *Pages 316–319*

For 65–66 refer: The current of Rock River flows at 4 miles per hour. John rows 1 mile upstream in the time he rows 5 miles downstream.

65. John's rate in still water is s. His rate downstream is _?_, and his rate upstream is _?_.

66. In terms of s, the time John travels downstream is _?_.

67. A pilot flies 600 miles with a tail wind of 25 m.p.h. Against the same wind, he flies only 450 miles in the same time. Find his rate in still air.

Cumulative Review: Chapters 1–8

In each case select the correct answer.

1. If $R = \{0, 1, 4, 9, \ldots\}$ is the set of squares of integers, then R is closed under (*addition*) (*multiplication*) (*neither*).

2. In the set of directed numbers $|y| < 3$ is equivalent to $(y < -3)$ $(y < 3)$ $(-3 < y < 3)$.

3. The solution set of $\frac{(r-2)(r+5)}{r-2} = 1$ is $(\{-4\})$ $(\{-5\})$ $(\{2, -4\})$.

4. If $a + 1 = b$, then $(a > b)$ $(a < 2b)$ $(a < b)$.

5. If c is a directed number, then $-c$ is (*always*) (*sometimes*) (*never*) a positive number.

6. $-(2a - k) = (2a + k)$ $(-2a - k)$ $(-2a + k)$.

7. $\frac{(10r)(5r^2)}{5r} = (2r)$ $(10r^2)$ $(2r^2)$.

8. The set of values of x for which $\frac{5(x-3)}{(x^2 + x - 12)}$ is undefined is $(\{-4\})$ $(\{3, -4\})$ $(\emptyset)$.

9. If $ax - a > 2a$ and $x - 1 < 2$, then $(a = 1)$ $(a < 0)$ $(a > 0)$.

Draw a conclusion about the value of t from each statement.

10. $2t = 0$ **11.** $5s(t - 3) \neq 0$ **12.** $6s^2t^3 < 0$ **13.** $\frac{t+1}{t+1} = 1$

Which statements are true and which are false? Justify your answers.

14. For every number m, $\frac{34 - m}{17} = 2 - m$.

15. For each number x, $5(6x^2 + x) - 2(4x^2 - 3x) = 22x^2 - x$.

16. $y + 2$ is a factor of $y^4 + 7y^3 + 10y^2$.

17. If A and B are polynomials of degree 3, then $A + B$ is a polynomial of degree 3.

18. The solution set of $2k^2 + k = 1$ is a subset of the set of integers.

Complete each statement.

19. In factored form the L.C.D. of $\frac{2}{2-y}$, $\frac{8}{y^2-4}$, and $\frac{4}{(2-y)^2}$ is _?_.

20. If $z \notin \{-3, 3\}$, $\frac{27(z-3)^2}{z^2-9} =$ _?_.

21. The ratio of acid to the total volume of a solution containing 12 cc. of acid in 16 cc. of water is _?_.

22. If $x - 2$ is a factor of $x^3 - 8$, another factor is _?_.

23. The ratio of $\frac{7xy}{11ab}$ to $\frac{11a^2}{7x}$ is _?_.

24. The numbers which equal their reciprocals are _?_ and _?_.

Find the solution set of each open sentence.

25. $\frac{2(n-3)}{7} - \frac{2-n}{3} = \frac{3n-2}{21}$

26. $\frac{3x-4}{x-4} - \frac{x+4}{x+1} = 2$

27. $1 + \frac{2}{t-3} = \frac{3}{(t-3)^2}$

28. $\frac{1}{y} - \frac{1}{y+1} = \frac{1}{y(y+1)}$

Express each polynomial as a constant multiplied by a product of irreducible polynomials with integral coefficients.

29. $7k^4 - 63k^2$

30. $60 + 7h - h^2$

31. $12cr - 8cs - 15dr + 10ds$

32. $4z^3t - 20z^2t^2 + 25zt^3$

33. $49(n-k)^2 - 25$

34. $x^2 - \frac{1}{6}x - \frac{1}{6}$

Express each exercise as a fraction in lowest terms.

35. $\frac{1}{y+2} - \left\{\frac{3y}{y^2-4} + \frac{2y}{2-y}\right\}$

36. $\frac{1}{(x-3)^2} + \frac{12}{(x+3)^2} - \frac{8}{x^2-9}$

37. $\frac{4a^2-9b^2}{a^2-2ab+b^2} \div \frac{2a^2+5ab+3b^2}{a^2-b^2}$

38. $\left(\frac{2q}{k}\right)^2 \cdot \frac{48k^2s}{(5tq^2)^2} \div \left(\frac{6qs}{5t^2}\right)^3$

39. (Optional) $\dfrac{\frac{1}{t-v} - \frac{1}{t+v}}{\frac{2}{t^2-v^2}}$

40. A dealer bought an air conditioner for \$150. By selling it at a discount of 25% on the list price, he made a profit of 20% on the cost. Find the list price.

41. Two circles whose areas differ by 36π square cm. have radii differing in length by 3 cm. Find the length of the radius of the larger circle.

42. What quantities of silver 70% pure and 75% pure mixed together make 20 pounds of silver which is 73% pure?

43. The average of a number and its reciprocal is $\frac{17}{15}$. Determine the smallest such number.

44. A plane traveled 1200 miles in $3\frac{1}{2}$ hours. Its average speed for the last 600 miles was 100 m.p.h. less than its average for the first 600. Find the average speed for the last 600 miles.

45. One pipe fills a tank in 40 minutes, and another fills it in 60 minutes. A third pipe empties the tank in 30 minutes. In what time will the tank fill if all three pipes are open?

Extra for Experts

Divisibility of Integers

Knowing whether one integer is an integral multiple of (is divisible by) another helps in factoring, reducing fractions, and checking computations.

An analysis of the general form of the decimal numeral of an integer will enable you to develop criteria of divisibility.

$$N = a_n 10^n + a_{n-1}10^{n-1} \cdots + a_3 10^3 + a_2 10^2 + a_1 10 + a_0$$

Here, the *a*'s represent the digits in the numeral, and the powers of ten correspond to place values. If you were discussing 8,037,291, then $a_6 = 8$, $a_5 = 0$, $a_4 = 3$, $\ldots a_1 = 9$, $a_0 = 1$.

Many of these criteria are based on the following property of integers:

> If $N = p + q$, and q is a multiple of r, then N is divisible by r if, and only if, p is a multiple of r.

You may be acquainted with some of these criteria of divisibility.

Divisor	Test
2	Is the last digit 0, 2, 4, 6, or 8?
3	Is the sum of the digits divisible by 3?
4	Is the integer named by the last two digits divisible by 4?
5	Is the last digit 0 or 5?
7	From the right, group the digits by threes, and mark these groups alternately *positive* and *negative;* then, total the signed groups. Is this sum divisible by 7?
9	Is the sum of the digits divisible by 9?
10	Is the last digit 0?
11	Mark the digits alternately *positive* and *negative* from the right; then, total the signed digits. Is this sum divisible by 11?
13	Compute the sum as in the test for 7. Is this sum divisible by 13?

EXAMPLE 1. **Is 72,135 ÷ 9 an integer?**

Solution: $7 + 2 + 1 + 3 + 5 = 18;\ 18 \div 9 = 2.$

$\therefore$ **72,135 ÷ 9 is an integer, Answer.**

EXAMPLE 2. **Is 386,749 ÷ 11 an integer?**

Solution: $9 - 4 + 7 - 6 + 8 - 3 = 11;\ 11 \div 11 = 1.$

$\therefore$ **386,749 ÷ 11 is an integer, Answer.**

EXAMPLE 3. **Is 296,348,026 ÷ 13 an integer?**

Solution: $026 - 348 + 296 = -26;\ -26 \div 13 = -2.$

$\therefore$ **296,348,026 ÷ 13 is an integer, Answer.**

EXAMPLE 4. **Prove the divisibility test for 3 and for 9.**

Solution: **Use the commutative property, and reverse the order of the terms in the general form of the integer N.**

$$N = a_0 + 10a_1 + 10^2a_2 + 10^3a_3 \ldots + 10^{n-1}a_{n-1} + 10^na_n$$

$$N = a_0 + (1 + 9)a_1 + (1 + 99)a_2 \ldots + (1 + \underbrace{99 \ldots 9}_{n\text{ nines}})a_n$$

$$N = (a_0 + a_1 + \ldots + a_n) + (9a_1 + 99a_2 + \ldots + \overbrace{99 \ldots 9}a_n)$$

$$N = \underbrace{(a_0 + a_1 + \ldots + a_n)}_{?} + \underbrace{9(a_1 + 11a_2 + \ldots + 11 \ldots 1a_n)}_{\text{multiple of 9 (and 3)}}$$

$\therefore$ N is divisible by 3 if, and only if, $a_0 + a_1 \ldots + a_n$ is divisible by 3.
N is divisible by 9 if, and only if, $a_0 + a_1 \ldots + a_n$ is divisible by 9.

EXAMPLE 5. **Prove the divisibility test for 4.**

Solution:

$$N = a_0 + 10a_1 + (2 \cdot 5)^2a_2 + (2 \cdot 5)^3a_3 \ldots + (2 \cdot 5)^na_n$$

$$N = a_0 + 10a_1 + 4 \cdot 25a_2 + 8 \cdot 125a_3 \ldots + 2^n \cdot 5^na_n$$

$$N = \underbrace{(a_0 + 10a_1)}_{?} + \underbrace{4(25a_2 + 250a_3 \ldots + 2^{n-2} \cdot 5^na_n)}_{\text{multiple of 4}}$$

$\therefore$ N is divisible by 4 if, and only if, $10a_1 + a_0$ is divisible by 4.

Questions

1. Test for integral quotients.

- **a.** 5208 ÷ 3
- **b.** 5208 ÷ 4
- **c.** 5208 ÷ 5
- **d.** 5208 ÷ 9
- **e.** 147,809 ÷ 7
- **f.** 9819 ÷ 9
- **g.** 367,892 ÷ 11
- **h.** 147,810 ÷ 13

2. Prove the rules for divisibility

- **a.** by 2
- **b.** by 5
- **c.** by 7
- **d.** by 11
- **e.** by 13

3. Explain why a number divisible by two different prime numbers is divisible by their product. Use this fact to devise a test for divisibility by 6.

4. Devise a rule to test for divisibility by 12, and test 1346 ÷ 12.

5. Devise a rule to test for divisibility by 25, and test 67,475 ÷ 25.

6. In this problem, the number N is in its general form.

- **a.** Show that N is divisible by 4 if $2a_1 + a_0$ is divisible by 4.
- **b.** Show that N is divisible by 8 if $4a_2 + 2a_1 + a_0$ is divisible by 8.

THE HUMAN EQUATION

Wedding Plans That Failed

"Salutation to the elephant-headed Being who infuses joy into the minds of his worshipers, . . . whose feet are reverenced by the gods."

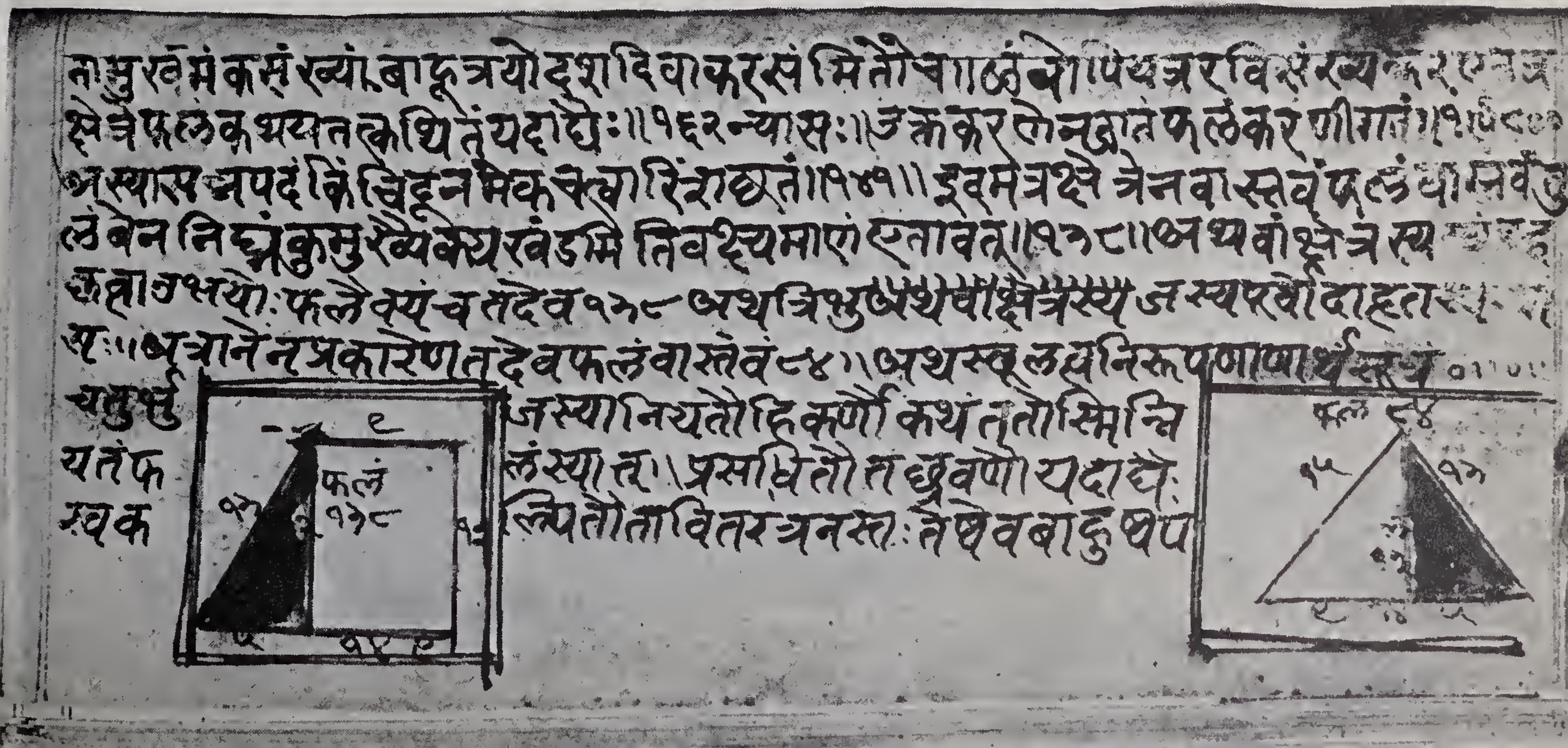
So begins a book written in India about the middle of the twelfth century. From the opening sentence you would scarcely suspect it to be a book about mathematics. But it is, and the man who wrote it, Bhaskara the Learned, was a great mathematician.

Bhaskara's most famous book is called *The Lilivati*. It is a great algebra book, named for the author's twelve-year-old daughter Lilivati. Bhaskara wrote it in an attempt to comfort his child in a great disappointment. The story is this:

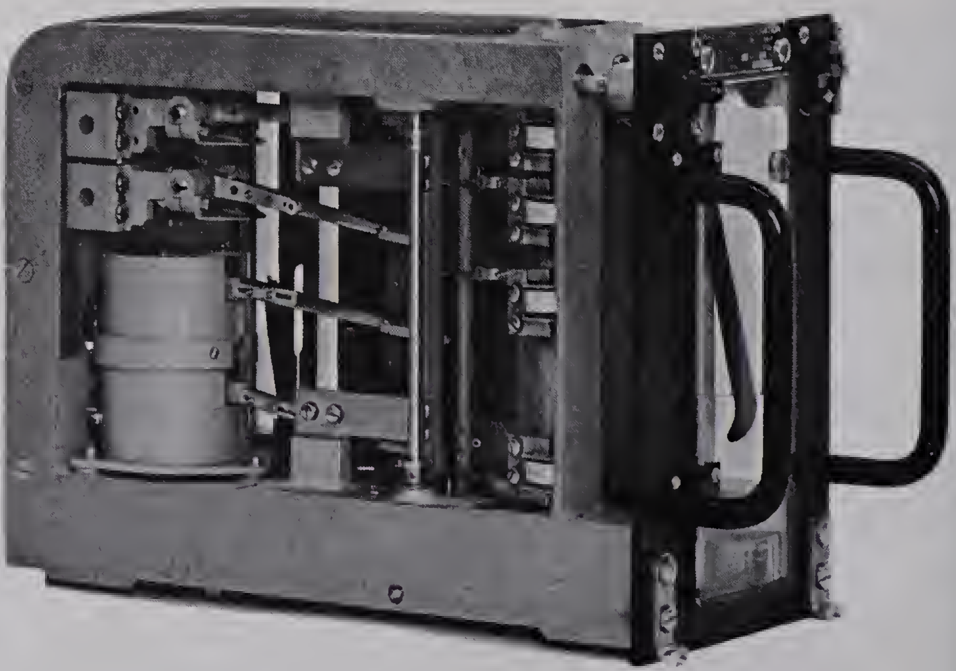
Astrologers had found that there was only one moment in all Lilivati's life when she could be married safely, on one particular day when she was twelve years old. So Bhaskara had arranged her wedding for that day. As the favorable moment approached, the bride, adorned for the wedding, happily but anxiously watched the hour cup, which floated on a vessel of water near her. (A Hindu hour cup had a small hole in the bottom; water trickled in and caused the cup to sink at the end of the hour.) Unnoticed, a pearl fell from Lilivati's headdress into the cup, and stopped the trickle of water. The wedding party waited. But the ceremony was never performed. Before the accident was discovered, the time for the wedding had passed. Lilivati never could be married.

But Lilivati's name lives on. For her father fulfilled the promise he made to her that day: "I will write a book of your name which shall remain to the latest times; for a good name is a second life and the groundwork of eternal existence."

A page from The Lilivati, *the algebra book that made a young girl famous.*



CHAPTER 9



Graphs

A picture is worth a thousand words. Reports on many subjects are illustrated by graphs, as you may know, but you may not be acquainted with the use of graphs in continuous recording devices.

The cardiogram, shown in the upper photo, is a vital tool in medical diagnosis because it is a record of a patient's heart action. A flight data recorder, lower left, notes the operations of vital parts of a jet aircraft. Although some devices, like the cardiograph, make ink traces on paper, the flight data recorder engraves its records in stainless steel tape with diamond styluses, assuring their survival in a crash.

Similar machine-made records report on the functioning of many parts of our industrial enterprises. Such graphs not only free men from dial watching and recording, but they can be filed as permanent records. The graphs are continuous, and they show immediately changes in variables which easily might be overlooked in long columns of figures. Designing machines to make graphs is an engineer's job; interpreting their reports is a part of the work of many people.

ORDERED PAIRS OF NUMBERS AND POINTS IN A PLANE

9–1 Open Sentences in Two Variables

See 9-1 T.M. pg. 32 for motivation.

Have you ever had this kind of problem? With \$1.20 Nancy wishes to buy two kinds of cookies which cost 2¢ and 3¢ each.

If she buys some of each kind, many combinations are possible:

Let x = number of 2¢ cookies,

and y = number of 3¢ cookies.

Then $2x$ = cost, in cents, of 2¢ cookies,

and $3y$ = cost, in cents, of 3¢ cookies.

120 = amount, in cents, available.

Total cost of two purchases = total amount spent.

$$2x + 3y = 120$$

x	y	$2x + 3y = 120$	
0	40	$2(0) + 3(40) = 120 = 120$	True
30	20	$2(30) + 3(20) = 120 = 120$	True
20	30	$2(20) + 3(30) = 130 = 120$	False
45	10	$2(45) + 3(10) = 120 = 120$	True
60	0	$2(60) + 3(0) = 120 = 120$	True

Nancy may replace x and y with various nonnegative integers. The pair (30, 20) makes the statement true, but the pair (20, 30) makes it false. Each pair is taken only in the order written, and is called an *ordered pair of numbers.* The first number is the **first coordinate**, and the second is the **second coordinate.** Two ordered pairs of numbers are equal when their first coordinates are equal and their second coordinates are equal. Thus, $(1, 5) = (\frac{6}{6}, \frac{15}{3})$ but $(1, 5) \neq (5, 1)$.

With the value of x given first, the ordered pair of numbers (30, 20) is a **root of the open sentence in two variables** ($2x + 3y = 120$) because for this pair the sentence is true. The **solution set of an open sentence in two variables** is the set of all roots of the sentence. What are other roots of $2x + 3y = 120$?

EXAMPLE 1. Find the solution set of $3x + y = 5$ when the replacement set for *both* x and y is $A = \{0, 1, 2, 3, 4, 5\}$.

Solution:

1. **Transform the given sentence into an equivalent one, solved for y.**

$$3x + y = 5$$
$$y = 5 - 3x$$

2. **Replace x by each member of its replacement set, in turn, and determine the corresponding value of y.**
3. **If the value of y determined in Step 2 belongs to the replacement set for y, then the pair of corresponding values is a root of the sentence.**

To be strictly precise, say "replace x by a numeral."

x	$5 - 3x$	$y = 5 - 3x$
0	$5 - 3(0)$	$5 \in A$
1	$5 - 3(1)$	$2 \in A$
2	$5 - 3(2)$	$-1 \notin A$
3	$5 - 3(3)$	$-4 \notin A$
4	$5 - 3(4)$	$-7 \notin A$
5	$5 - 3(5)$	$-10 \notin A$

$\therefore$ The solution set is $\{(0, 5), (1, 2)\}$, Answer.

EXAMPLE 2. Find the solution set of $2y - x \geq 4$

if $x \in \{-2, 0, 2\}$ and $y \in \{0, 1, 2, 3\}$.

Solution:

$$2y - x \geq 4$$
$$2y \geq 4 + x$$
$$y \geq 2 + \frac{x}{2}$$

x	$2 + \frac{x}{2}$	$y \geq 2 + \frac{x}{2}$	y
-2	$2 + \frac{-2}{2}$	$y \geq 1$	1, 2, 3
0	$2 + \frac{0}{2}$	$y \geq 2$	2, 3
2	$2 + \frac{2}{2}$	$y \geq 3$	3

$\therefore$ The solution set is $\{(-2, 1), (-2, 2), (-2, 3), (0, 2), (0, 3), (2, 3)\}$, **Answer.**

ORAL EXERCISES

Is the given ordered pair of numbers a root of the open sentence? Why? Assume that the set of directed numbers is the replacement set of each variable.

SAMPLE. $5x - 2y = 7; (3, 11)$

What you say: Not a solution, because $5 \cdot 3 - 2 \cdot 11 \neq 7$

1. $x + y = 9; (5, 4)$ Soln.
2. $2x - y = 7; (4, 1)$ Soln.
3. $3x + y = 8; (4, -2)$ Not soln.
4. $x + 5y = 4; (-9, 1)$ Not soln.
5. $2x - 3y = 8; (4, 0)$ Soln.
6. $7x - y > -2; (1, -9)$ Soln.
7. $-2x + 5y \geq -7; (-1, -2)$ Not soln.
8. $-x - 4y \geq 0; (-2, -1)$ Soln.
9. $x^2 - y = 7; (-3, 2)$ Soln.
10. $xy - x = 2; (-3, 1)$ Not soln.

Transform each open sentence into an equivalent one having y as one member.

SAMPLE. $2x + y = 3$ *What you say:* $y = 3 - 2x$

11. $x + y \leq 1$ $y \leq 1 - x$
12. 4 3x $3x + y \geq -4$
13. $5y = 6x - 10$ $y = \frac{6}{5}x - 2$
14. $6x - y = 0$ $y = 6x$
15. $7x - y = 0$ $y = 7x$
16. $x - y = -2$ $y = x + 2$
17. $5x + 2y = 8$ $y = 4 - \frac{5}{2}x$
18. $12x + 4y = 6$ $y = \frac{3}{2} - 3x$
19. $x - y > 2$ $y < x - 2$

WRITTEN EXERCISES

Find all values of *a* and *b* for which these ordered pairs are equal.

SAMPLE. $(a + 3, 4b + 7) = (2a - 5, b + 4)$

Solution: $a + 3 = 2a - 5$ and $4b + 7 = b + 4$

$8 = a$ $\qquad 3b = -3$

$b = -1$

Check: $[8 + 3, 4(-1) + 7] \stackrel{?}{=} [2(8) - 5, -1 + 4]$

$(11, -4 + 7) \stackrel{?}{=} (16 - 5, 3)$

$(11, 3) = (11, 3)$ ✓

$\therefore a = 8, b = -1$, **Answer.**

A

1. $(a - 11, b + 8) = (12a, 3b)$
2. $(3a, 2b - 1) = (a - 6, 3b)$
3. $(1 - a, 3) = (5 - 7a, |b|)$
4. $(2a + 4, b^2) = (3a + 7, -b)$
5. $(2 - a^2, b^2) = (a, 36)$
6. $(|a + 1|, b^3) = (2, 0)$

Find the solution set of each sentence having {−6, 0, 6} as the replacement set of *x* and {directed numbers} as that of *y*.

7. $y = x + 3$
8. $y = 4 - x$
9. $5x + y = 8$
10. $y - 3x = 7$
11. $7x - 3y = 30$
12. $3x + 2y = 12$

Find the solution set.

13. $-x + 6y = 10;\ x \in \{-10, 2\},\ y \in$ {positive numbers}
14. $x - 8y = 15;\ x \in \{-1, 15\},\ y \in$ {negative numbers}
15. $10 - 5y = 2x;\ x \in \{0, 5\},\ y \in \{0\}$
16. $21 + 3y = 8x;\ x \in \{0, 3\},\ y \in \{1\}$

B

17. $x + 5y = 7 - x;\ x \in \{-4, -1\},\ y \in$ {integers}
18. $1 - x = 3y + 4x;\ x \in \{1, 2\},\ y \in$ {integers}
19. $y + 1 \geq 2x;\ x \in \{0, 1\},\ y \in \{-1, 1\}$
20. $y - 1 < 3x;\ x \in \{-1, 0\},\ y \in \{-3, 0\}$
21. $|x| - y < 3;\ x \in \{-1, 3, 4\},\ y \in \{0, 1, 2\}$
22. $|x| - y \geq 2;\ x \in \{-2, -1, 3\},\ y \in \{0, 1, 6\}$

23. $|x| + 2y = y + 4;\ x \in \{-3, 0, 4\},\ y \in \{\text{positive integers}\}$

24. $|x| - 3y = y + 12;\ x \in \{-1, 0, 4\},\ y \in \{\text{negative integers}\}$

25. $2x + y^2 = 8y - 10;\ x \in \{-5, 1\},\ y \in \{\text{prime numbers}\}$

26. $3x + y^2 = 7y - 3;\ x \in \{-1, 3\},\ y \in \{\text{prime numbers}\}$

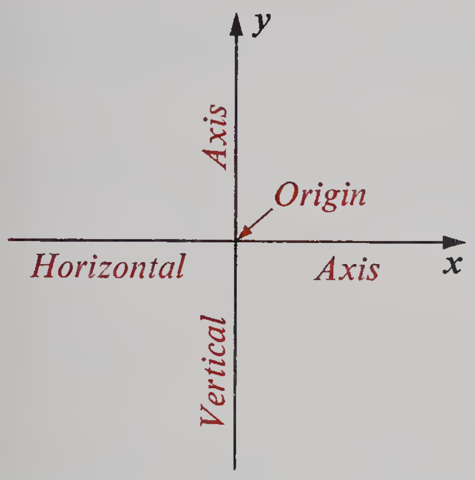
Write an equation with two or more variables for each problem. Give appropriate replacement sets and solution sets.

27. If baseballs cost \$1.50 apiece and bats cost \$3.50 apiece, how many of each can a team purchase for \$50?

28. Cupcakes cost 5 cents each, ice cream bars, 10 cents, and punch, 25 cents a quart. How can Jo spend her \$7.50 for refreshments if she buys the same number of cupcakes as ice cream bars, at least 20 of each, and at least 12 quarts of punch?

9–2 Coordinates in a Plane

Each root of an open sentence in one variable is one number, which you can graph on a number line. Each root of an open sentence in two variables is a pair of numbers, which you graph on two number lines intersecting at right angles. Choose a horizontal line called the *horizontal axis* (x-axis) and a vertical line, the *vertical axis* (y-axis). Their point of intersection is the *origin*. Next, select a scale to make each axis a number line whose zero-point is the origin, and indicate the scale on the axes.



Point out that labeling the axes x and y is just a convenience. See T.M. pg. 32 (1).

To locate the *graph of the ordered pair* (3, −2) (Figure 9–1), draw a vertical line through the graph of 3 on the x-axis and a horizontal line through the graph of −2 on the y-axis. The point of intersection of these lines is the graph of (3, −2). Mark the point with a dot or cross; this is called *plotting the point*.

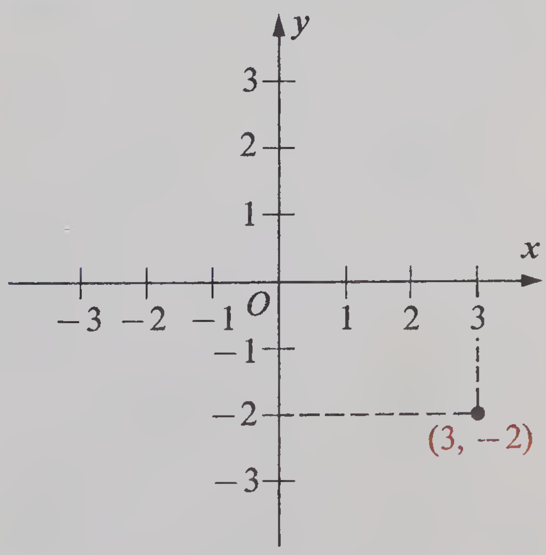


Figure 9–1

Emphasize that the numeral written first is associated with the horizontal number line, and the one written second, with the vertical number line.

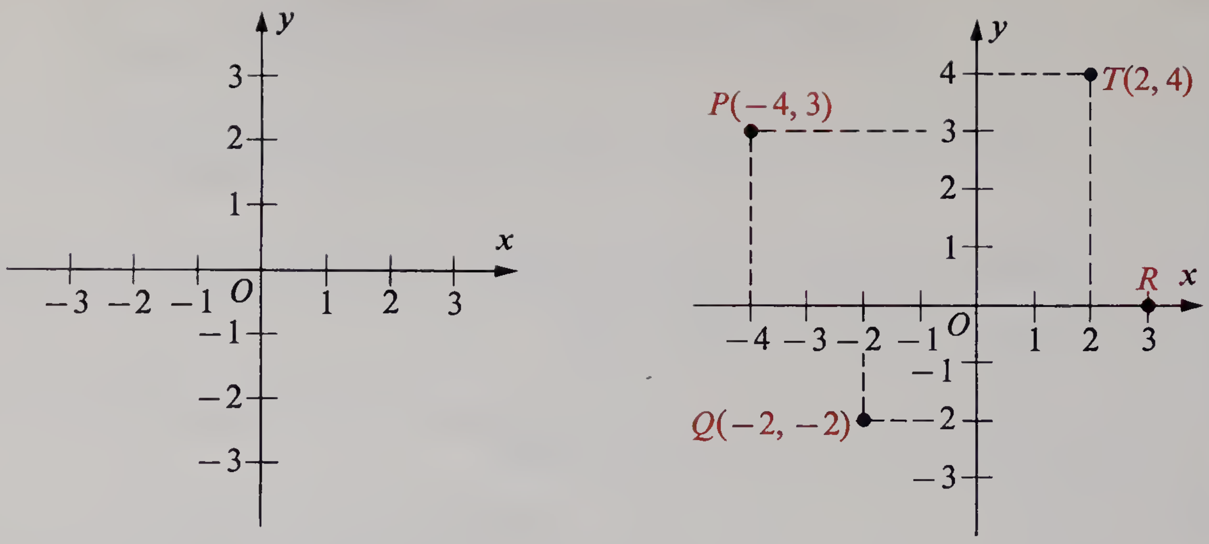


Figure 9–2

Stress that every point in the plane has *two* coordinates. See T.M. pg. 32 (2).

Positive numbers are paired with points on the x-axis to the right of the origin and with points on the y-axis above the origin (drawing above left). Negative numbers relate to points to the left of the origin on the x-axis and below the origin on the y-axis.

The surface on which the axes lie is a set of points called a plane. With every point in this plane you can associate a particular ordered pair of numbers.

From point P in Figure 9–2 (above right) draw a vertical line to the x-axis; the coordinate of the point where it meets the axis is the *abscissa* (ab-**sis**-a) of P, -4.

Draw a horizontal line from P to the y-axis; the coordinate of this meeting point is the *ordinate* (**or**-de-nate) of P, 3.

The *coordinates* of P always are written with the abscissa first, $(-4, 3)$.

In Figure 9–2, verify the coordinates of: $Q(-2, -2)$, $R(3, 0)$, $T(2, 4)$, and the origin $(0, 0)$.

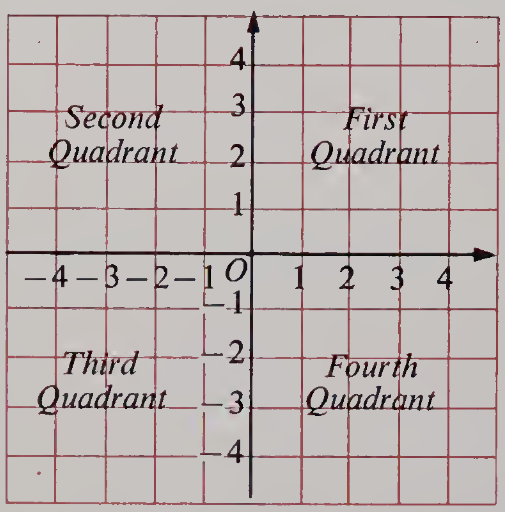


Figure 9–3

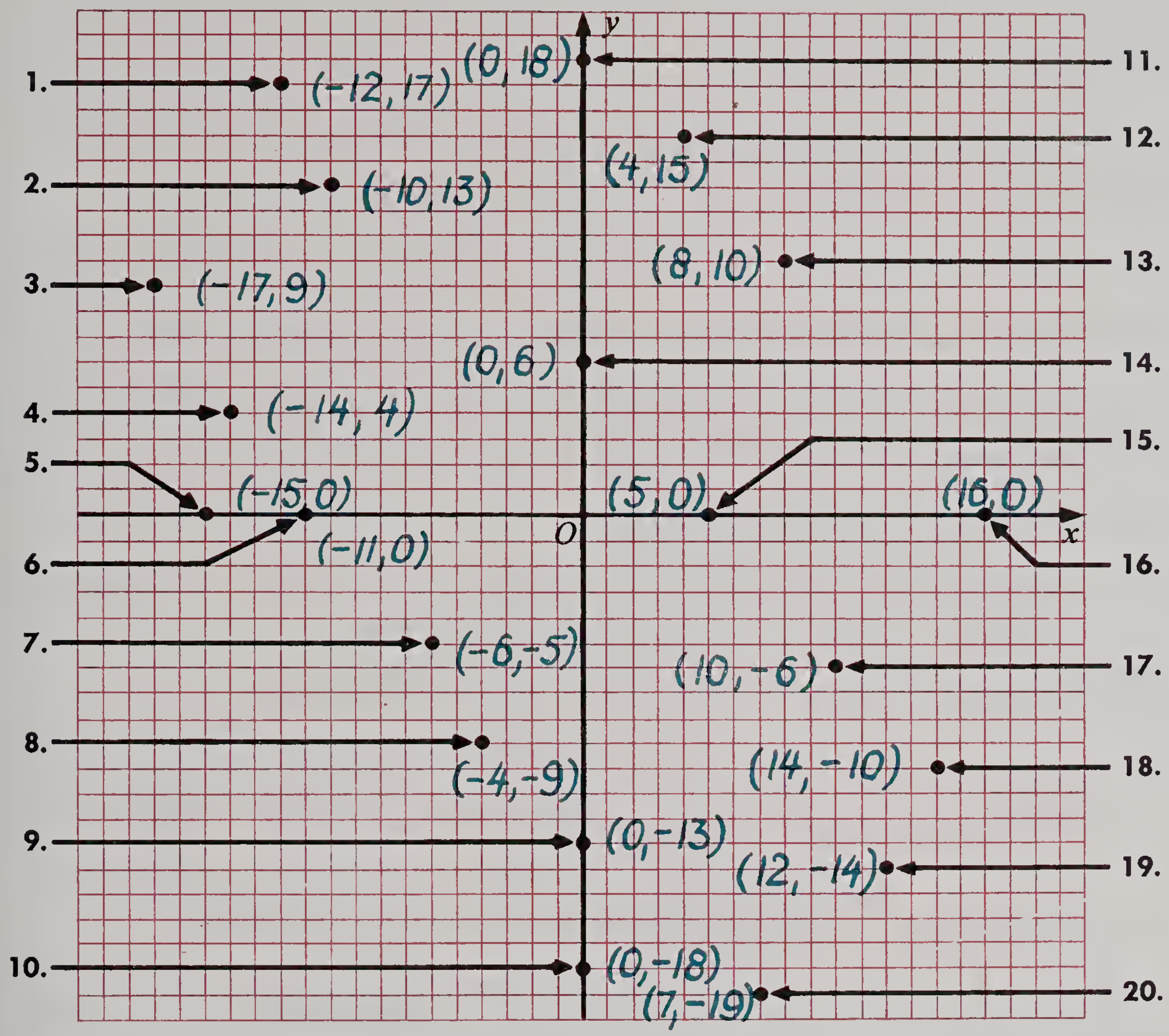
The one-to-one correspondence between points and number pairs is called a *plane coordinate system* or a *coordinate plane.* Because of this correspondence you can think of a point as an ordered pair of numbers and can picture an ordered pair as a point.

The axes of the coordinate system divide the plane into four regions, called *quadrants*, numbered as shown in Figure 9–3.

Ask the class to characteriz the coordinates in each quadrant; e.g. in Quadrant I both coordinates are positive.

ORAL EXERCISES

Give the coordinates of each numbered point.


y
1.
(-12,17)
(0,18)
11.
12.
(4,15)
2.
(-10,13)
3.
(-17,9)
(8,10)
13.
(0,6)
14.
4.
(-14,4)
5.
15.
(-15,0)
(5,0)
(16,0)
O
x
(-11,0)
6.
16.
7.
(-6,-5)
(10,-6)
17.
8.
(14,-10)
18.
(-4,-9)
9.
(0,-13)
(12,-14)
19.
10.
(0,-18)
(7,-19)
20.

In Exercises 21–30 name the quadrants containing the points described.

21. The abscissa is 3. Q1, Q4

22. The ordinate is 2. Q1, Q2

23. The ordinate is −5. Q3, Q4

24. The abscissa is −3. Q2, Q3

25. The abscissa equals the ordinate. Q1, Q3

26. The abscissa equals the negative of the ordinate. Q2, Q4

27. The abscissa is −1, and the ordinate is positive. Q2

28. The abscissa is positive, and the ordinate is −4. Q4

29. The absolute value of the abscissa is 5. All

30. The absolute value of the ordinate is 2. All

31. What is the ordinate of every point on the x-axis? 0

32. What is the abscissa of every point on the y-axis? 0

WRITTEN EXERCISES

Plot the graph of each of the following.

A

1. (3, 4)
2. (0, 9)
3. (−8, 9)
4. (−7, 6)
5. (−4, 4)
6. (9, 2)
7. (12, −7)
8. (7, −10)
9. (−6, −10)
10. (−11, −8)
11. (8, 0)
12. (−8, 0)

Exercises 13–18 list three vertices of a rectangle. Find the fourth vertex.

B

13. (0, 0), (0, −4), (6, 0)
14. (0, 0), (−2, 0), (0, 3)
15. (2, 2), (−3, −1), (2, −1)
16. (3, 4), (−1, 1), (3, 1)
17. (−8, 5), (−2, −3), (−2, 5)
18. (−3, −3), (−7, −5), (−7, −3)

Plot three points in at least two quadrants whose coordinates are integers satisfying the given requirement.

19. The abscissa is twice the ordinate.
20. The abscissa is two less than the ordinate.
21. The ordinate is three more than half the abscissa.
22. The ordinate is twice the absolute value of the abscissa.

LINEAR EQUATIONS AND STRAIGHT LINES

9–3 The Graph of a Linear Equation in Two Variables

Every root of $y = 1 - 2x$ is an ordered pair of numbers represented by (x, y). The graph of one such root, $(2, -3)$, is the point P in Figure 9–4. To find other roots of this equation, substitute values for x and obtain those for y, as shown in the table.

$y = 1 - 2x$		
x	$1 - 2x$	y
−3	1 − 2(−3)	7
−2	1 − 2(−2)	5
−1	1 − 2(−1)	3
0	1 − 2(0)	1
1	1 − 2(1)	−1
2	1 − 2(2)	−3
3	1 − 2(3)	−5

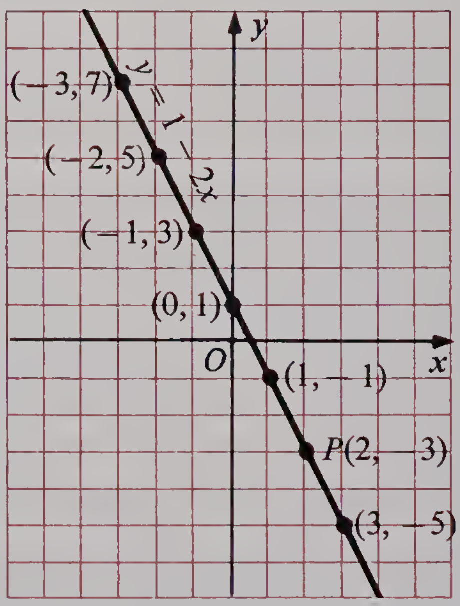


Figure 9–4

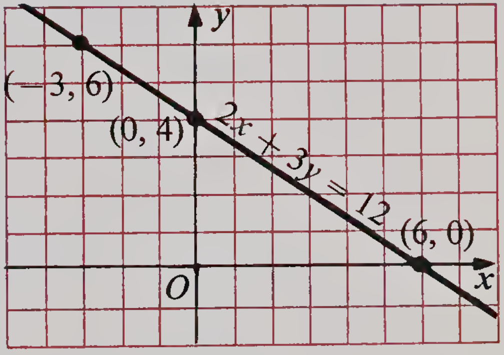
The points are said "to belong to," "be on," or "be points of" the line.

T.M. 32 Figure 9–4 suggests that the points corresponding to these roots lie on an unending straight line. If {directed numbers} is the replacement set for both x and y, each root of the equation gives the coordinates of a point on this line, and the coordinates of each point satisfy the given equation. Because this line is the set of *all those points* and *only those points* whose coordinates satisfy the equation, the line is called the **graph of the equation**, and $y = 1 - 2x$ is an *equation of the line.*

Since its graph is a straight line, this is a *linear equation in two variables.* In a linear equation, each term is a constant or a monomial of degree 1. Thus, $5x - 4y = 3$ is linear, but, $\frac{1}{x} + y = 3$, $x^2 - 4y = 3$, and $xy = 6$ are not. Although you need plot only two points to graph a linear equation, it is good practice to plot a third point, as a check.

Each line has many equivalent equations; $y = 1 - 2x$, $2y = 2 - 4x$, and $-3y = -3 + 6x$ are all equations of the line in Figure 9-4.

EXAMPLE 1. **Graph $2x + 3y = 12$.**



Solution:

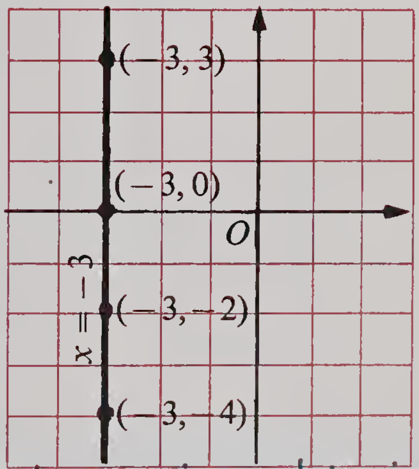
$$2x + 3y = 12$$
$$3y = 12 - 2x$$
$$y = 4 - \tfrac{2}{3}x$$

Recall that two points determine a line.

x	$4 - \frac{2}{3}x$	y
−3	$4 - \frac{2}{3}(-3)$	6
0	$4 - \frac{2}{3}(0)$	4
6	$4 - \frac{2}{3}(6)$	0

EXAMPLE 2. **Graph $x = -3$.**

Solution: **Since the equation places no restrictions on y, all those points having abscissa -3 correspond to its roots, regardless of their second coordinate.**



Point out that these drawings are incomplete pictures. See T.M. pg. 32 (4).

ORAL EXERCISES

State whether or not each equation is linear.

1. $\frac{x}{2} - 5y = 1$ Yes
2. $3x = 2y$ Yes
3. $x^2 + y^2 = 9$ No
4. $\frac{2x}{3} - \frac{y}{4} = 3$ Yes
5. $xy = 6$ No
6. $3x^2 - y = 4$ No
7. $\frac{x}{2} = \frac{y}{8}$ Yes
8. $y = 2x^2$ No
9. $x^2 = 5x + 4$ No

Solve each equation for y in terms of x.

10. $2x - y = 6$ $y = 2x - 6$ **13.** $x - y = 3$ $y = x - 3$ **16.** $2y - 4x = 0$ $y = 2x$

11. $2x + y = 7$ $y = 7 - 2x$ **14.** $x = 10y$ $y = \frac{x}{10}$ **17.** $4y - 2x = 0$ $y = \frac{1}{2}x$

12. $\frac{3x}{2} + y = 1$ $y = 1 - \frac{3x}{2}$ **15.** $\frac{y}{6} = x$ $y = 6x$ **18.** $\frac{y}{5} = 2x$ $y = 10x$

State the relation between the ordinate and abscissa of points on the graph of each equation.

SAMPLE. $y = 4x + 7$

What you say: The ordinate is seven more than four times the abscissa.

19. $y = 3x$ Ord. = 3 × abs. **22.** $y = 2x - 7$ Ord. = 2 × abs. − 7. **25.** $xy = 6$ Abs. × ord. = 6

20. $y = -2x$ Ord. = −2 × abs. **23.** $y = 3x + 4$ Ord. = 3 × abs. + 4. **26.** $y = x^2 - 3$ Ord. = (abs

21. $y = -7x$ Ord. = −7 × abs. **24.** $2x + 3y = 9$ Ord. = $-\frac{2}{3}$ × abs. + 3. **27.** $y = |x| - 2$ Ord. = |ab

Tell which of the given pairs of points belong to the graph of the equation.

28. $x - y = 9$; (4, 13), (−4, −13) (−4, −13)

29. $x - 4y = 13$; (3, 4), (1, −3) (1, −3)

30. $2y + 7x = 0$; (0, 0), (−35, 10) (0, 0)

31. $4x - 3y - 1 = 0$; (1, 1), (0, 0) (1, 1)

These exercises emphasize tha a point is on the graph if, and only if, its coordinates satisfy equation.

WRITTEN EXERCISES

Be sure that students extend their graphs to a reasonable le h.

Graph each equation.

A

1. $y = 3x$ **4.** $y = -2x$ **7.** $2y = x$ **10.** $2x + y = 4$

2. $y = 4x$ **5.** $y = -3x$ **8.** $4y = -x$ **11.** $6x - 2y = 1$

3. $x = 4$ **6.** $y = -2$ **9.** $2x - y = 1$ **12.** $2x + 3y = 6$

Find the coordinates of the point where the graph of each equation crosses (a) the x-axis and (b) the y-axis. A graphical solution or a solution by substitution i accept e.

B

13. $5x - 3y - 30 = 0$ **15.** $12x - 60 = 5y$ **17.** $3y = 7x$

14. $7y + 4x - 28 = 0$ **16.** $9y - 36 = 2x$ **18.** $y = |x|$

Graph each pair of equations in the same coordinate plane. Name the coordinates of the point where the graphs intersect, and show by substitution that they satisfy both equations.

C

19. $x = y$; $3x + y = 4$ **21.** $x + y = 4$; $2x - y = 5$

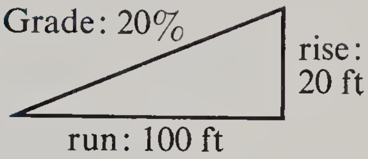
20. $y = -x$; $y - 4x = 5$ **22.** $x - y = 5$; $7x + y = 3$

Graph the following equations.

23. $|x| = 2$ **24.** $y = |x|$ **25.** $|y| = 4$ **26.** $y = |x| - 2$

9–4 Slope of a Line

See 9-4 T.M. pg. 32 for motivation.



To describe the steepness, or grade, of a hill, you determine the vertical *rise* for every 100 feet of horizontal *run*. If a hill rises 20 feet for every 100 feet of horizontal distance, its grade is the ratio $\frac{20}{100}$, or 20%. To describe the steepness, or *slope*, of a straight line you choose two points on it, such as $P(2, 1)$ and $Q(4, 7)$ in Figure 9–5, and compute a similar quotient:

$$\text{slope} = \frac{\textbf{rise}}{\textbf{run}} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{6}{2} = 3.$$

Because the horizontal change in moving from P to Q is the change in the abscissa,

$$\textbf{horizontal change} = \textbf{abscissa of } Q - \textbf{abscissa of } P = 4 - 2.$$

The vertical change is the *corresponding* difference of ordinates:

$$\textbf{vertical change} = \textbf{ordinate of } Q - \textbf{ordinate of } P = 7 - 1.$$

Thus, using m to denote slope, you have

$$m = \frac{\text{difference of ordinates}}{\text{difference of abscissas}} = \frac{7 - 1}{4 - 2} = 3.$$

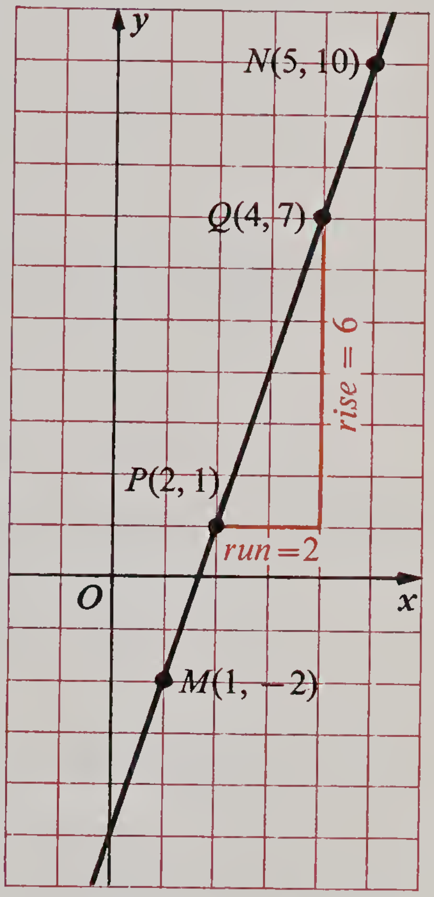


Figure 9–5

To be consistent, you always move on a line from left to right. On a line with slope 3, each horizontal change of 1 unit produces a positive change of 3 units in the vertical direction. For the line joining S and T:

$$m = \frac{0 - 3}{4 - (-2)} = \frac{-3}{6} = -\frac{1}{2}.$$

Check by counting boxes; from S to T are 6 units of horizontal change and -3 units of vertical change. For each positive

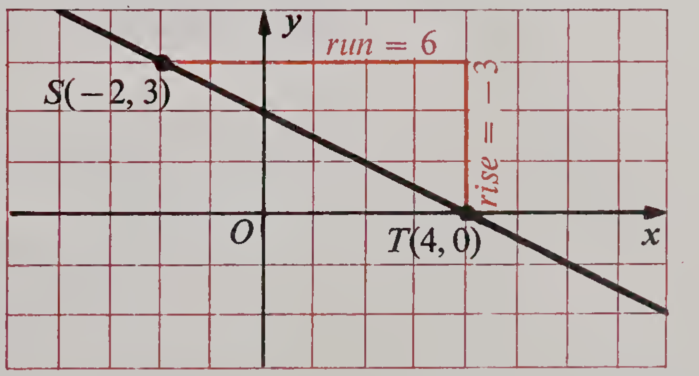


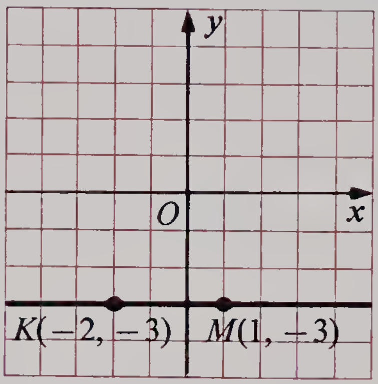
Figure 9–6

change of one horizontal unit, therefore, there is a negative change of half a vertical unit, a rate of change equal to $-\frac{1}{2}$.

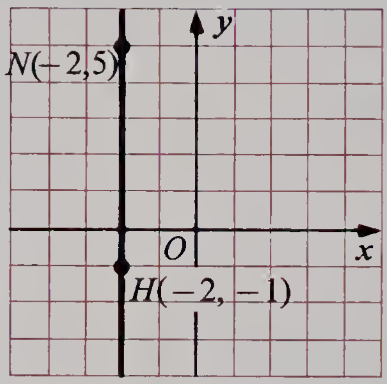
Whenever a line falls from left to right, its slope is a negative number; when it rises from left to right, its slope is a positive number. Can the slope of a line be 0? The slope of the line joining $K(-2, -3)$ and $M(1, -3)$ in Figure 9–7 is

$$m = \frac{-3 - (-3)}{1 - (-2)} = \frac{-3 + 3}{1 + 2} = \frac{0}{3} = 0.$$

Do you see that the slope of every horizontal line is 0?



· *Figure 9–7* ·



· *Figure 9–8* ·

The slope computation for Figure 9–8 is

$$\frac{5 - (-1)}{-2 - (-2)} = \frac{5 + 1}{-2 + 2} = \frac{6}{0}.$$

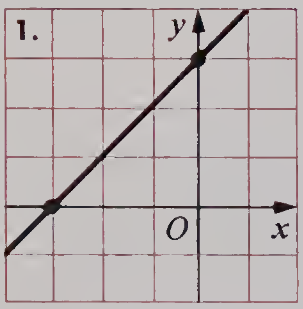
Since you may not divide by 0, this line, like *every vertical line, has no slope.*

A basic property of a line is that its slope is constant. Thus, you may use *any* two of its points in computing its slope.

ORAL EXERCISES

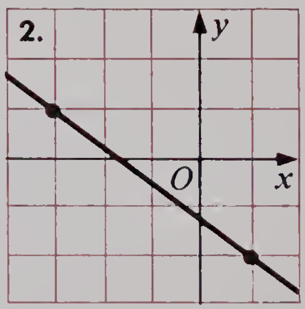
Give the slope of each line in Exercises 1–8.

1.



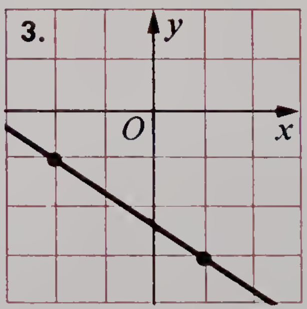
Slope: 1

2.



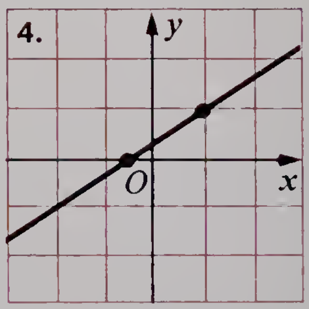
$-\frac{3}{4}$

3.



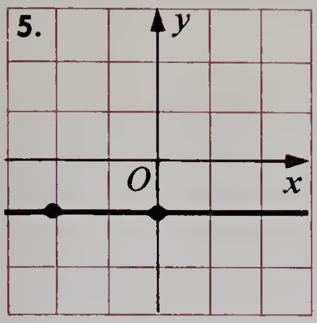
$-\frac{2}{3}$

4.

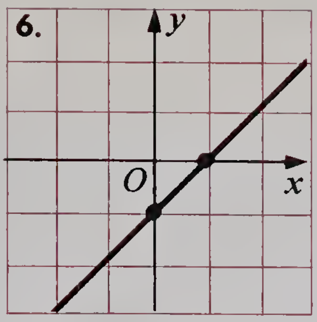


$\frac{2}{3}$

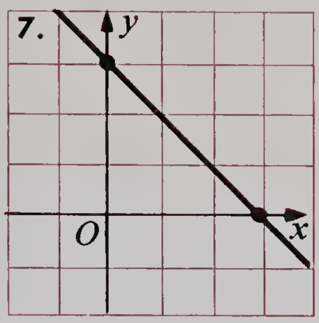
5.



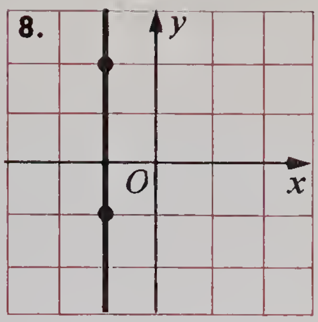
6.



7.



8.



Slope: 0   1   -1   No slope

Do the points given in each table lie on a line? If so, tell its slope.

SAMPLE.

2 2 2

x	0	2	4	6
y	5	3	0	-1

-2 -3 -1

What you say: The points do not lie on a line, because equal increases in the value of x do not produce equal changes in the value of y.

These exercises emphasize that the slope gives the change in y produced by an increase of one in x.

9.

x	0	1	2	3
y	1	3	5	7

Yes; 2

10.

x	0	1	2	3
y	1	3	6	11

No

11.

x	-1	0	1	2
y	0	1	0	-3

No

12.

x	5	6	7	8
y	2	1	0	-1

Yes; -1

13.

x	0	2	4	6
y	-1	2	5	8

Yes; $\frac{3}{2}$

14.

x	0	2	4	6
y	-3	1	5	9

Yes; 2

15.

x	1	3	6	10
y	-2	0	3	7

Yes; 1

16.

x	9	5	-3	-9
y	5	3	-1	-4

Yes; $\frac{1}{2}$

WRITTEN EXERCISES

The line should extend beyond the two points.

Plot each pair of points, draw a straight line through them, and determine the slope of the line from the graph. Check by finding the slope algebraically.

A

1. $(2, 3); (4, 5)$

2. $(6, 1); (10, 3)$

3. $(1, -2); (4, -6)$

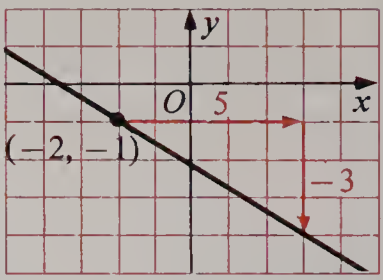
4. $(-2, 1); (2, -2)$

5. $(-1, 5); (-1, -5)$

6. $(5, -1); (-5, -1)$

Through the given point, draw a line with the given slope.

SAMPLE. $(-2, -1)$; slope $= -\frac{3}{5}$



Solution: From $(-2, -1)$ measure 5 units to the right and 3 units down. The point reached together with the point $(-2, -1)$ determine the line.

7. $(1, 2)$; slope $= 1$

8. $(-2, 0)$; slope $= \frac{3}{2}$

9. $(3, -2)$; slope $= -\frac{1}{3}$

10. $(-1, -3)$; slope $= -2$

11. $(-3, 2)$; slope $= 0$

12. $(1, -4)$; no slope

Determine a so that the slope m of the line through each pair of points has the given value. Check your solution by graphing the points.

B **13.** $(-3, 2a)$; $(-1, 3a)$; $m = -\frac{1}{2}$

14. $(3, a)$; $(-5, 5a)$; $m = \frac{1}{2}$

15. $(3, 5)$; $(4, a)$; $m = 2$

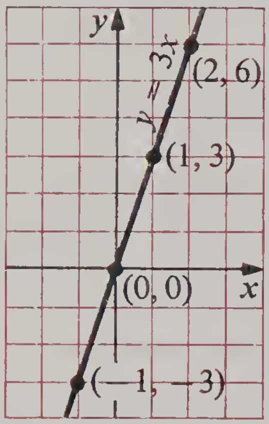
16. $(2, 0)$; $(7, a)$; $m = -1$

17. $(-2, 6)$; $(1, 3a)$; $m = -2$

18. $(1, -4)$; $(3, 2a)$; $m = 5$

9-5 The Slope-Intercept Form of a Linear Equation

The graph of $y = 3x$ is the straight line containing the points whose coordinates are given in the table:

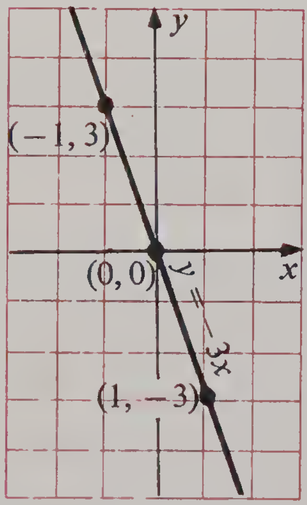


1 1 1

x	-1	0	1	2
y	-3	0	3	6

3 3 3

Do you see that when the abscissas of two points on the line differ by 1 their ordinates differ by 3, the slope of the line? Notice that the line passes through the origin



Can you guess the slope of the line whose equation is $y = -3x$? It is -3, because an increase of 1 in the abscissa produces a change of -3 in the ordinate. This line also contains the origin.

In general:

A line whose equation is of the form

$$y = mx$$

has slope m and passes through the origin.

Emphasize that the y-intercept is a number, not a point.

Do you see that the graph of $y = 3x + 4$ is a straight line of slope 3? In Figure 9–9, compare the graphs of $y = 3x + 4$ and $y = 3x$. They have equal slope, but they cross the y-axis at different points. The ordinate of the point where a line crosses the y-axis is called the line's **y-intercept**. To determine the y-intercept, replace x by 0 in the equation of each line:

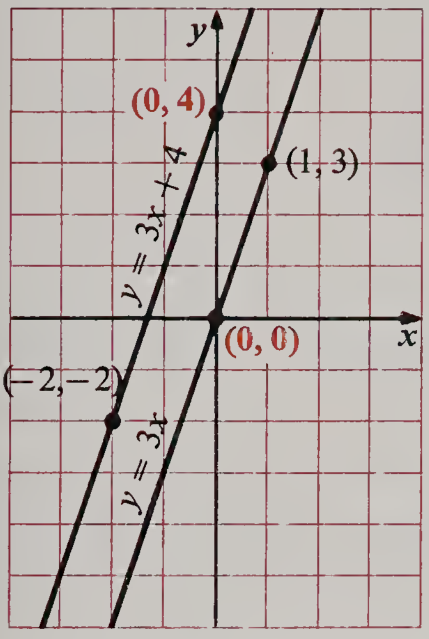


Figure 9–9

$$\begin{array}{lcl} y = 3x & & y = 3x + 4 \\ y = 3 \cdot 0 & & y = 3 \cdot 0 + 4 \\ y = 0 & \leftarrow y\text{-intercepts} \rightarrow & y = 4 \end{array}$$

If you write $y = 3x$ as $y = 3x + 0$, you can see that the constant term is the y-intercept of each graph, $y = 3x + 0$ and $y = 3x + 4$.

An equation of the form $y = mx + b$, called the *slope-intercept* form, is the equation of a line whose slope is m and whose y-intercept is b.

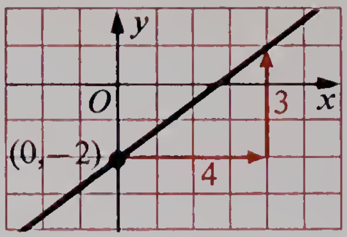
In describing a straight line, write its equation in the form $y = mx + b$. Then read the values of the slope m, and the y-intercept b.

Equation	**Transforming to $y = mx + b$**	**Describing the line**	
		Slope	**y-intercept**
$3x - 3y = 7$	$3y = 3x - 7;\ y = 1x - 2\frac{1}{3}$	1	$-2\frac{1}{3}$
$4x + 2y = 0$	$2y = -4x;\quad y = -2x + 0$	−2	0
$3y - 9 = 0$	$3y = 9;\quad y = 0 \cdot x + 3$	0	3

Stress that the form of the equation is changed, but its solution set (and, therefore, its graph) is not. In each case, the equations are equivalent.

EXAMPLE. Draw the line with $m = \frac{3}{4}$, $b = -2$; then find its equation.

Solution: The y-intercept is -2, so mark $(0, -2)$. As the slope is $\frac{3}{4}$, from this point move 4 to the right and 3 up, to locate a second point on the line. Then draw a line through the points.



$$\begin{array}{l} y = mx + \quad b \\ \quad\;\;\downarrow \qquad\; \downarrow \\ y = \frac{3}{4}x + (-2) \text{ or } 4y = 3x - 8, \text{ Answer.} \end{array}$$

ORAL EXERCISES

State the slope and y-intercept of each line.

$m=2; b=5$ **1.** $y = 2x + 5$

$m=3; b=4$ **2.** $y = 3x + 4$

$m=-1; b=10$ **3.** $y = -x + 10$

$m=-2; b=-3$ **4.** $y = -2x - 3$

$m=0; b=-8$ **5.** $y + 8 = 0$

6. $y + 2 = 0$ $m=0; b=-2$

7. $2y = -6x + 1$ $m=-3; b=\frac{1}{2}$

8. $3y = -12x + 2$ $m=-4; b=\frac{2}{3}$

9. $7 - x = y$ $m=-1; b=7$

10. $x + y = 5$ $m=-1; b=5$

11. $3y - x = -9$ $m=\frac{1}{3}; b=3$

12. $2x - 3y = 0$ $m=\frac{2}{3}; b=$

13. $3x + 2y = 0$ $m=-\frac{3}{2}; b$

14. $x + 1 = 0$ } no slop

15. $x - 3 = 0$ } no y-interc

WRITTEN EXERCISES

Write a linear equation with integral coefficients and the given slope and y-intercept.

A

1. $m = 3; b = 1$

2. $m = -2; b = 5$

3. $m = -\frac{1}{4}; b = -2$

4. $m = \frac{1}{3}; b = -3$

5. $m = -\frac{4}{7}; b = 0$

6. $m = 0; b = 7$

7. $m = 0; b = -10$

8. $m = -1; b = 2$

9. $m = 1; b = -1$

10. $m = 7; b = 0$

Use only the y-intercept and slope to graph each equation.

11. $y - 3x = 4$

12. $7x + y = 5$

13. $x + 2y - 3 = 0$

14. $x + 3y + 6 = 0$

15. $2x - 5y = 0$

16. $4x - 7y = 0$

17. $15x + 3y + 4 = 0$

18. $6y - 7x - 9 = 0$

9–6 Determining the Equation of a Line

The line in Figure 9–10 has slope $\frac{5}{2}$ and passes through the point $(-4, -3)$. The slope-intercept form of the equation of this line is

$$y = \frac{5}{2}x + b.$$

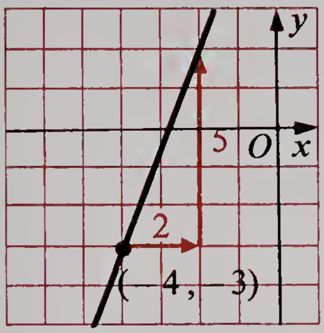


Figure 9–10

Since the point $(-4, -3)$ is on the line, its coordinates must satisfy the equation; that is,

Emphasize this reasoning.

$$-3 = \frac{5}{2}(-4) + b, \text{ or } -3 = -10 + b, \text{ or } 7 = b.$$

Thus, an equation of the line is

$$y = \frac{5}{2}x + 7 \qquad \text{or} \qquad 2y = 5x + 14.$$

To determine an equation of a line containing two given points, find the slope of the line, and then find the y-intercept, as above. The following example illustrates the method.

EXAMPLE. **Find an equation of the line which passes through the points whose coordinates are (2, 4) and (3, 7).**

Solution:

1. Slope $= \dfrac{7 - 4}{3 - 2} = \dfrac{3}{1} = 3$

2. The slope-intercept form of the equation is

$$y = mx + b \qquad \text{or} \qquad y = 3x + b$$

Choose one point, say (2, 4). Since it lies on the line:

$$4 = 3 \cdot 2 + b \qquad \text{or} \qquad 4 = 6 + b$$
$$\therefore -2 = b$$

3. To check, show that the coordinates of the other point (3, 7) satisfy the equation:

$$y = 3x - 2$$
$$7 \stackrel{?}{=} 3 \cdot 3 - 2$$
$$7 \stackrel{?}{=} 9 - 2$$
$$7 = 7 \checkmark$$

$\therefore$ An equation of the line is: $y = 3x - 2$, Answer.

WRITTEN EXERCISES

Find an equation of the line through the given point, having the given slope.

Ⓐ
1. $(-3, -2); 3$
2. $(-1, 0); 5$
3. $(2, 8); -2$
4. $(-5, 0); \frac{2}{7}$
5. $(-2, -5); \frac{7}{2}$
6. $(0, 0); -\frac{8}{3}$
7. $(0, 0); -\frac{3}{8}$
8. $(1, -5); 0$
9. $(-2, 3); 0$

Find an equation of the line through the given points.

10. $(2, 3); (5, 6)$
11. $(1, 4); (3, 6)$
12. $(0, 1); (0, 0)$
13. $(0, 0); (0, -1)$
14. $(0, 0); (-1, 2)$
15. $(5, -3); (0, 0)$
16. $(-4, -4); (-2, 1)$
17. $(-5, 1); (0, -2)$
18. $(0, -1); (4, -3)$

Determine *a*'s value so that the equation's graph passes through the given point.

Ⓑ
19. $ax + 3y = 5; (1, 4)$
20. $2x + ay = 4; (3, 1)$
21. $5x - 2y + a = 0; (-1, 3)$
22. $4x - ay = 2; (-3, -2)$

Find an equation of the line parallel to the given line, through the given point.

23. $x + y = 5; (3, 4)$
24. $x - y = 3; (-2, 1)$
25. $x + 2y = 6; (2, -3)$
26. $x - 3y = 6; (-6, -2)$

Determine the coordinates of the point where the lines intersect. Pupils may solve these graphically; a few may use substitution.

27. $5x + 7y - 35 = 0$; the x-axis
28. $7y - 2x + 14 = 0$; the x-axis
29. $6x - 5y - 3 = 0; x = 3$
30. $3x - 8y + 27 = 0; x = -1$

INEQUALITIES AND SPECIAL GRAPHS

9–7 Graph of an Inequality in Two Variables

In Figure 9–11 the graph (line l) of $y = 3$ divides the coordinate plane into two regions. If you start at any point on line l, say (1, 3), and move vertically upward, the y-coordinate *increases* as you move. If you move vertically downward from this point, the value of y *decreases*. In either case, the value of x remains 1.

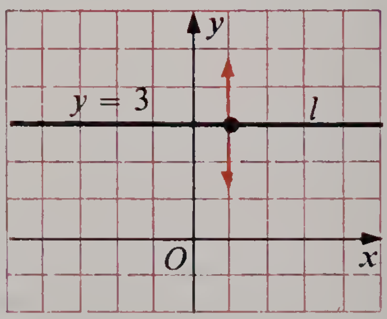


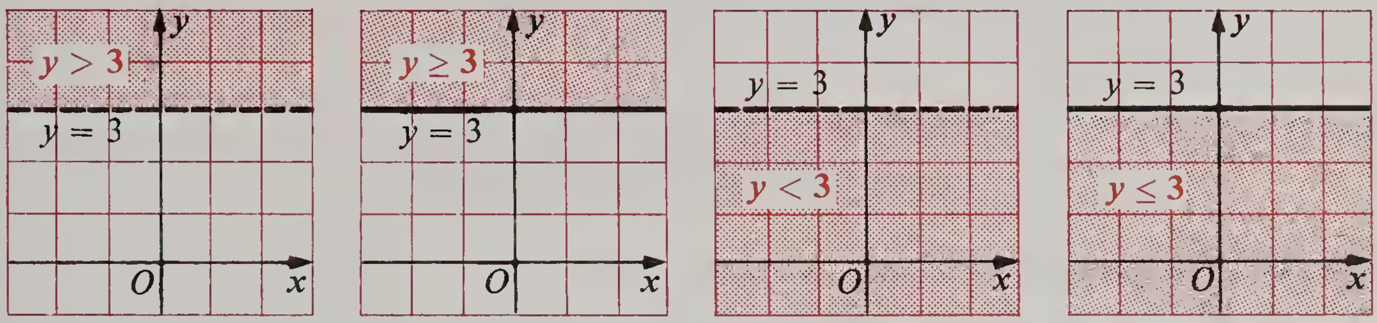
Figure 9–11

See 9-7 T.M. pg. 33.

A half-plane may or may not contain its boundary line. If it does, it may be called a "closed half-plane"; otherwise, an "open half-plane."

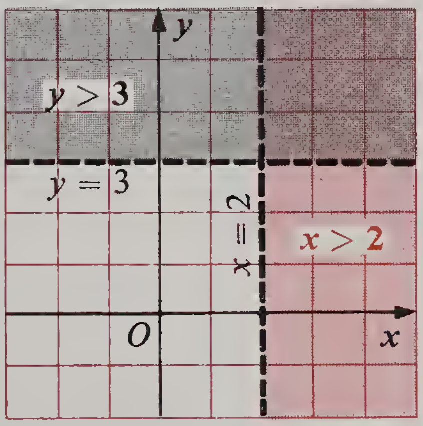
The equation $y = 3$ is the boundary of two *half-planes.* In Figure 9–12 the half-plane above the line consists of all points for which $y > 3$, and is the graph of that inequality. The half-plane below the line is the graph of $y < 3$. The half-plane above the line along with the *boundary line* forms the graph of $y \geq 3$, while the boundary line and the half-plane below it is the graph of $y \leq 3$.

These graphs are indicated by shading. If the boundary is part of the region, the line is a solid line; if it is not, a dashed line is used with appropriate shading to distinguish the half-planes above and below the dashed line.

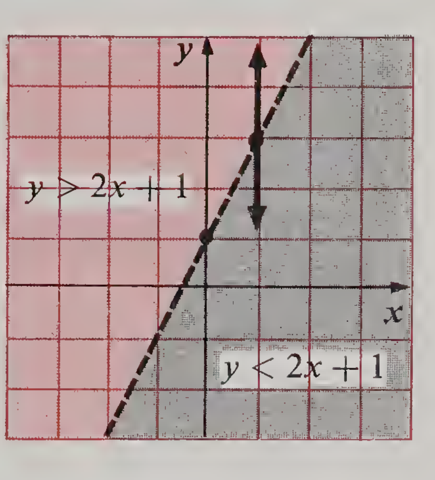


· *Figure 9–12* ·

The inequalities $y > 3$ and $x > 2$ are graphed on the same coordinate plane in Figure 9–13. The points in the upper right-hand section of the plane represent those points whose coordinates satisfy both inequalities. That is, these are the points for which $y > 3$ and $x > 2$.



· *Figure 9–13* ·



· *Figure 9–14* ·

Figure 9–14 shows the graph of $y = 2x + 1$ dividing the plane into two half-planes. For each x, all the points on the line satisfy the equation $y = 2x + 1$; all the points in the region above the line satisfy the inequality $y > 2x + 1$; and all the points in the region below the line satisfy the inequality $y < 2x + 1$.

x	3	2	1	0	-1	-2	-3
y	7	5	3	1	-1	-3	-5

EXAMPLE. Graph the inequality $x - y < 2$.

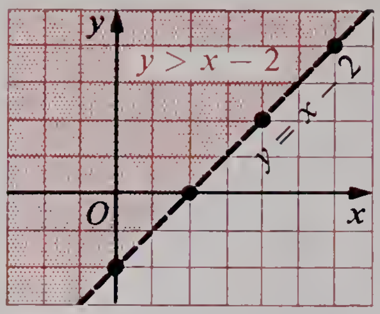
Solution: 1. Transform the inequality into one having y as one member.

Have pupils state the process used to obtain each equivalant inequality.

$$x - y < 2$$
$$-y < 2 - x$$
$$y > x - 2$$

2. Graph $y = x - 2$, and show it as a dashed line.

3. Shade the half-plane above the line.



ORAL EXERCISES

Transform each open sentence into one having y alone as one member.

1. $x + y > 6$ $y > 6 - x$
2. $y - x < 2$ $y < x + 2$
3. $y - 3x \le 4$ $y \le 3x + 4$
4. $5x + y \ge -1$ $y \ge -5x - 1$
5. $2x + 6y < 0$ $y < -\frac{1}{3}x$
6. $9x + 3y \le 0$ $y \le -3x$
7. $x < 4y$ $\frac{1}{4}x < y$
8. $x > 7y$ $\frac{1}{7}x > y$
9. $8x - 2y \ge 0$ $4x \ge y$
10. $10x - 5y < 0$ $2x <$
11. $x - y > -1$ $x + 1 >$
12. $2x - y \ge 3$ $2x - 3 \ge$

Which indicated points belong to the graph of the given inequality?

13. $x - y \le 0$; (7, −3), [✓ (2, 2)]
14. $y - x \ge 0$; [✓ (1, 1)], [✓ (−2, 5)]
15. $2y - x \ge 2$; [✓ (2, 3)], (6, 2)
16. $2x - 3y > -1$; $(\frac{1}{2}, 1)$, [✓ (0, 0)]

Emphasize that a point belongs to the graph if, and only if, its coordinates satisfy
inequality.

WRITTEN EXERCISES

Graph each inequality in two variables.

A

1. $y \ge -3$
2. $y \le 2$
3. $y < 0$
4. $x < 1$
5. $x > -3$
6. $y > x$
7. $y \le 2x$
8. $y < -\frac{x}{4}$
9. $y > -\frac{x}{2}$
10. $x \ge 0$
11. $y + 5x \le 0$
12. $y - 3x \ge -1$

In a coordinate plane indicate the region consisting of all points whose coordinates satisfy both inequalities.

B

13. $x \ge -2$ and $y \le 5$
14. $x < 4$ and $y > -4$

15. $x + y \leq -2$ and $y > 0$

16. $x + y \geq 3$ and $y < 0$

17. $x - y > -1$ and $x \leq 0$

18. $x - y < 2$ and $x \geq 0$

C **19.** $|y| > 2$

20. $|y| \leq 1$

21. $y \leq |x|$

22. $y + |x| = 0$

23. $-1 < y \leq 3$

24. $-2 \leq x < 4$

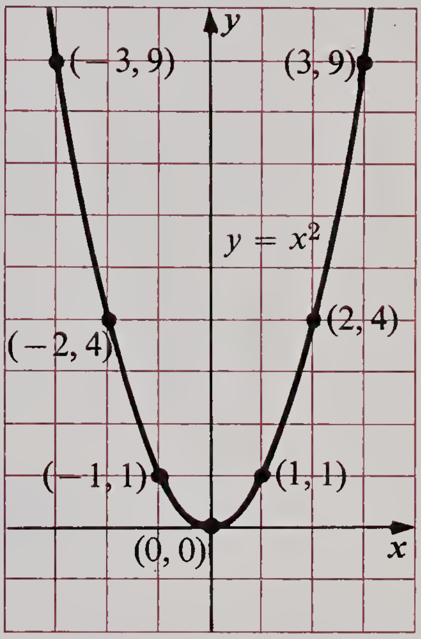
9–8 Graphs That Are Parabolas

Several points on the graph of the quadratic equation $y = x^2$ have been plotted in Figure 9–15, and connected by a smooth curve.

$y = x^2$

x	x^2	y
0	0^2	0
1	1^2	1
2	2^2	4
3	3^2	9
-1	$(-1)^2$	1
-2	$(-2)^2$	4
-3	$(-3)^2$	9

See 9-8 T.M. pg. 33.

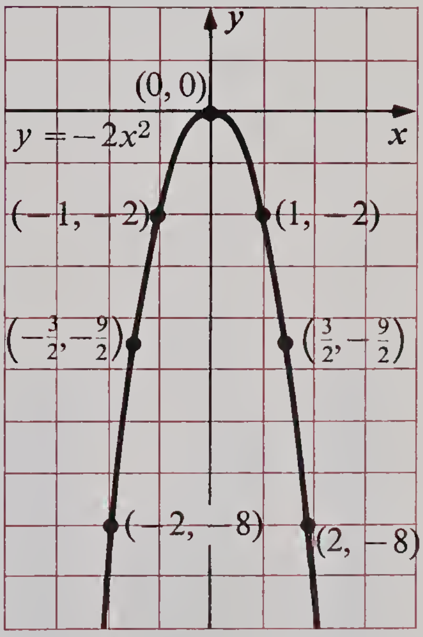


· *Figure 9–15* ·

In Figure 9–16 you see the graph of another quadratic equation, $y = -2x^2$.

$y = -2x^2$

x	$-2x^2$	y
0	$-2(0)^2$	0
1	$-2(1)^2$	-2
$\frac{3}{2}$	$-2(\frac{3}{2})^2$	$-\frac{9}{2}$
2	$-2(2)^2$	-8
-1	$-2(-1)^2$	-2
$-\frac{3}{2}$	$-2(-\frac{3}{2})^2$	$-\frac{9}{2}$
-2	$-2(-2)^2$	-8



· *Figure 9–16* ·

To construct such graphs, plot enough points to enable you to draw a smooth curve. A curve, such as those in Figures 9–15, 9–16, and 9–17 is called a *parabola* (pa-**rab**-oh-luh). The path of a projectile moving in a vacuum, the cable supporting a suspension bridge, and certain cross sections of the reflector on a searchlight are examples of parabolas.

The graph of every quadratic equation of the form $y = ax^2 + bx + c$ for $a \neq 0$ is a parabola.

$y = x^2 + 2x - 8$		
x	$x^2 + 2x - 8$	y
-5	$25 - 10 - 8$	7
-4	$16 - 8 - 8$	0
-3	$9 - 6 - 8$	-5
-2	$4 - 4 - 8$	-8
-1	$1 - 2 - 8$	-9
0	$0 + 0 - 8$	-8
1	$1 + 2 - 8$	-5
2	$4 + 4 - 8$	0
3	$9 + 6 - 8$	7

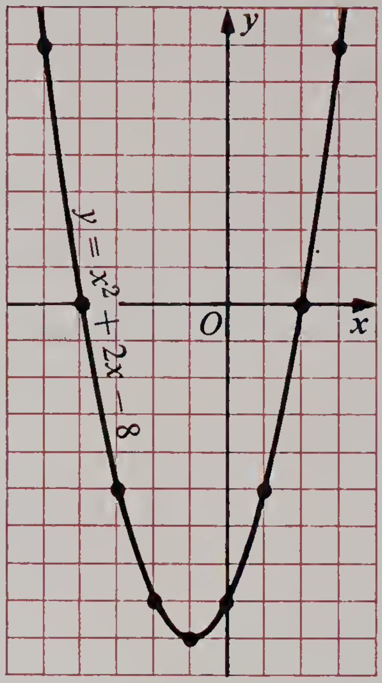


Figure 9–17

WRITTEN EXERCISES

Graph the equations, and draw a smooth curve through the points.

A

1. $y = 2x^2$
2. $y = 3x^2$
3. $4y = x^2$
4. $3y = x^2$
5. $3y = -x^2$
6. $2y = -x^2$
7. $y = x^2 + 1$
8. $y = x^2 + 2$
9. $y = x^2 - 4$
10. $y = x^2 - 9$
11. $y = 2(x^2 - 1)$
12. $y = -2(x^2 - 1)$

B

13. $y = x^2 - 2x - 3$
14. $y = x^2 + 4x - 3$
15. $y = x^2 + 2x$
16. $y = x^2 - 4x$
17. $y = -x^2 - x$
18. $y = x^2 + x + 2$
19. $y = -1 - x^2$
20. $y = -1 + 4x - x^2$

STATISTICAL GRAPHS

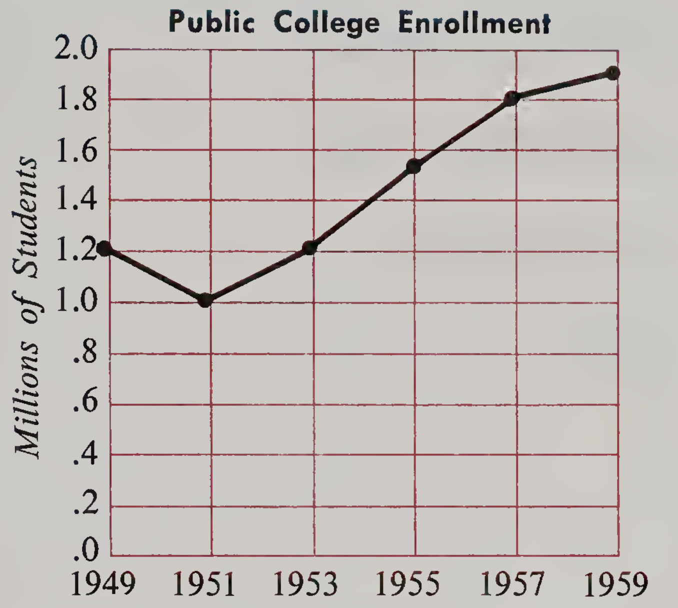
9–9 Broken-Line and Bar Graphs

This topic is not new for most pupils.

Pictures in the form of *broken-line* and *bar graphs* provide quick understanding of statistical data. The points at the breaks in a broken-line graph and the points at the ends of the bars in a bar graph represent ordered pairs of numbers.

A broken-line graph results when you plot ordered pairs of numbers and join them by a series of straight line segments. In Figure 9–18 these line segments do not show the college enrollment for intermediate years, but help you visualize the changes from year to year. You can see readily, for example, that while enrollment was up from 1951 to 1959, the rise was very sharp from 1953 to 1957. Make it clear that broken-line graphs, unlike linear graphs, do not permit interpolation.

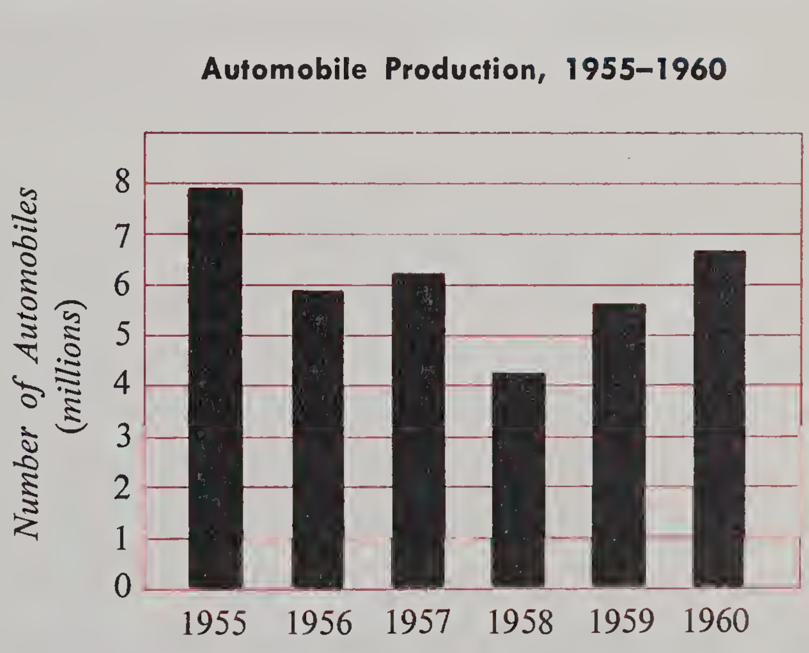
Year	Public College Enrollment (nearest 100,000)
1949	1,200,000
1951	1,000,000
1953	1,200,000
1955	1,500,000
1957	1,800,000
1959	1,900,000



· *Figure 9–18* ·

The graph in Figure 9–19 uses parallel bars of varying lengths to represent measurements of quantities. The scale of the bars must start at zero, so that their relative lengths will be correct. Show a misleading bar graph. See 9-9 T.M. pg. 33.

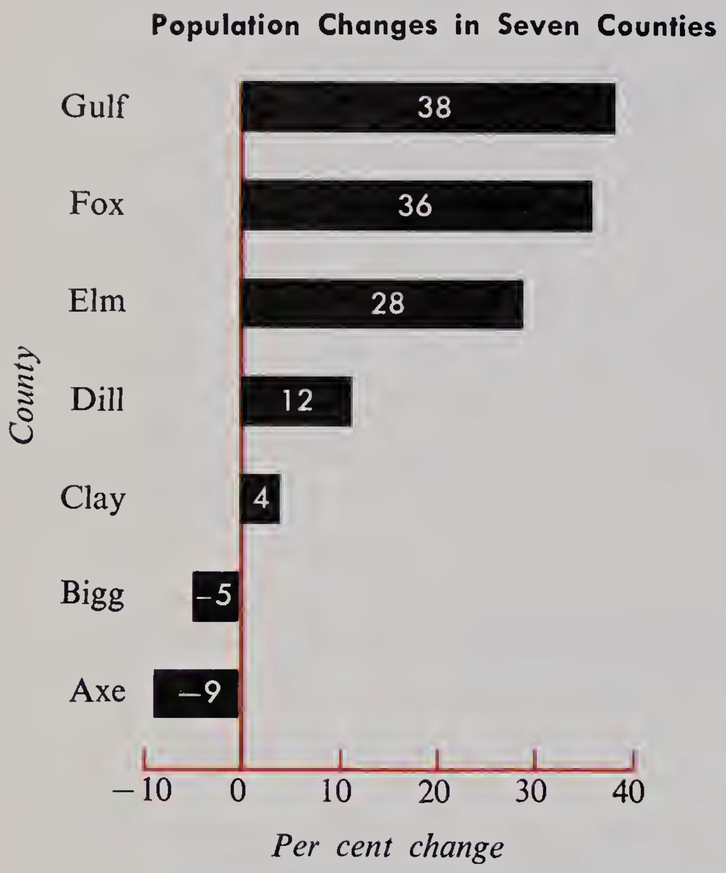
Year	Number of Cars (nearest 100,000)
1955	7,900,000
1956	5,800,000
1957	6,100,000
1958	4,200,000
1959	5,600,000
1960	6,700,000



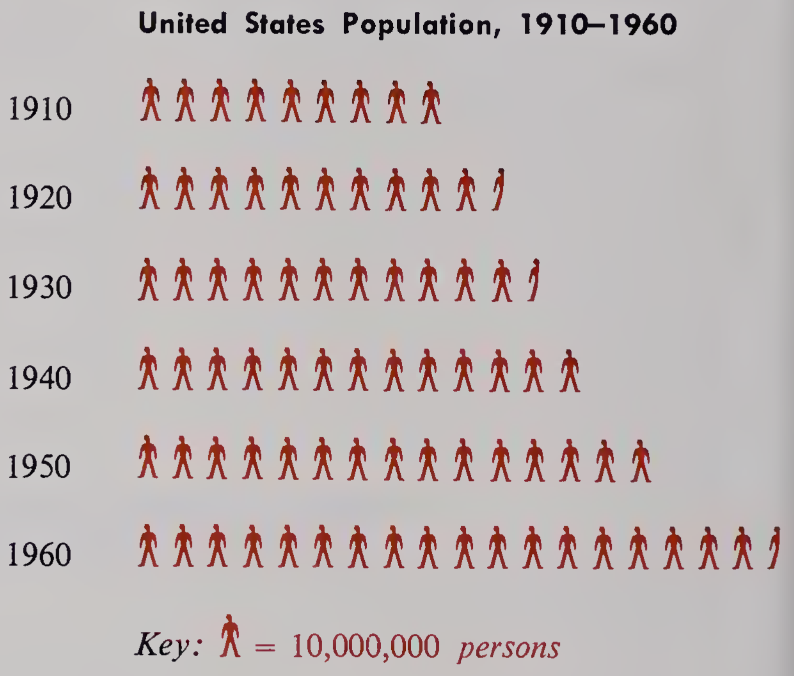
· *Figure 9–19* ·

Such bar graphs compare facts clearly; here, you can see that more cars were produced in 1959 than in 1958, and that more were produced in 1955 than in any of the other years. Horizontal bars in Figure 9–20 show the positive or negative per cent change in population in several counties over a ten-year period.

Pictographs or *pictorial graphs* like Figure 9–21 are special bar graphs in which rows of uniform symbols replace the bars.



· *Figure 9–20* ·



· *Figure 9–21* ·

WRITTEN EXERCISES

Solve, and give each graph a title.

A

1. Make a bar graph to show the oceans' areas, in millions of square miles: Pacific, 69; Atlantic, 41; Indian, 28; Antarctic, 8; Arctic, 5.
2. Make a bar graph showing these heights: Eiffel Tower, 1000 feet; Washington Monument, 550 feet; Empire State Building, 1250 feet; St. Peter's Church, 450 feet; and the Great Pyramid, 480 feet.
3. Draw a bar graph of the per cent change in cost of these items in one year: food, 3; housing, 1; clothing, −2; medical care, 4; utilities, −1.
4. Show on a broken-line graph beginning with January the average monthly temperature in Galveston: 54°; 56°; 62°; 69°; 75°; 81°; 83°; 83°; 80°; 73°; 63°; 56°.

(continued on page 357)

The following insert significantly aids pupils in visualizing solutio
sets of systems of inequalities. Encourage better students to
study the discussion of convex polygons, especially in connect
with the Extra on linear programing, pages 461-462.

Graphs of Inequalities

The following pages show the steps in graphing three separate sets of simultaneous inequalities. As you read the following paragraphs, refer to the diagrams in sequence.

The top sequence of diagrams graphs the set:

$$\begin{aligned} y &\geq 2x \\ 3x + 2y &\geq 6 \\ y &< -2 \end{aligned}$$

The line $y = 2x$ drawn on page H separates the plane into two half-planes: the region above the line (hatched) and the region below (unhatched). The coordinates of the points above and on the line satisfy the inequality $y \geq 2x$; below the line, $y < 2x$. Since the graph of $y \geq 2x$ includes the line $y = 2x$, that line is drawn as a solid boundary.

On page I the line $3x + 2y = 6$ or $y = 3 - \frac{3}{2}x$ separates the plane into two half-planes: the red region above the line and the uncolored region below it. Above or on the line $3x + 2y \geq 6$; below, $3x + 2y < 6$. The set of points whose coordinates satisfy both of the inequalities $y \geq 2x$ and $3x + 2y \geq 2$ consists of the points of the region that is both hatched and red.

The line $y = -2$ on page J produces another separation of the plane: the green region below ($y < -2$) and the region above ($y > -2$). Since the graph of $y < -2$ does not include the line $y = -2$, that line is shown as a dashed boundary.

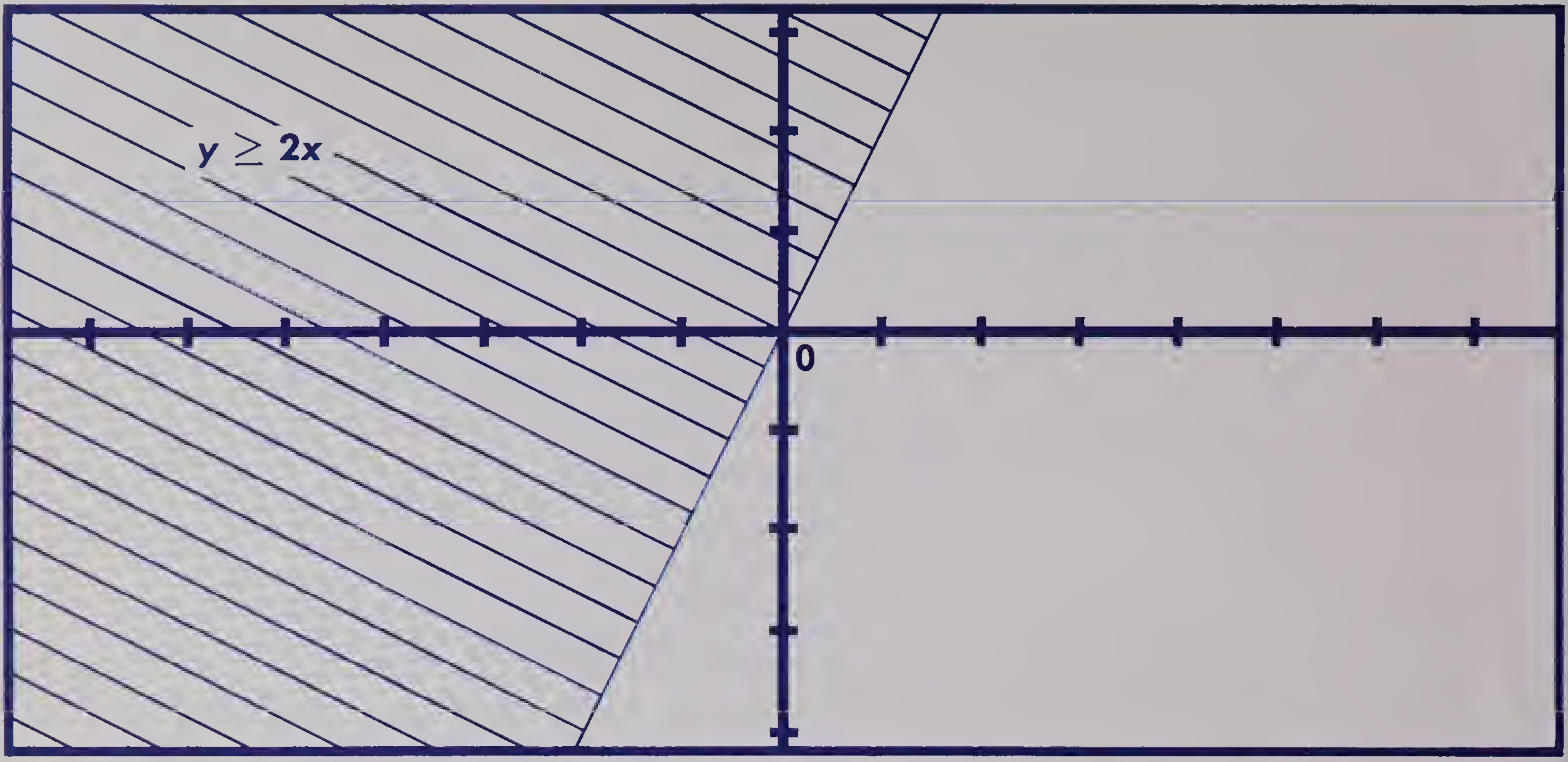
The points whose coordinates satisfy all three inequalities $y \geq 2x$, $3x + 2y \geq 6$, and $y < -2$ should be seen on page J in a region that is hatched and brown. But there is no such region! This means that the three inequalities have no common root; that is, the solution set of this simultaneous system is $\emptyset$.

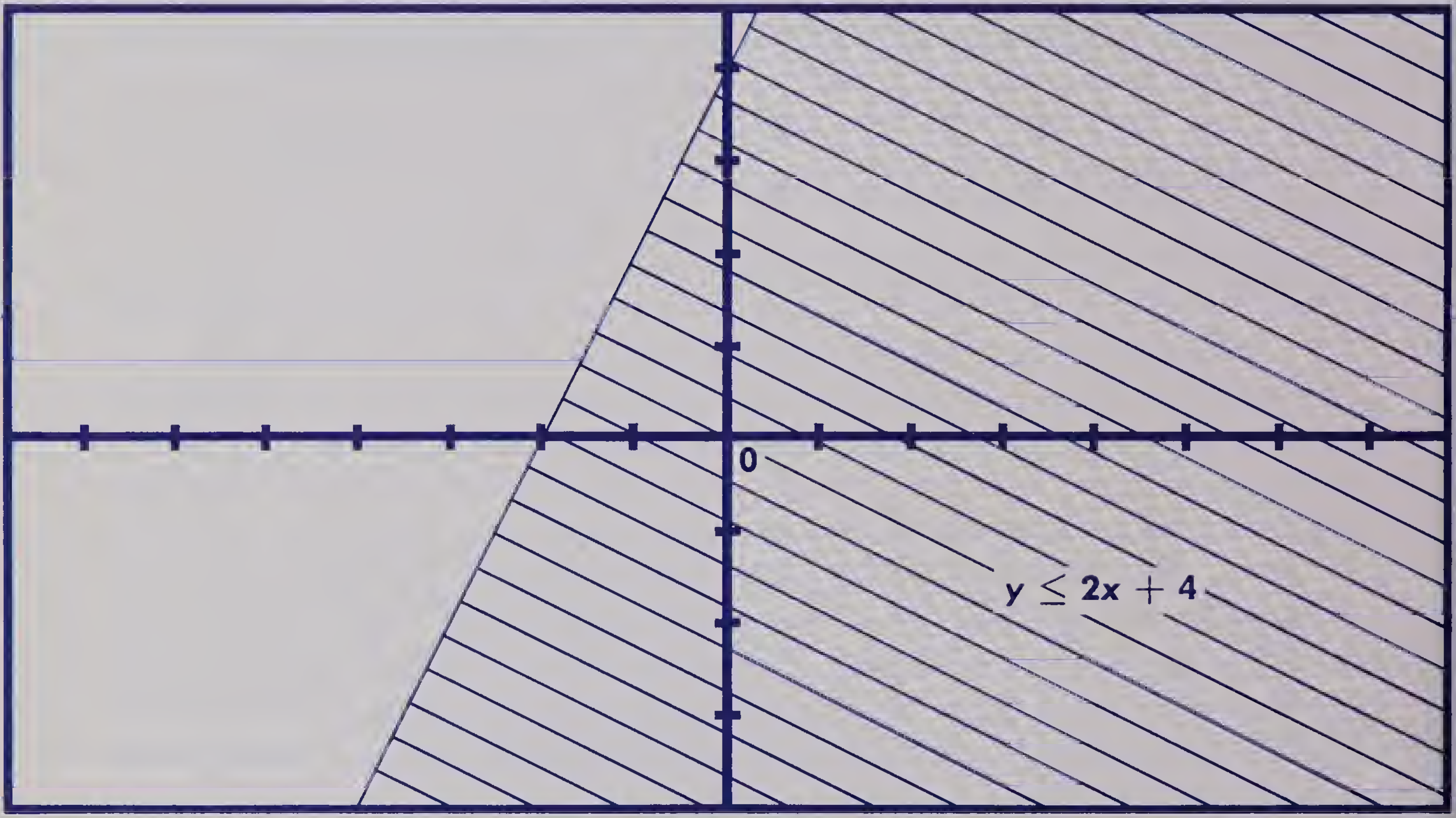
Can you use inequalities to describe the unhatched and uncolored region in the top diagram on page J? If you assume that this region includes its green boundary line, but not its blue or red boundaries, then the region is the graph of the simultaneous system: $y < 2x$, $3x + 2y < 6$ and $y \geq -2$. Can you similarly describe the other six regions shown in the plane? Be sure to decide whether or not you are to include the boundaries in describing the regions.

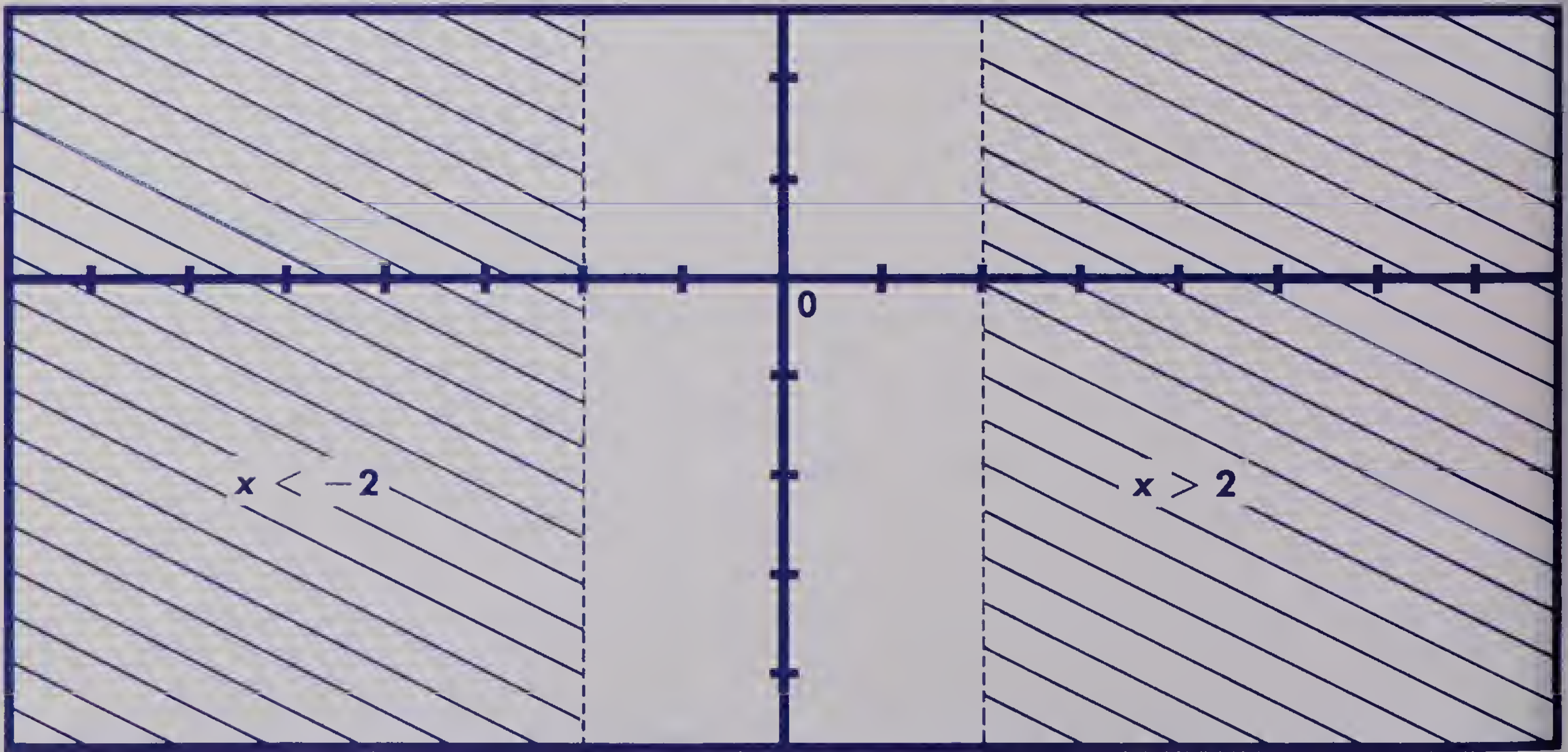
The second sequence of diagrams beginning on page H builds up the graph of the solution set of the simultaneous system: $y \leq 2x + 4$, $x + y + 2 \geq 0$, and $y + 4x < 4$. This system is equivalent to: $y \leq 2x + 4$, $y \geq -x - 2$, and $y < 4 - 4x$. The graph is shown on page J as the region that is hatched and brown. Note that it includes its blue and red boundaries, but not its green one.

The final sequence of diagrams begins on page H with the graph of the inequality $|x| > 2$. Since a point belongs to the graph of this inequality if, and only if, its abscissa satisfies *either* of the inequalities $x > 2$ or $x < -2$, the hatched region consists of two parts: the region to the right of the vertical line $x = 2$ and the region to the left of the line $x = -2$.

Page I introduces the graph of $-3 < y \leq 1$. To belong to the graph, a point must have its ordinate satisfy *both* of the inequalities $-3 < y$ and $y \leq 1$. Hence, the graph consists of the red horizontal strip above the line $y = -3$, and on or below the line $y = 1$.


$y \geq 2x$
0

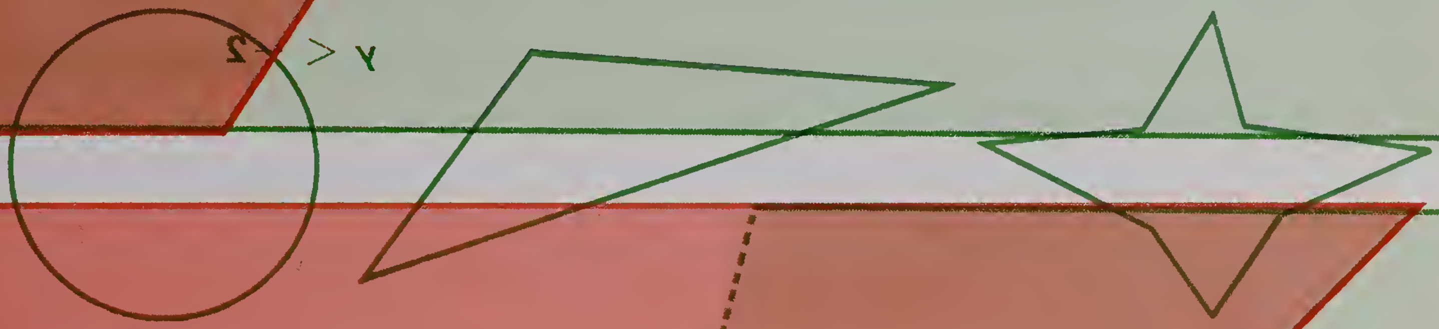

0
$y \leq 2x + 4$


0
$x < -2$
$x > 2$

Do you see that together the inequalities $|x| > 2$ and $-3 < y \leq 1$ separate the plane into nine regions? By turning to page J you find each of these regions identified as the graph of a pair of inequalities. Note: (*i*) a hatched region does not include the points for which $|x| = 2$; (*ii*) a red region includes all points for which $y = 1$, but no points for which $y = -3$.

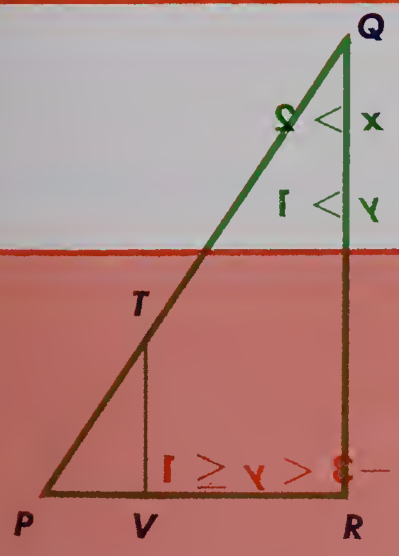
CONVEX POLYGONS

The regions into which a plane is separated by a finite number of straight lines are called *convex* regions. When a region is convex, it contains every line segment joining any two of its points. For example, in the following figure the circle and the triangle enclose convex regions, but the star does not.



Convex regions in the plane are important in studying the values of linear expressions such as $2x + 3y$. Let us call this expression f and write $f = 2x + 3y$. Suppose you want to find the greatest and least values of f over the set of ordered pairs (x, y) satisfying the inequalities $y \leq 2x + 4$, $x + y + 2 \geq 0$, and $y + 4x \leq 4$. This means that you want to maximize and minimize $2x + 3y$ over the closed triangle shown as the hatched, brown region in the middle diagram of page J.

A remarkable fact is that over closed, convex polygons the maximum and minimum values of any linear expression occur at vertices (corner points) of the region. To see that this is true, you must first carefully examine the coordinates of points on the line segment joining any two points in the plane.

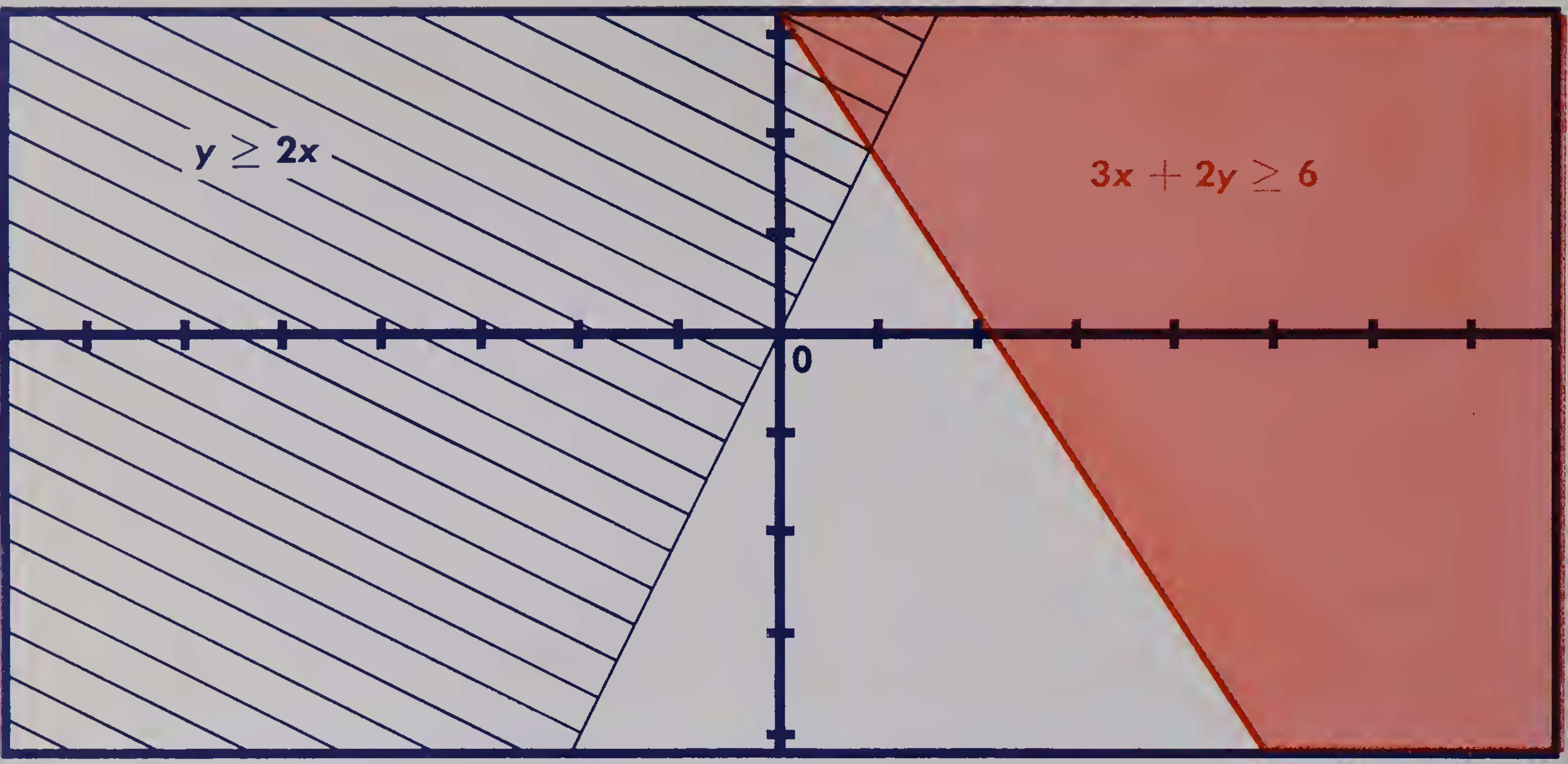


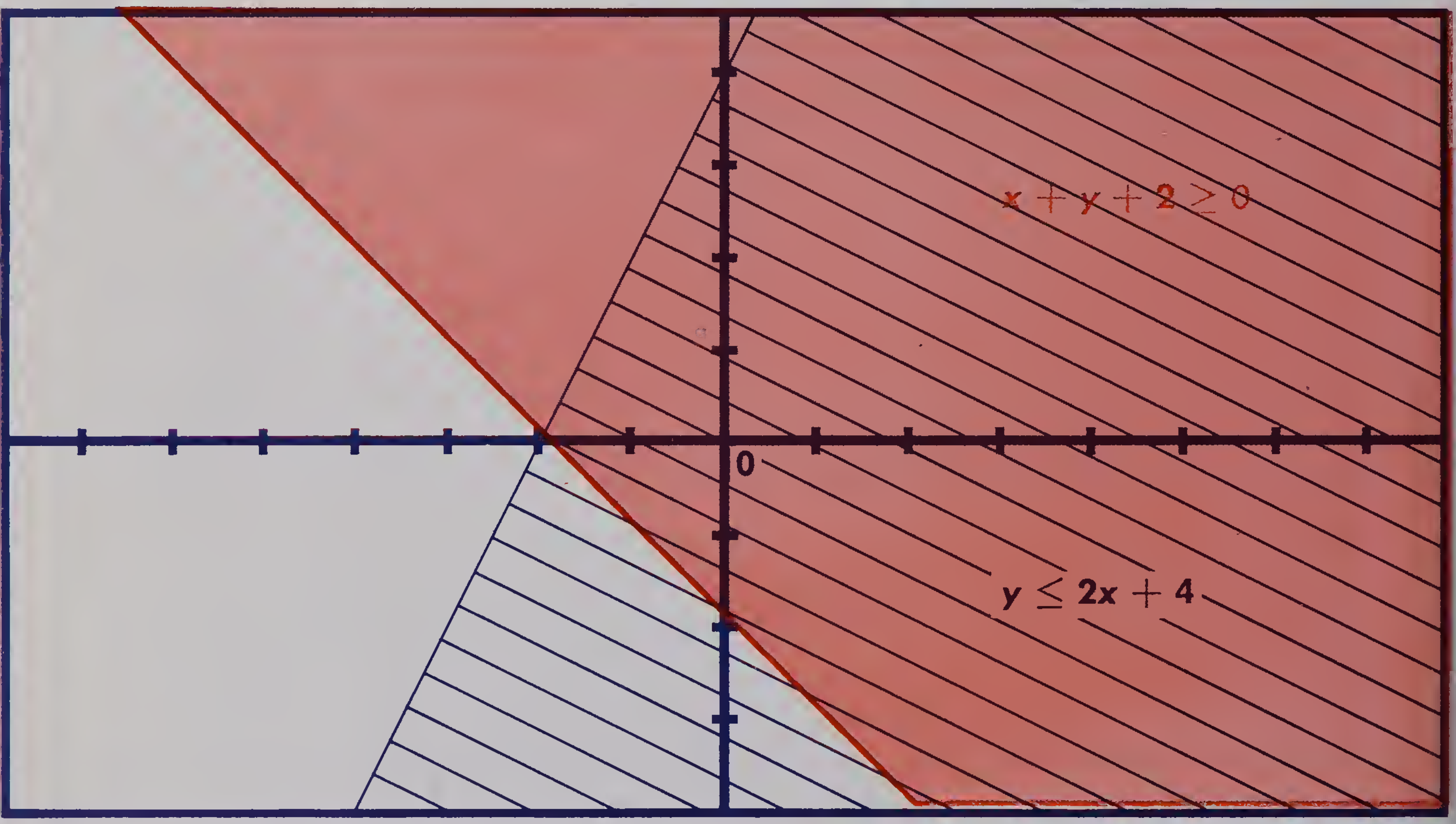
Let $P(a, b)$ and $Q(c, d)$ be two points in the plane, and let T be a point on line segment PQ. Suppose that the distance from P to T ($\overline{PT}$) is one-third of the distance from P to $Q(\overline{PQ})$. Can you guess what x_T, the abscissa of T, might be? It seems reasonable to guess that since $\overline{PT} = \frac{1}{3}\overline{PQ}$, x_T might be the number one-third of the way from a (the abscissa of P) to c (the abscissa of Q). This amounts to guessing that in the right triangles PTV and PQR the following ratios of lengths of corresponding sides are equal: $\dfrac{\overline{PV}}{\overline{PR}} = \dfrac{\overline{PT}}{\overline{PQ}} = \dfrac{1}{3}$. Because these triangles are similar, these ratios are, indeed, equal, so your guess is correct. Thus,

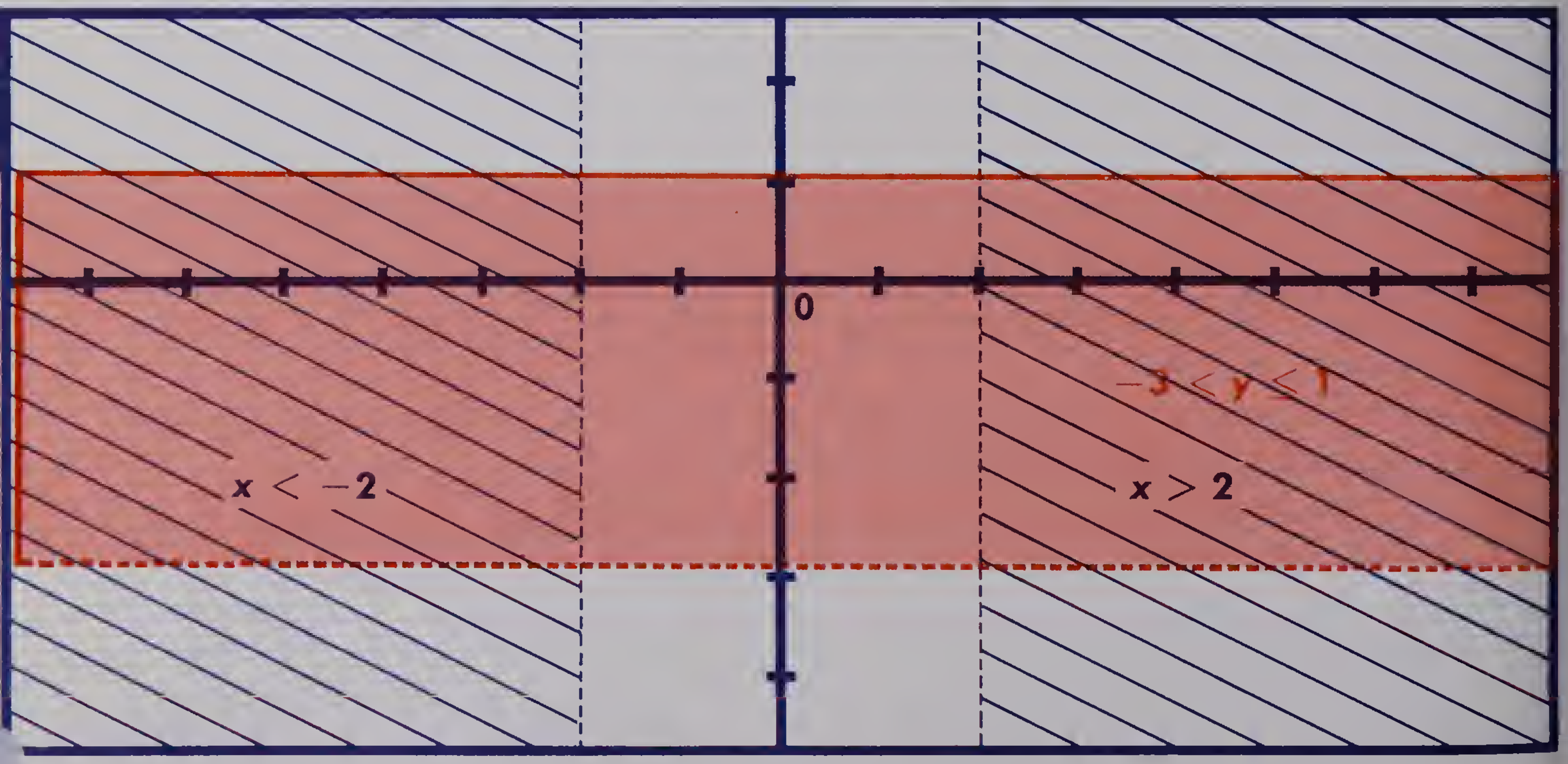
$$\underbrace{x_T}_{\text{abscissa of } T} \overset{\textstyle =}{\underset{\text{is}}{\downarrow}} \underbrace{a + \tfrac{1}{3}(c - a)}_{\tfrac{1}{3} \text{ of the way from } a \text{ to } c.}$$

This means

$$x_T = (1 - \tfrac{1}{3})a + \tfrac{1}{3}c.$$


$y \geq 2x$
$3x + 2y \geq 6$
0

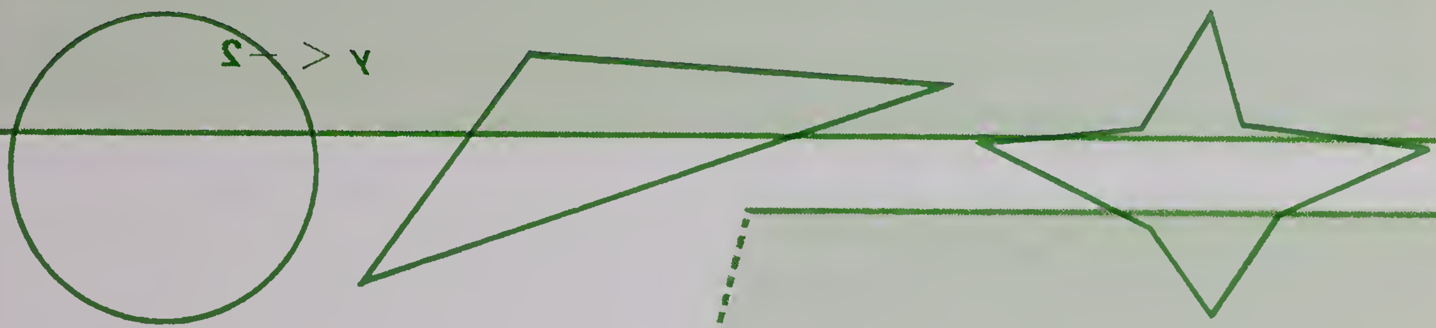

$x + y + 2 \geq 0$
0
$y \leq 2x + 4$


0
$-3 < y \leq 1$
$x < -2$
$x > 2$

Do you see that together the inequalities $|x| > 2$ and $-3 < y \leq 1$ separate the plane into nine regions? By turning to page J you find each of these regions identified as the graph of a pair of inequalities. Note: (*i*) a hatched region does not include the points for which $|x| = 2$; (*ii*) a red region includes all points for which $y = 1$, but no points for which $y = -3$.

CONVEX POLYGONS

The regions into which a plane is separated by a finite number of straight lines are called *convex* regions. When a region is convex, it contains every line segment joining any two of its points. For example, in the following figure the circle and the triangle enclose convex regions, but the star does not.



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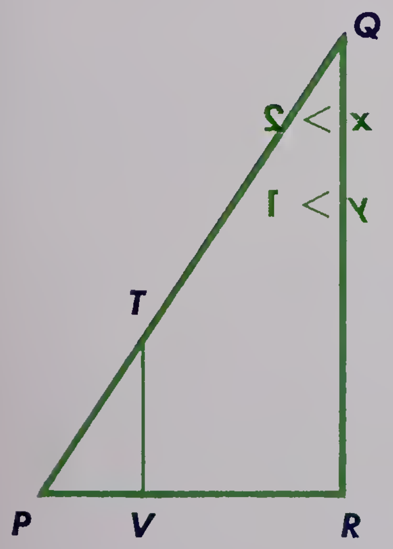

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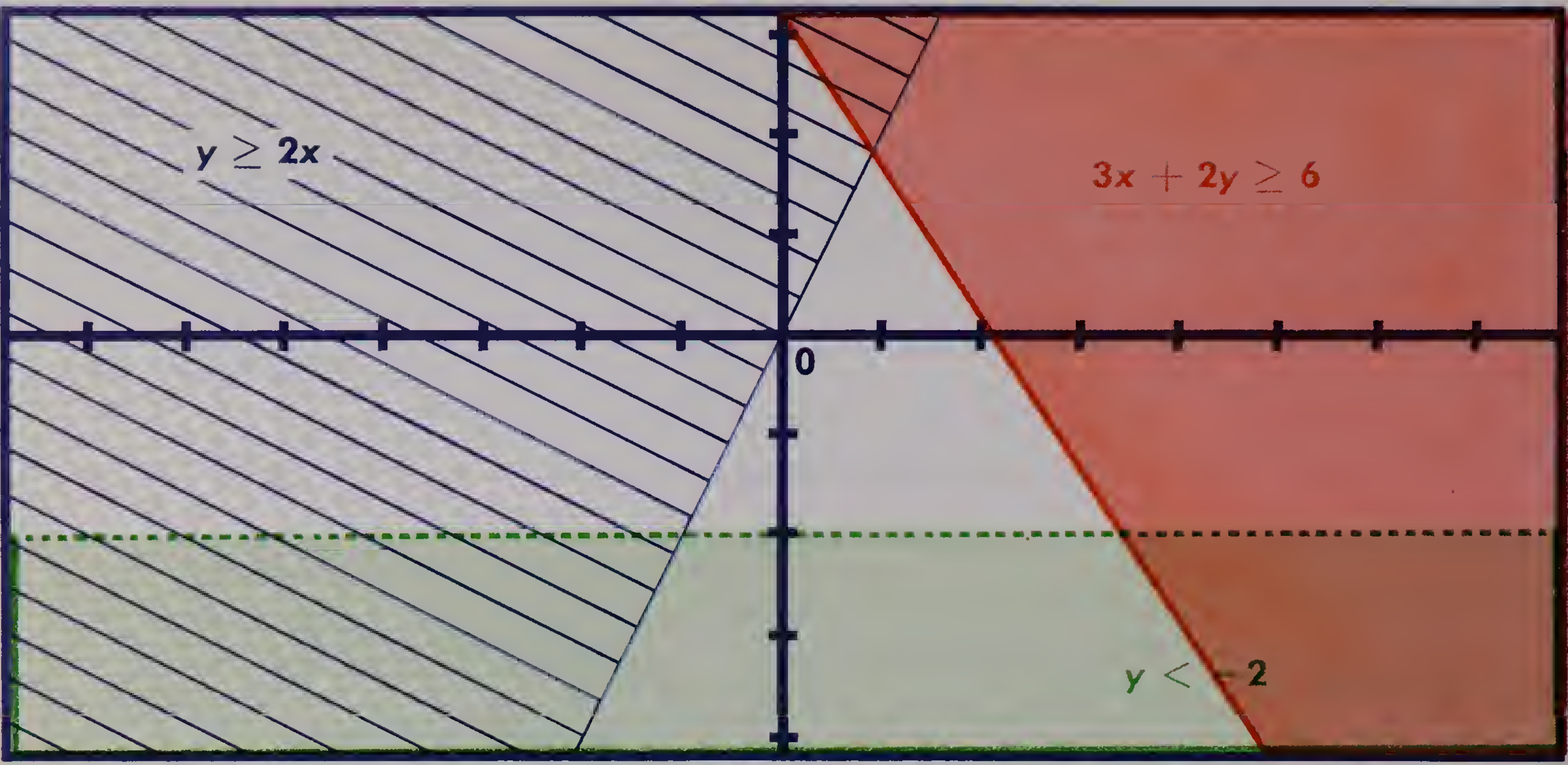
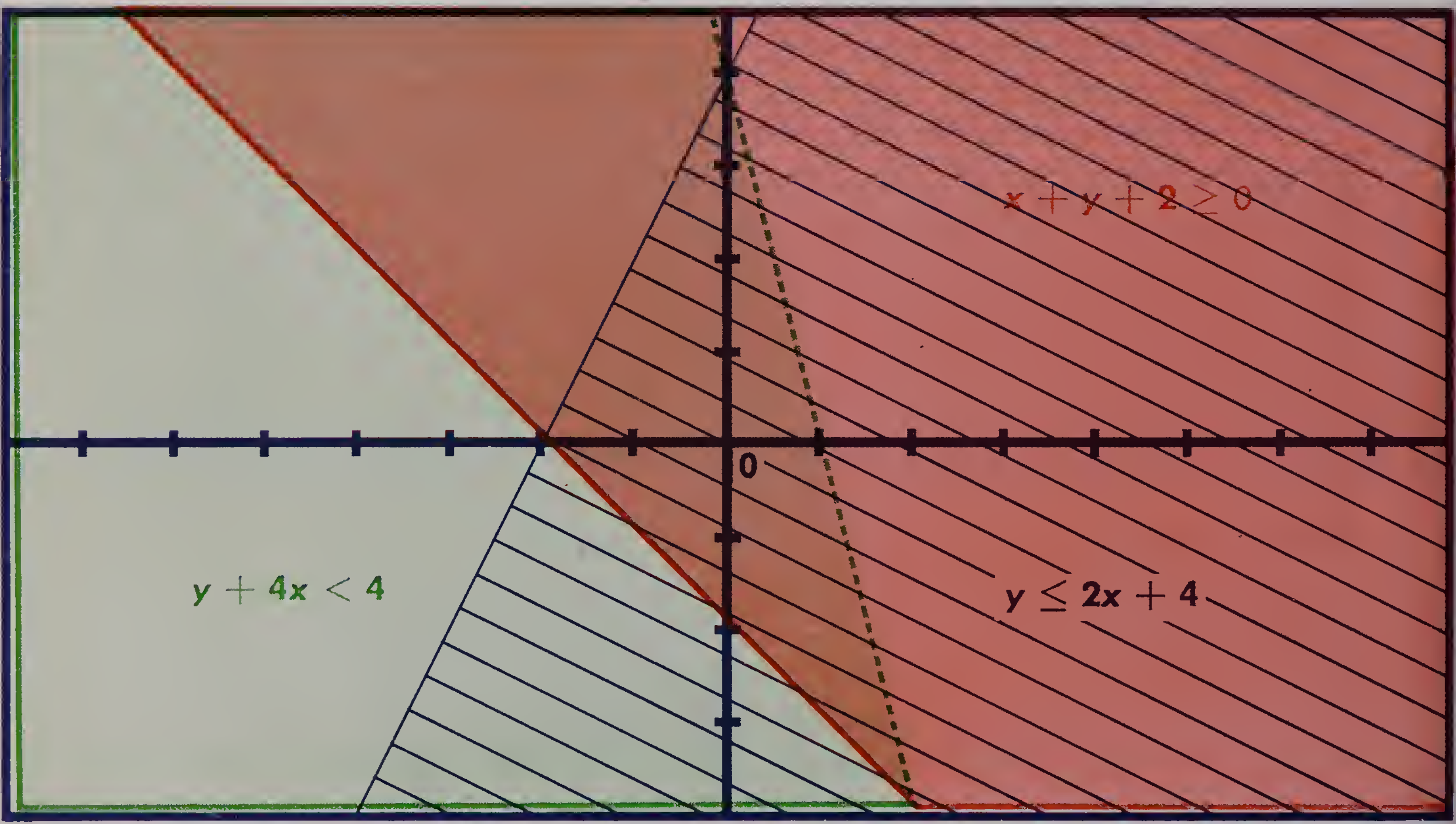
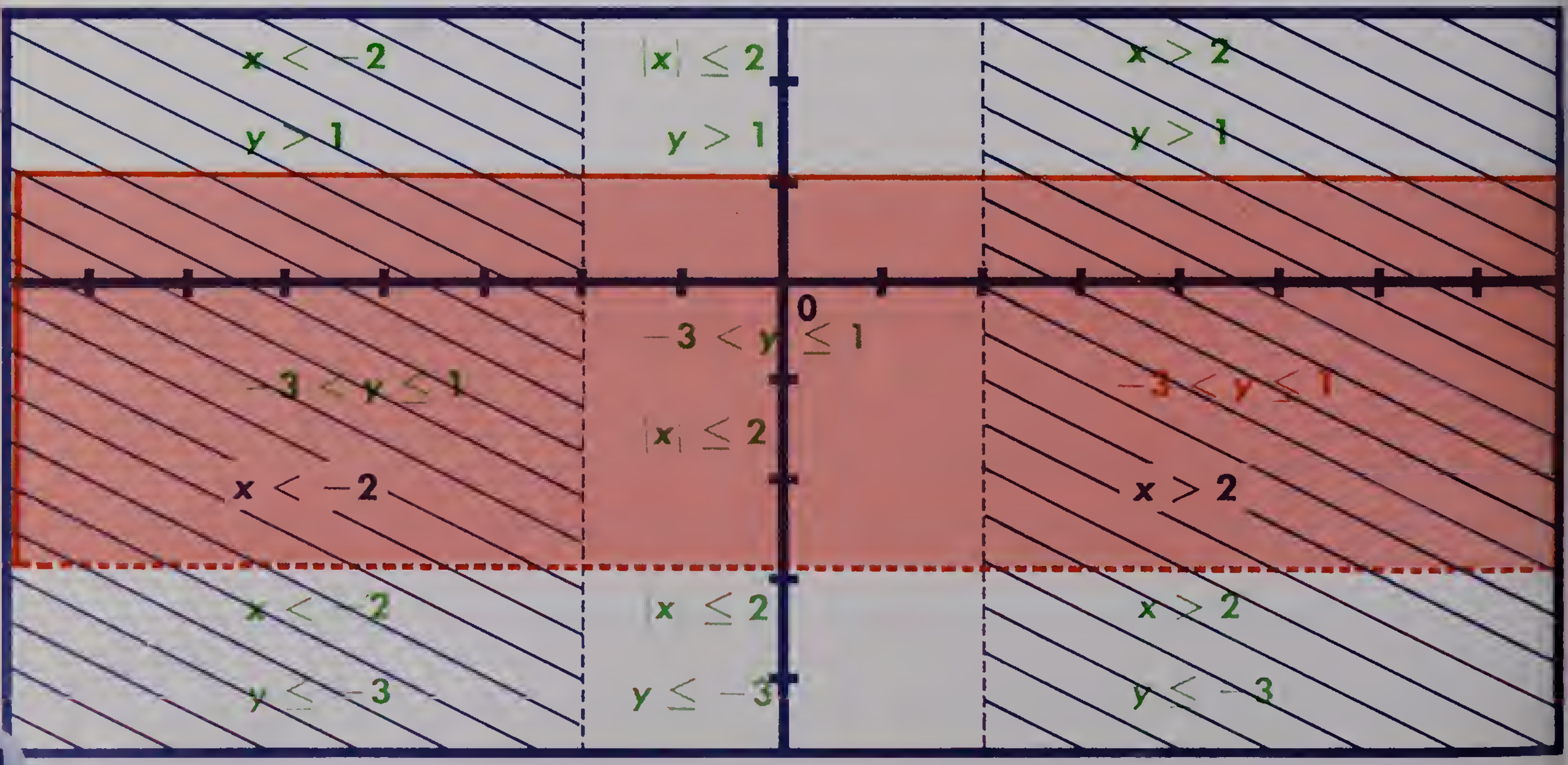

y ≥ 2x
3x + 2y ≥ 6
0
y < −2


x + y + 2 ≥ 0
0
y + 4x < 4
y ≤ 2x + 4


x < −2
|x| ≤ 2
x > 2
y > 1
y > 1
y > 1
0
−3 < y ≤ 1
−3 < y ≤ 1
−3 < y ≤ 1
|x| ≤ 2
x < −2
x > 2
x < −2
|x| ≤ 2
x > 2
y ≤ −3
y ≤ −3
y ≤ −3

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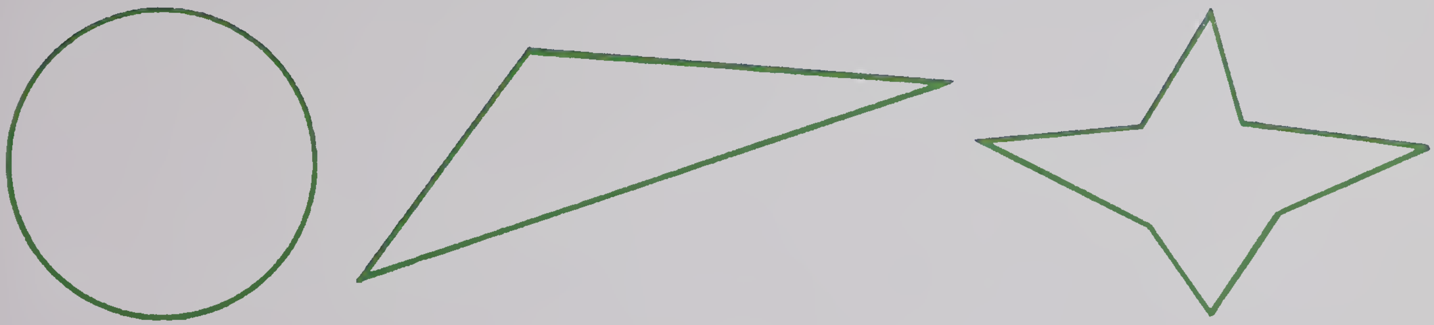
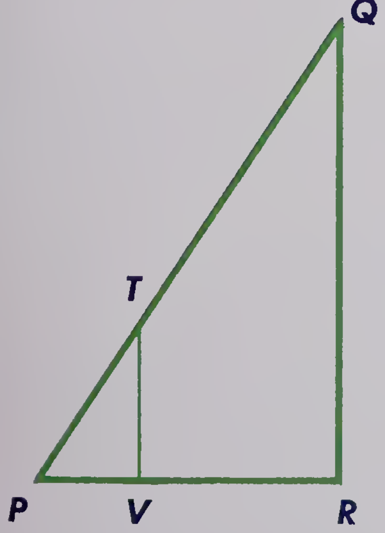
The regions into which a plane is separated by a finite number of straight lines are called convex regions. When a region is convex, it contains every line segment joining any two of its points. For example, in the following figure the circle and the triangle enclose convex regions, but the star does not.



Convex regions in the plane are important in studying the values of linear expressions such as $2x + 3y$. Let us call this expression f and write $f = 2x + 3y$. Suppose you want to find the greatest and least values of f over the set of ordered pairs (x, y) satisfying the inequalities $y \leq 2x + 4$, $x + y + 2 \geq 0$, and $y + 4x \leq 4$. This means that you want to maximize and minimize $2x + 3y$ over the closed triangle shown as the hatched, brown region in the middle diagram of page J.

A remarkable fact is that over closed, convex polygons the maximum and minimum values of any linear expression occur at vertices (corner points) of the region. To see that this is true, you must first carefully examine the coordinates of points on the line segment joining any two points in the plane.



Let $P(a, b)$ and $Q(c, d)$ be two points in the plane, and let T be a point on line segment PQ. Suppose that the distance from P to T ($\overline{PT}$) is one-third of the distance from P to Q($\overline{PQ}$). Can you guess what x_T, the abscissa of T, might be? It seems reasonable to guess that since $\overline{PT} = \frac{1}{3}\overline{PQ}$, x_T might be the number one-third of the way from a (the abscissa of P) to c (the abscissa of Q). This amounts to guessing that in the right triangles PTV and PQR the following ratios of lengths of corresponding sides are equal: $\frac{\overline{PV}}{\overline{PR}} = \frac{\overline{PT}}{\overline{PQ}} = \frac{1}{3}$. Because these triangles are similar, these ratios are, indeed, equal, so your guess is correct. Thus,

$$\underbrace{x_T}_{\text{abscissa of } T} \quad \underset{\text{is}}{\underset{\downarrow}{=}} \quad \underbrace{a + \tfrac{1}{3}(c - a)}_{\frac{1}{3} \text{ of the way from } a \text{ to } c.}$$

This means

$$x_T = (1 - \tfrac{1}{3})a + \tfrac{1}{3}c.$$

By a similar argument you discover that

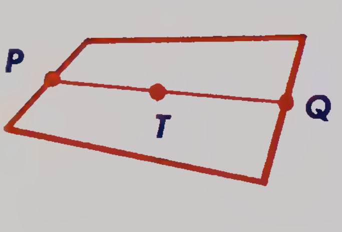
$$y_T = (1 - \tfrac{1}{3})b + \tfrac{1}{3}d.$$

In general, if T is between P and Q, there is a number t between 0 and 1 such that $\frac{\overline{PT}}{\overline{PQ}} = t$ and

$$x_T = (1 - t)a + tc$$

$$y_T = (1 - t)b + td.$$

Now suppose that P and Q are *any* points on the boundary of a closed convex polygon.
Let $f_P = 2a + 3b$ be the value of f at P and $f_Q = 2c + 3d$ be the value of f at Q. Then, f_T (the value of f at T) $= 2x_T + 3y_T$.



$$\therefore\ f_T = 2[(1 - t)a + tc] + 3[(1 - t)b + td]$$

$$\therefore\ f_T = (1 - t)(2a + 3b) + t(2c + 3d)$$

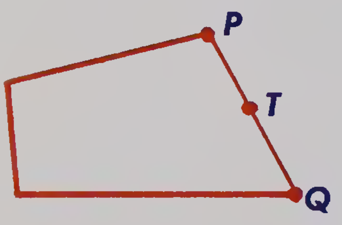
$$\therefore\ f_T = (1 - t)f_P + tf_Q$$

Thus, the value of f at T is between the values of f at P and Q. Of course, if $f_Q = f_P$ then $f_T = f_P$.

Do you see the significance of this discovery? It means that at T the value of f can be neither greater nor less than it is at P or Q.

But, P and Q are any points on the boundary of a closed convex polygon. Now, given any point T in the interior of such a region, there is a line segment containing T that intersects the boundary of the region in just two points and that otherwise lies wholly within the region. This means that the value of f at any interior point T can be neither larger nor smaller than the values of f on the boundary of the region.

Now suppose that P and Q are vertices of the polygon. It is still true that the value of f at T, a nonvertex point on the boundary, can be neither greater nor less than its values at P and Q.



We may, therefore, conclude that $2x + 3y$ (or any linear expression $mx + ny$) assumes its maximum and minimum values over a closed convex polygon at vertices of that region.

In particular, over the hatched, brown triangle in the middle diagram of page J, the greatest and least values of $2x + 3y$ must occur at one or another of the three vertices: $A(2, -4)$, $B(-2, 0)$, $C(0, 4)$. Evaluating $2x + 3y$ at each of these points, you see that over the triangle the maximum 12 occurs at C and the minimum -8 occurs at A.

Point	$2x + 3y$
$A(2, -4)$	-8
$B(-2, 0)$	-4
$C(0, 4)$	12

It is possible that the maximum and minimum values occur at other points besides vertices. What matters is that by testing values at the vertices you can in a finite number of steps determine the greatest and least values of any linear expression over any closed convex polygon.

5. Make a broken-line graph of the number of scholarships awarded in each decade from 1920 to 1960: 700, 800, 1000, 1600, 7600.
6. Make a broken-line graph of the inches of rainfall in New York City in each month from January to December: 5.3; 2.2; 6.6; 5.1; 1.5; 3.4; 2.2; 3.1; 8.6; 3.4; 7.1; 3.2.
7. Make a pictograph of the membership distribution in the fifteenth assembly of the United Nations: Western bloc, 23; Latin American bloc, 20; Asian-African bloc, 41; Soviet bloc, 9; others, 3.
8. Make a pictograph of the median price of used cars in these years: 1951, \$600; 1952, \$850; 1953, \$900; 1954, \$700; 1955, \$600; 1956, \$650; 1957, \$700; 1958, \$600.

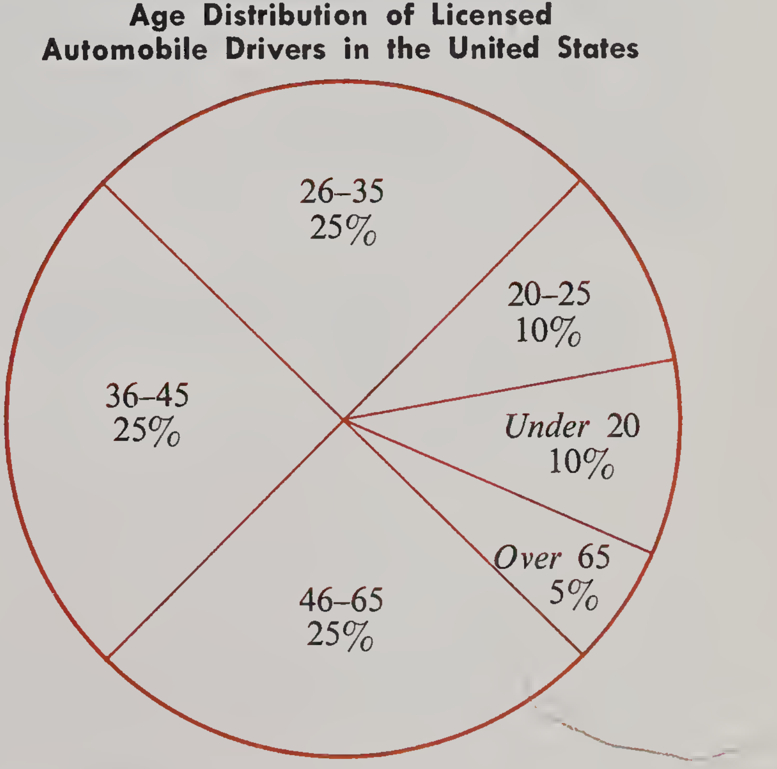
9–10 Circle Graphs

A *circle graph* is used to compare the parts of a whole with each other and with the total. The size of each sector depends on the angle at the center of the circle, so you must know the number of degrees in this *central angle* for your comparison.

Since the circle graph in Figure 9–22 shows that 10% of the drivers are under 20 years of age, the central angle of this sector contains 10% of the 360 degrees around the center of the circle,

or $\frac{10}{100} \times 360 = 36°.$

In like manner, you can compute the number of degrees in the other central angles.



Some pupils may need instruction in the use of a protractor.

Figure 9–22

WRITTEN EXERCISES

1. Bonville's taxes are spent thus: $33\frac{1}{3}\%$ for schools, libraries, and museums; 25% for police, fire, and sanitation; 15% for streets, parks, and recreation; $16\frac{2}{3}\%$ for jails, homes for the destitute, and hospitals; 10% for administration. (a) Make a circle graph. (b) If the third listed item costs \$31,500, find Bonville's total budget.

2. Of Midville Boys High School graduates, 40% join the army, 25% join the navy, 20% enter college, and 15% get jobs in town. (a) Make a circle graph. (b) If 80 boys join the army, how many graduate?

Illustrate each exercise with a circle graph.

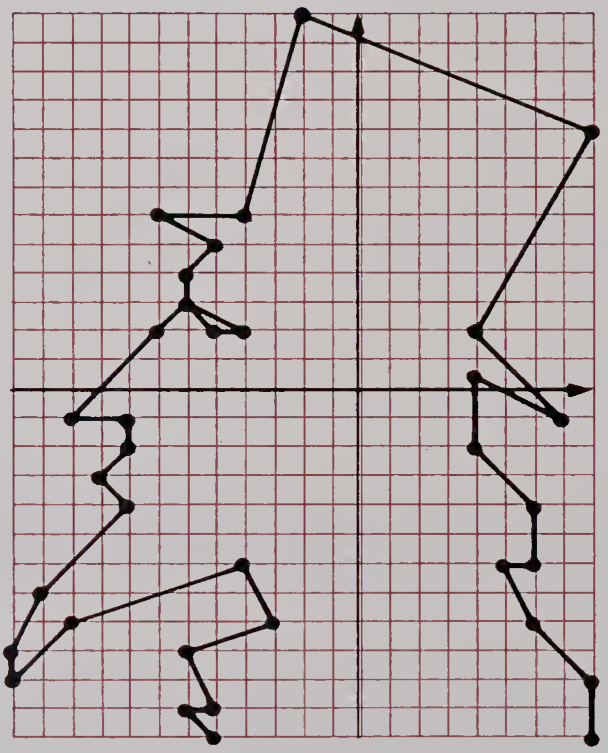
3. The average American diet is: 46% carbohydrates, 14% proteins, 23% unsaturated fats, 17% saturated fats.
4. An ideal diet is: 69% carbohydrates, 16% proteins, 11% unsaturated fats, 4% saturated fats.

Just for Fun

Coordinated Pictures

By drawing a silhouette on a sheet of graph paper and by noting the points at which the outline changes direction, you can make a list of coordinates which represent the picture. Someone else can take your list, plot your number pairs, and connect them with lines to reproduce your picture. If you plot and connect each of these points in turn, you will see such a picture develop:

(−5, −10), (−5, −8), (−8, −4), (−8, 0), (−5, 4), (−3, 5), (1, 5), (3, 3), (2, 2), (5, −1), (4, −2), (3, −2), (2, −3), (2, −4), (4, −6), (4, −7), (3, −8), (1, −8), (1, −10)



You need not confine your artistry to human profiles; animals, plants, and cartoon characters also are suitable subjects for coordinated pictures.

Chapter Summary

Inventory of Structure and Method

1. To set up a **rectangular coordinate system in a plane**, choose a vertical and a horizontal line, and scale them as number lines intersecting at zero. The ordered pair (a, b) corresponds to the point whose directed distance from the vertical axis is a and from the horizontal axis is b.
2. A plane coordinate system enables you to picture the **solution set of an open sentence in two variables** as the set of points whose coordinates satisfy the open sentence. Thus, to graph a linear equation in two variables, each having the set of directed numbers as its replacement set, draw the straight line determined by plotting any two roots of the equation.
3. To measure the **slope** of a straight line, choose two different points on the line, and compute the ratio of the difference between the ordinates of the points to the corresponding difference between the abscissas of the points. It is a property of a straight line that this ratio is the same for every pair of distinct points on the line.
4. A line with slope m and y-intercept b is the graph of the equation $y = mx + b$. This **slope-intercept form** of a linear equation can be used to find an equation for a line
   - **a.** with given slope and passing through a given point;
   - **b.** passing through two different points.
5. To **graph an inequality in two variables** graph the related equality, and then shade the half-plane defined by the inequality.
6. The graph of an equation of the form $y = ax^2 + bx + c$, $a \neq 0$, is a curve called a **parabola**.
7. Broken-line, bar, and circle graphs are employed for the visual presentation of statistics to display comparisons and trends in data.

Vocabulary and Spelling

ordered pair of numbers (*p. 334*)
first coordinate (*p. 334*)
second coordinate (*p. 334*)
root of an open sentence in two variables (*p. 334*)
solution set of an open sentence in two variables (*p. 334*)
horizontal axis (*p. 337*)
vertical axis (*p. 337*)
origin (*p. 337*)
graph of an ordered pair (*p. 337*)
plotting a point (*p. 337*)
abscissa (*p. 338*)
ordinate (*p. 338*)

coordinates of a point (*p. 338*)
plane coordinate system (*p. 338*)
coordinate plane (*p. 338*)
quadrant (*p. 338*)
graph of an equation (*p. 341*)
an equation of a line (*p. 341*)
linear equation in two variables (*p. 341*)
slope of a line (*p. 343*)
y-intercept (*p. 347*)
slope-intercept form (*p. 347*)
graph of an inequality in two variables (*p. 351*)
half-plane (*p. 351*)
boundary line (*p. 351*)
parabola (*p. 353*)
broken-line graph (*p. 354*)
bar graph (*p. 354*)
pictograph (*p. 356*)
circle graph (*p. 357*)

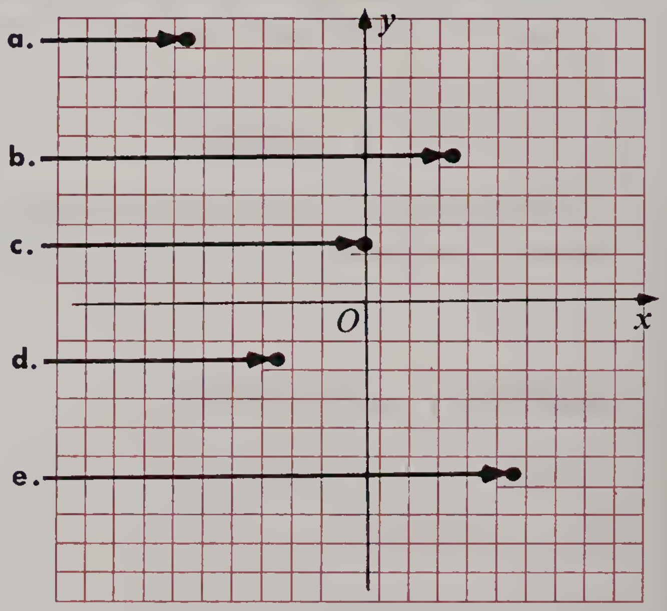
Chapter Test

9–1 **1.** For what value of c does $(c, -4)$ belong to the solution set of $3x + 4y = 17$?

2. If $(3a - 1, b^2) = (5a - 7, 2b - 1)$, find the values of a and b.

3. If $x \in \{-1, 0, 1\}$ and $y \in \{0, 1\}$, give the solution set of $2x + y < 3$.

9–2 **4.** Give the coordinates of each indicated point in the adjoining figure.



5. Graph these points in a coordinate plane:

a. $(5, 2)$
b. $(-3, -4)$
c. $(2, -3)$
d. $(-2, 0)$
e. $(-4, 4)$

9–3 **6.** Graph the one linear equation:

$x^2 + y^2 = 4, \qquad xy - 3x = 2, \qquad x - 2y = 11$

7. Do the points $(2, 7)$ and $(7, 2)$ belong to the graph of $2y - 3x = 8$? Justify your answer.

9-4 **8.** Find the slope of the line joining $(-7, 2)$ and $(1, 18)$.

9. Draw a line with slope 3 through $(5, -2)$.

10. If $(3, -1)$ is on the graph of $2x + ay = 4$, find the value of a.

9-5 **11.** Give the slope and y-intercept of the line $5x - 3y = 24$.

12. Write an equation of the line through $(0, -4)$ with slope 10.

9-6 **13.** Find an equation of the line containing $(8, 0)$ with slope $-\frac{7}{4}$.

14. Write an equation of the line joining $(5, 1)$ and $(7, -3)$.

9-7 **15.** Graph $y + 2x < 6$. **16.** Graph $2y \geq 3x + 4$.

9-8 **17.** Graph $y = 2x^2 - 4$.

9-9 **18.** Make a bar graph of the sale of twenty-eight $25 bonds, thirty-five $50 bonds, fourteen $100 bonds, and seven $500 bonds.

19. Construct a broken-line graph of:

Daily Calorie Needs of Children

Age in Years	3	6	9	12
Calories	1200	1600	2000	2500

9-10 **20.** On a circle graph show how each dollar was used in a business if 30¢ was spent for labor, 10¢ for operations, and 60¢ for materials.

Chapter Review

9-1 **Open Sentences in Two Variables** *Pages 333–337*

1. A root of an open sentence in two variables is an __?__ __?__ of numbers for which the sentence is true.

2. $(a, b) = (c, d)$ if __?__ and __?__.

3. (3, __?__) is a root of the equation $10x + y = 7$.

4. If $x \in \{-1, 0, 1\}$ and $y \in \{\text{integers}\}$, the solution set of $5x - 2y = 11$ is __?__.

9-2 **Coordinates in a Plane** *Pages 337–340*

5. To associate a point with each ordered pair of numbers, draw two __?__ at right angles whose point of intersection is the __?__.

6. The first coordinate of $P(3, -2)$ is the __?__ of P; the second coordinate is the __?__ of P.

7. A point whose ordinate is -3 lies __?__ units below the __?__ axis.

8. Points within the __?__ and __?__ __?__ have positive abscissas.

9. When both coordinates are negative, a point lies in the __?__ quadrant.

In a coordinate plane plot the points in Exercises 10–13.

10. $(3, 1)$ **11.** $(-4, 3)$ **12.** $(-3, -5)$ **13.** $(4, -2)$

9–3 The Graph of a Linear Equation in Two Variables *Pages 340–343*

14. The graph of an equation is the set of points whose coordinates __?__ the equation.

15. The point (5, __?__) belongs to the graph of $x + 2y = 7$.

16. In a linear equation each term is a __?__ or a monomial of degree __?__.

17. The graph of a linear equation in two variables, each having the set of directed numbers as replacement set, is a __?__ __?__.

18. Graph $y = 2x - 4$ and $5x + 3y = 10$.

9–4 Slope of a Line *Pages 343–346*

19. The slope of a line measures its __?__.

20. Given two points on a line, the slope of the line is the ratio of the difference of their __?__ to the corresponding difference of their __?__.

21. The slope of the line containing $(5, 4)$ and $(7, -6)$ is __?__.

22. Every horizontal line has slope __?__.

23. A vertical line has __?__ slope.

24. When a line rises from left to right, its slope is a __?__ number.

25. A line whose slope is a __?__ number falls from left to right.

9–5 The Slope-Intercept Form of a Linear Equation *Pages 346–348*

26. The line with equation $y = mx + b$ has slope __?__ and y-intercept __?__.

27. The y-intercept of a line is the __?__ of the point where the line intersects the __?__ -axis.

28. The line $y - 4x + 5 = 0$ crosses the y-axis at (0, __?__); its slope is __?__.

29. An equation of the line with slope -7 and y-intercept 1 is __?__.

9-6 Determining the Equation of a Line *Pages 349–350*

30. If $y = 2x + b$ passes through the point $(-5, 1)$, then $b =$?.

31. Give an equation of the line through $(2, -6)$ with slope -4.

32. Give an equation of the line through $(-4, -2)$ and $(2, 1)$.

33. Give an equation of the line parallel to $4x - 2y = 1$ and passing through $(-1, -4)$.

9-7 Graph of an Inequality in Two Variables *Pages 350–353*

34. The graph of $y = 3x - 1$ divides the coordinate plane into two ?. Points above the line belong to the graph of y ? $3x - 1$; points below the line belong to the graph of y ? $3x - 1$.

35. To graph $y \leq 5 - 2x$, draw the graph of y ? $5 - 2x$. Then shade the region ? that line and show the boundary as a ? line.

36. To graph $5x + 3y > 6$, make ? the left member of the inequality.

37. Graph $5x + 3y > 6$ and $5x - 3y > 6$.

9-8 Graphs That Are Parabolas *Pages 353–354*

38. Calculate the values of y in this table if $y = -x^2$.

x	−3	−2	−1	0	1	2	3
y	−9	?	?	0	?	?	?

39. Graph $y = -x^2$ from the preceding table.

40. On the parabola $y = x^2 + 2$ name the coordinates of the point having the least ordinate.

41. On the parabola $y = 3 - x^2$ name the coordinates of the point having the greatest ordinate.

9-9 Broken-Line and Bar Graphs *Pages 354–357*

42. On a broken-line graph, the only significant points are those that have been ?.

43. Make a broken-line graph, and make a bar graph of the average price of admission to movies over forty years: 1919, 30¢; 1929, 75¢; 1939, 40¢; 1949, 75¢; 1959, 95¢.

9-10 Circle Graphs *Pages 357–358*

44. A circle graph is used to compare the ? of a whole with each other and with the ?.

45. The size of a sector of a circle depends on the size of the ? angle.

46. Make a circle graph of Jane's budget: savings, 20%; school expenses, 25%; personal needs, 20%; entertainment and gifts, 25%; charity, 10%.

Extra for Experts

The Sieve of Eratosthenes

Eratosthenes (air-a-**tos**-tha-neez) was a Greek scholar who lived about 200 B.C.; his sieve was not a kitchen utensil, but a scheme for identifying prime numbers. You, like Eratosthenes, will find it useful to be able to sift prime numbers from products; so, in order to learn how, write all the numbers from 1 to 100. (For convenience, arrange them in ten rows.)

Do you remember the definition of a prime — an integer greater than 1, divisible only by itself and 1? By definition, then, 2 is a prime number. But every second number after 2 is divisible by 2; so, beginning with 4, cross out every second number in your table of 100. The number 3 is divisible only by itself and 1; so leave it untouched. Every third number after 3 is divisible by 3. Some of these numbers, such as 6 and 12, already have been crossed out; cross out all the rest of them.

The number 4 is a product, and so is every fourth number thereafter. Go on to 5, which is a prime. Cross out every fifth number not already crossed out after it.

By continuing in this manner, you can exhaust the nonprimes for any given range of integers.

Questions

1. Construct a sieve of Eratosthenes for the integers from 1 to 200.

2. What is the largest prime number you need to consider as a factor before you eliminate all the nonprimes (a) from 1 to 100 (b) from 1 to 200? Explain your answer.

3. How many primes did you find between (a) 1 and 50, (b) 50 and 100, (c) 100 and 150, (d) 150 and 200?

4. Do you see any regular pattern in the distribution of prime numbers?

5. Are any even numbers prime? Explain.

The Power of Thought

"My advice," said the school principal to frail, young René Descartes, "is to lie in bed as late as you like each morning." Descartes acted on this advice, not only while he was a sickly boy in school, but all the rest of his life.

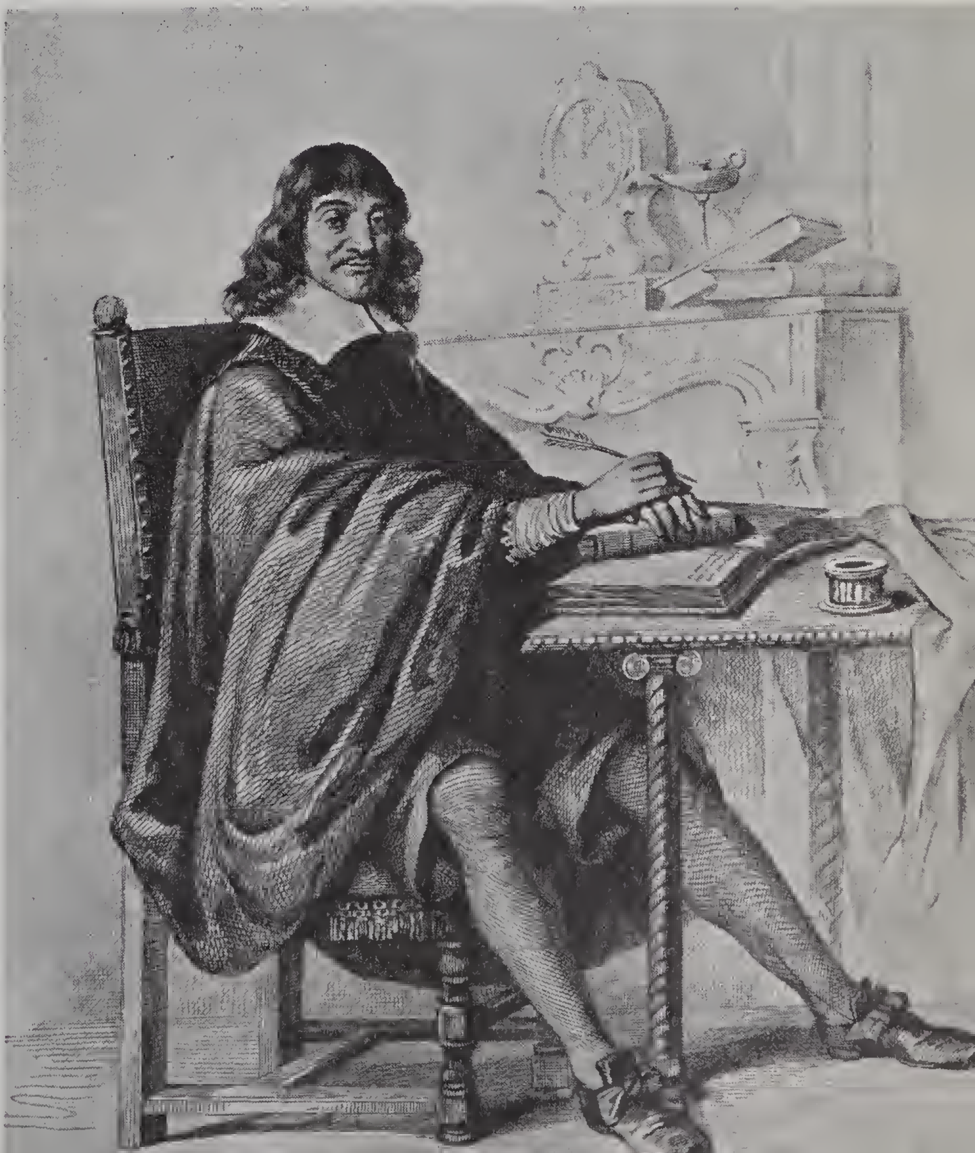
As a young man, Descartes drifted. After several years in Paris, he entered the Dutch army — a fashionable and not very strenuous career for a seventeenth-century French gentleman.

But in Holland something happened. One day, Descartes saw a poster in Flemish, a language he could not read. Curious, he asked a passer-by to translate it to him. And he received a reply like this: "The poster bears a challenge," said the man, who happened to be a college president. "It asks whether anyone can solve a certain problem in mathematics." He stopped. "I'll read you the problem on one condition. You must promise me you will try to solve it."

Descartes was intrigued. He promised to try. And he found a solution. It was work, and took time, but he enjoyed it.

When Descartes discovered the pleasure in study, he left the army, and for the rest of his life devoted himself to learning. He became a great scientist, a great philosopher, and a great mathematician. He still stayed in bed in the morning. There he was undisturbed, and he could use this time to think. He watched a fly crawl across the ceiling, and figured out how to describe its path by an equation. He thought of ways to apply algebra to geometry, and of how to apply geometry to algebra. He invented a branch of mathematics: coordinate geometry.

It is not where a person is that counts so much as what he does!



René Descartes after he had gained renown as scientist, philosopher, and mathematician.

CHAPTER 10

HOUR-BY-HOUR FORECAST
TIME
10:30
PRECINCTS REPORTING
21%
ELECTORAL VOTES
537
500
450
400
350
150
100
50
0
311
ELECTED
226
7:00 7:30 8:00 8:30 9:00 9:30 10:00 10:30 11:00 11:30 12:00

Sentences in Two Variables

Although computers solve many problems for scientific research, their use to predict presidential election results (top picture) is more familiar to you. From previous election patterns, statisticians develop prediction equations containing variables whose values are determined by the early returns. Substituting these values in the equations it has been given, the machine makes its predictions. As later returns come in (bottom picture), new values are substituted and revised predictions made. Without the machine, the calculation of one prediction would take longer than counting the ballots.

In this chapter, you will learn methods used by some digital computers for solving two equations containing two variables. However, the machines are capable of solving thirty-six equations in thirty-six variables in less time than it may take you to solve one of the problems here.

SOLVING SYSTEMS OF LINEAR OPEN SENTENCES

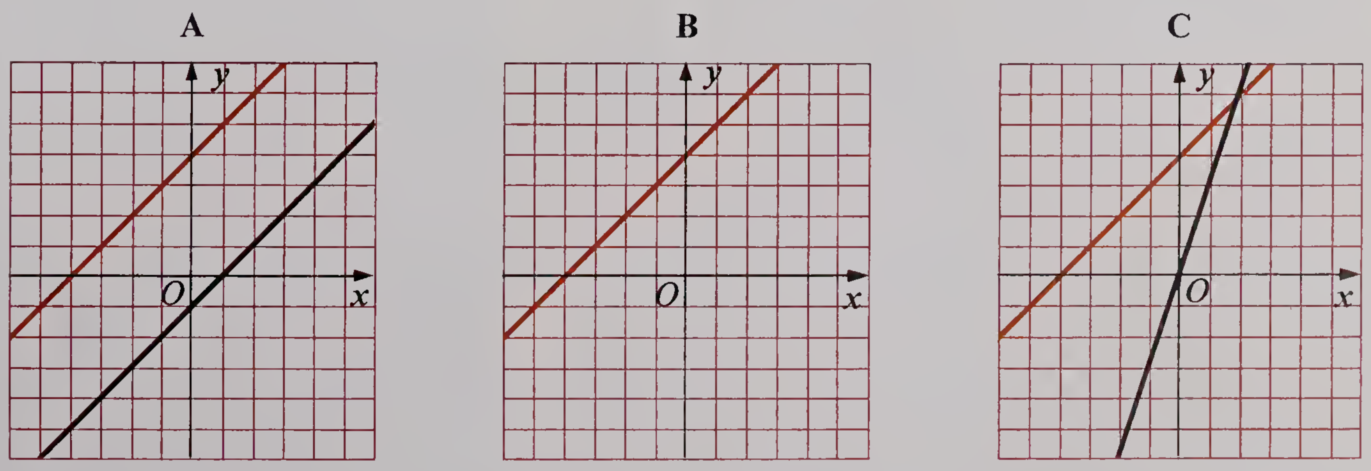
10–1 The Graphic Method

The graph of a linear equation in two variables is a straight line. When the graphs of two such equations in the same variables are drawn on the same axes, the resulting lines may have in common:

A. no point — the lines are *parallel;*

B. all their points — the lines *coincide;*

C. just one point — the lines *intersect.*



A: $y = x + 4$, $y = x - 1$

B: $y = x + 4$, $2y = 2x + 8$

C: $y = x + 4$, $y = 3x$

Figure 10–1

Because two equations impose two conditions on the variables at the same time, they are called a *system* of **simultaneous equations**. To solve such a system, you seek the ordered pairs of numbers that satisfy *both* equations of the system. Stress that a root (or solution) is an ordered pair of numbers.

The graphs in Figure 10–1 show that:

A. The system $\begin{aligned} y &= x + 4 \\ y &= x - 1 \end{aligned}$ has no solution; the graphs do not intersect.

B. The system $\begin{aligned} y &= x + 4 \\ 2y &= 2x + 8 \end{aligned}$ has an unlimited number of solutions; the graphs coincide.

C. The system $\begin{aligned} y &= x + 4 \\ y &= 3x \end{aligned}$ has just one solution, (2, 6); the graphs intersect at one point.

To understand why the equations of System **A** *cannot* have a common root, notice that if $y = x + 4$ and $y = x - 1$ were both true statements for some ordered pair (x, y), then by substitution

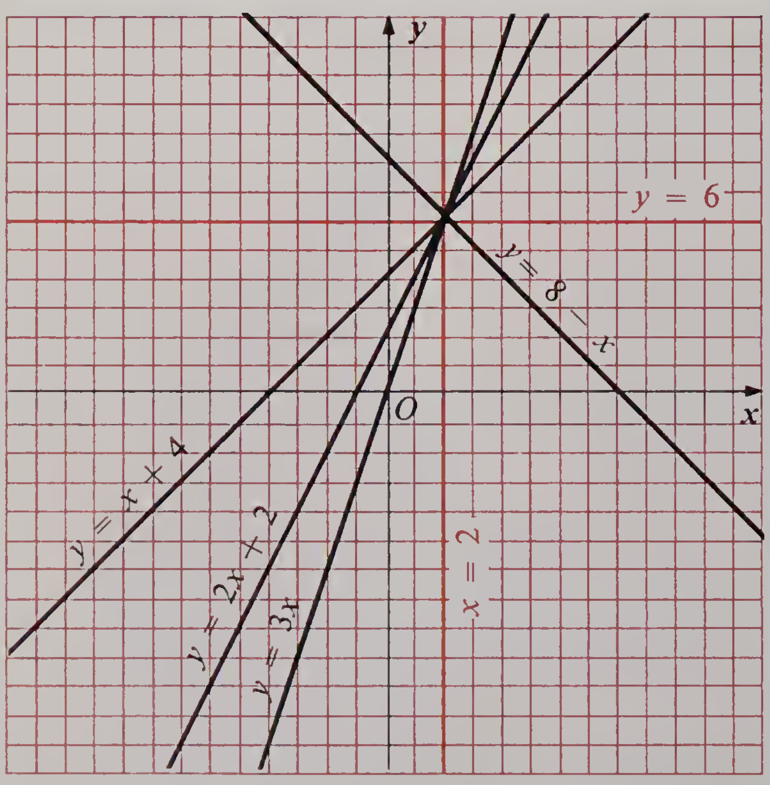
$$\begin{aligned} \boldsymbol{x + 4} &= \boldsymbol{x - 1}, \text{ and} \\ \mathbf{4} &= \mathbf{-1}, \text{ a false statement.} \end{aligned}$$

Simultaneous equations having no common root are called **inconsistent**.

Because the equations of Systems **B** and **C** do have common roots, they are called *consistent equations.* In System **B** the equations are equivalent: multiply each member of $y = x + 4$ by 2, and you obtain $2y = 2x + 8$. Such equivalent equations are also said to be **dependent**.

Neither equation of System **C** can be obtained from the other by multiplication. They are **independent equations** and have just one common root.

A pair of linear equations can be solved by graphing both on the same axes and determining the coordinates of the point of intersection.



Examine Figure 10–2. Do you see that (2, 6) is the common root of any pair of linear equations whose graphs pass through that point? In particular, the pair of red lines in the figure pass through it. One of the red lines is the horizontal line whose equation is $y = 6$, and the other is the vertical line $x = 2$.

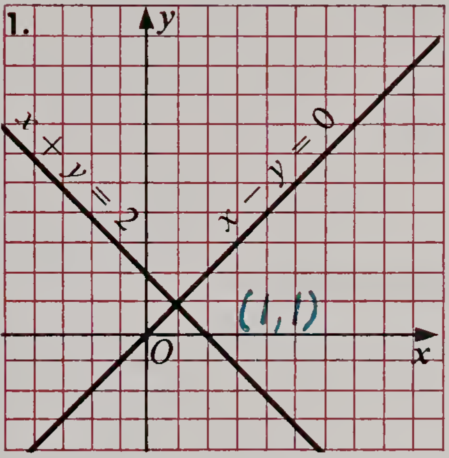
· *Figure 10–2* ·

Because the system of equations $x = 2$, $y = 6$ has the same solution set as the system $y = x + 4$, $y = 3x$ the systems are said to be *equivalent* to each other. Emphasize that this equivalent system contains two equations. See 10-1 T.M. pg. 33.

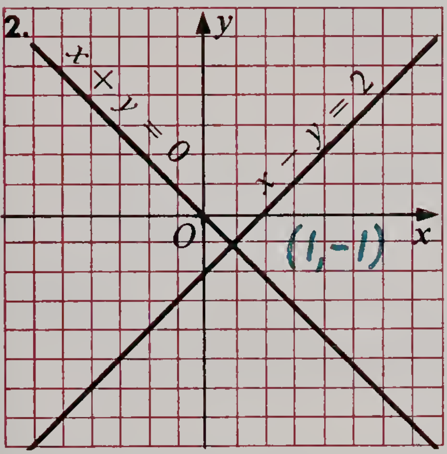
ORAL EXERCISES

Determine the coordinates of the point of intersection.

1.



2.



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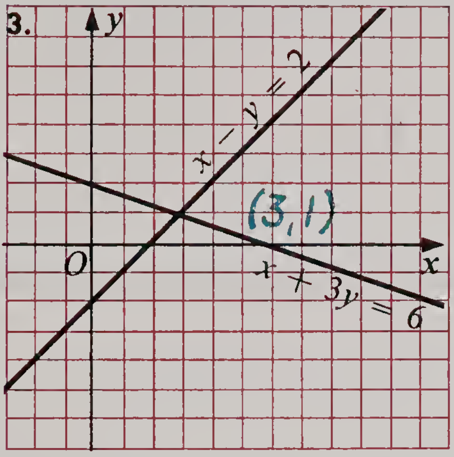

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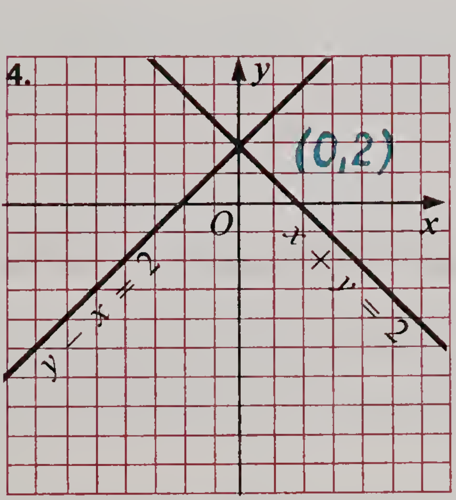

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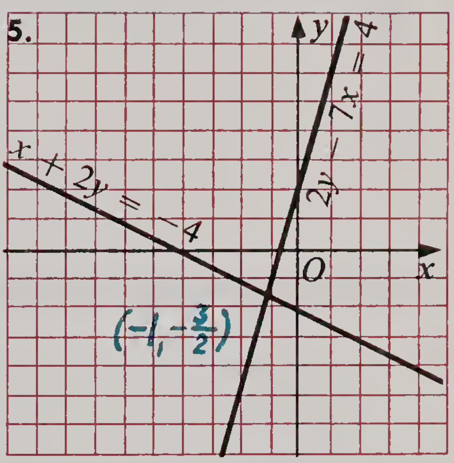

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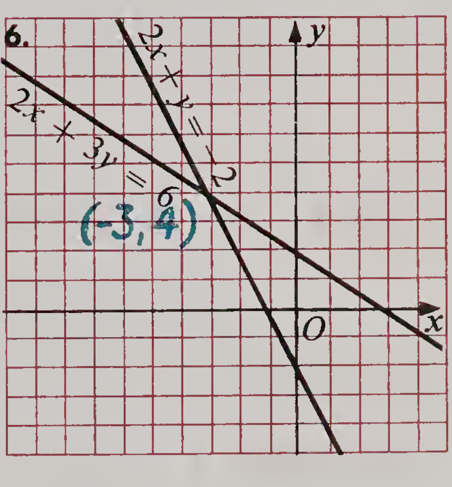


Do the equations in each pair have only one common root? Explain.

SAMPLE. $x + y = 1$
$x + y = 2$

What you say: No common root, because they are inconsistent; the sum of two numbers cannot be both 1 and 2.

C, Inc., D, Ind. stand for: equations are consistent, inconsistent, dependent, independent, respectively.

7. $x + y = 4$, $2x + y = 5$ Yes; C, Ind.
8. $2x + y = 5$, $2x + y = 8$ No; Inc.
9. $x - y = 3$, $2x - 2y = 6$ No; D
10. $x - y = 7$, $2x - y = 16$ Yes; C; Ind.
11. $x - 2y = 5$, $x - 2y = 10$ No; Inc.
12. $x + 2y = 13$, $2x + 4y = 26$ No; D
13. $x + y = 4$, $2x + 2y = 6$ No; Inc.
14. $3x - 6y = 3$, $4x - 8y = 4$ No; D
15. $2x - 4y = 10$, $3x - 6y = 15$ No; D

WRITTEN EXERCISES

Graph the equations. When the equations are consistent and independent, find their common root.

Ⓐ

1. $y = 3 - x$
$y = 1 + x$

2. $y = 5 - x$
$y = 3 + x$

3. $y = 2x + 4$
$x + y = 1$

4. $x - y = 1$
$y = 2x + 1$

5. $-x - y = 0$
$x + y = -4$

6. $x + 2y = 3$
$2x + 4y = 6$

7. $2y - 3x = 6$
$y = 3x$

8. $x + 2y = 5$
$x = 3y$

9. $x = 1 - y$
$y = -1 - x$

10. $x = y$
$3y = 0 + 3x$

11. $y = 2x - 4$
$6x + y = 0$

12. $y = \frac{3}{2}x + 4$
$y = -\frac{3}{2}x - 5$

Solve graphically, and estimate the answer to the nearest tenth.

Ⓑ

13. $2x - y = 7$
$x + 3y = 2$

14. $x + 2y = 6$
$y - 2x = 5$

15. $3x + 2y = 5$
$-6y - 9x = 7$

16. Where on the graph of $2x - 5y = 3$ is the abscissa twice the ordinate?

17. Find the area of a triangle whose vertices are determined by the graphs of $y = 2x + 4$, $x + y = 10$, and $y = -2$.

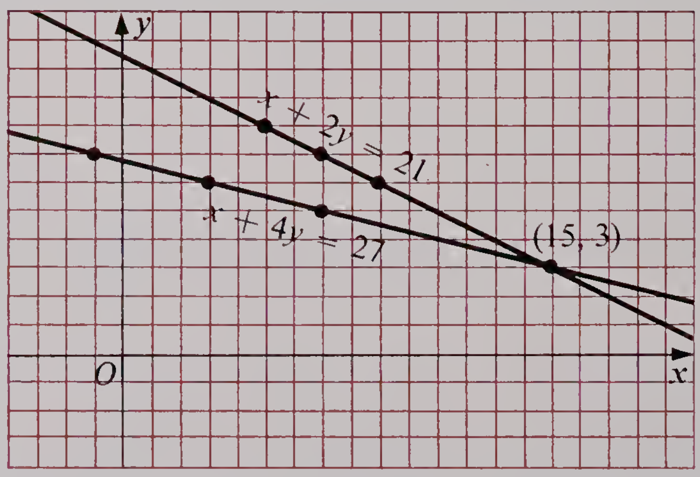
10–2 The Addition and Subtraction Method

This is also called elimination.

The algebraic method of solving simultaneous equations avoids inaccuracies which can occur in the drawing and reading of graphs.

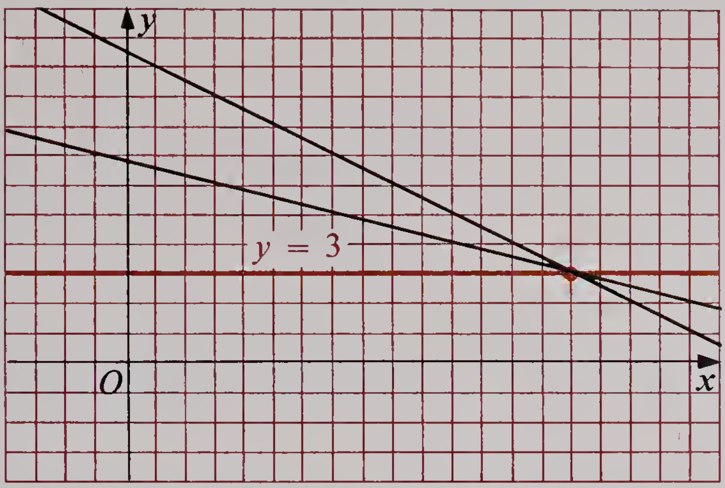
EXAMPLE 1. **Solve $x + 4y = 27$ and $x + 2y = 21$.**

Solution:



The algebraic solution provides a systematic method for finding the equivalent system made up of the equations of the horizontal and vertical lines through the point of intersection.

You can use the addition or subtraction property of equality to obtain such equations.



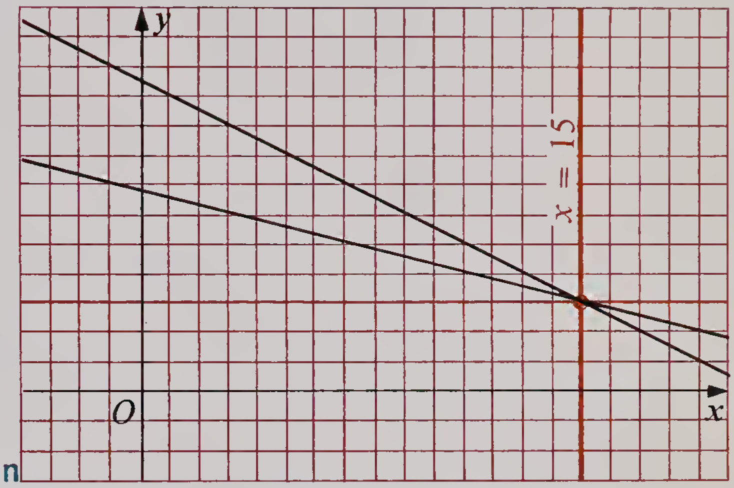
$$\begin{array}{rcl} x + 4y &=& 27 \\ x + 2y &=& 21 \\ \hline 2y &=& 6 \\ y &=& 3 \end{array}$$

(**horizontal** line through the intersection)

Now, substitute 3 for y in either of the original equations:

$$\begin{array}{rcl} x + 4y &=& 27 \\ x + 4(3) &=& 27 \\ x + 12 &=& 27 \\ x &=& 15 \end{array}$$

(**vertical** line through the intersection)



Be sure that at least your better students can explain why $x = 15$ and $y = 3$ is equivalent to the original system. See T.M. pg. 33 (1).

Check: Substitute 15 for x and 3 for y in both original equations.

$\therefore$ The solution set is $\{(15, 3)\}$, Answer.

Do not permit pupils to write $x = 15$, $y = 3$ as their answer. The root is an ordered pair of numbers, *not* a pair of equations.

EXAMPLE 2. Solve $5m + 2n = 20$ and $3m - 2n = 4$ simultaneously.

Solution:

$$\begin{array}{rcl} 5m + 2n &=& 20 \\ 3m - 2n &=& 4 \\ \hline 8m &=& 24 \\ m &=& 3 \end{array} \qquad \begin{array}{rcl} 5m + 2n &=& 20 \\ 5 \cdot 3 + 2n &=& 20 \\ 2n &=& 5 \\ n &=& \frac{5}{2} \end{array}$$

Check:

$$\begin{array}{rcl} 5m + 2n &=& 20 \\ 5(3) + 2(\frac{5}{2}) &\stackrel{?}{=}& 20 \\ 15 + 5 &\stackrel{?}{=}& 20 \\ 20 &=& 20 \ \checkmark \end{array} \qquad \begin{array}{rcl} 3m - 2n &=& 4 \\ 3(3) - 2(\frac{5}{2}) &\stackrel{?}{=}& 4 \\ 9 - 5 &\stackrel{?}{=}& 4 \\ 4 &=& 4 \ \checkmark \end{array}$$

$\therefore$ The solution set is $\{(3, \frac{5}{2})\}$, Answer.

WRITTEN EXERCISES

For use as oral exercises see T.M. pg. 34 (2).

Solve by addition or subtraction.

Ⓐ

1. $x + y = 74$
$x - y = 16$

2. $w + z = 122$
$w - z = 28$

3. $2r - s = 20$
$2r + s = 48$

4. $3m - n = 18$
$3m + n = 60$

5. $3p - q = 10$
$2p - q = 7$

6. $5A - B = 7$
$3A - B = 5$

7. $3s + 2t = 17$
$s + 2t = 5$

8. $4w - 5z = 10$
$2w + 5z = -10$

9. $5m - 3n = 19$
$2m + 3n = -5$

10. $2p - 5t = 12$
$2p - 3t = 12$

11. $3r - 4s = 1$
$3r - 2s = -1$

12. $2x - 3y = -4$
$4x - 3y = -6$

13. $2 = p - 3n$
$12 = p - n$

14. $6 = s - 3t$
$20 = s - t$

15. $3s - 5t = -38$
$3s + 5t = 2$

16. $11x - y = 42$
$3x + y = 0$

17. $3w + 8z = 158$
$7w + 8z = 230$

18. $2m + 11n = 265$
$5m + 11n = 316$

Clear the equations of fractions before adding or subtracting.

Ⓑ

19. $y + x = 8$
$\frac{1}{4}(x - y) = 1$

20. $2x - 3y = 16$
$\frac{1}{2}(3y + 2x) = -4$

21. $\frac{r}{5} - \frac{s}{3} = 0$
$\frac{s}{6} + \frac{r}{10} = -2$

22. $\frac{m}{2} - \frac{n}{3} = 3$
$\frac{n}{6} - \frac{m}{8} = -1$

23. $.7(r - c) = 7$
$.5(r + c) = 7$

24. $.75(r - c) = 3$
$.50(r + c) = 3$

25. $\frac{2p}{5} - \frac{3q}{10} = .1$
$\frac{3p}{2} - \frac{3q}{4} = .5$

26. $\frac{5u}{3} - v = 0$
$\frac{5u}{3} + 2v = -1$

10–3 Problems with Two Variables

Problems concerning two numbers can be solved by using one or two variables. A solution using two variables to form two open sentences is usually the more direct.

EXAMPLE A lending library has a fixed charge for the first three days and an additional charge for each day thereafter. Tom paid 27 cents for a book he kept seven days, while Sue paid 21 cents for one she kept five days. Find the fixed charge and the charge for each extra day.

Solution:

1	Choose two variables to represent the desired numbers.	Let x = the fixed charge and y = the extra daily charge.
2	Form two open sentences using the facts in the problem.	$x + 4y = 27$ (Ted's fee) $x + 2y = 21$ (Sue's fee)
3	Solve the equations. Same system as in 10-2, Example 1.	$$\begin{array}{rcl} x + 4y &=& 27 \\ x + 2y &=& 21 \\ \hline 2y &=& 6 \\ y &=& 3 \end{array} \qquad \begin{array}{rcl} x + 4(3) &=& 27 \\ x + 12 &=& 27 \\ x &=& 15 \end{array}$$
4	*Check* your results in the words of the problem.	Ted's fee: fixed charge of 15¢ plus 4 days at 3¢ a day is 27¢. √ Sue's fee: fixed charge of 15¢ plus 2 days at 3¢ a day is 21¢. √ Fixed charge is 15 cents Extra daily charge is 3 cents } Answer.

ORAL EXERCISES

Translate into a pair of equations with two variables.

1. One number is three times another. Their sum is 48. $x=3y$; $x+y=48$
2. The sum of two numbers is 42. One number is five times the other. $x+y=42$; $x=5y$
3. One number is three times the other. Their difference is 26. $x=3y$; $x-y=26$
4. One number is five times another. Their difference is 28. $x=5y$; $x-y=28$
5. A number is 8 more than another. Their sum is 164. $x=y+8$; $x+y=164$
6. A number is 7 less than another. Their sum is 75. $x=y-7$; $x+y=75$
7. One of two complementary angles exceeds the smaller by 18°. $x+y=90°$; $x-y=18°$
8. One of two supplementary angles exceeds twice the smaller by 36°. $x+y=180°$; $x-2y=36°$
9. The length of a rectangle is 4 times its width. Its perimeter is 280 cm.
10. A melon costs 25¢ more than half a pineapple, and both cost $1.60.

9. $l=4w$; $2l+2w=280$

10. $m-25=p/2$; $m+p=160$

PROBLEMS

Use two variables and two equations to solve each problem.

A

1. Half the perimeter of a house, which is 12 feet longer than it is wide, is 72 feet. What are the dimensions of the house?

2. Mr. Alden's farm is 82 acres larger than Mr. Bradford's. The two farms together contain 276 acres. How large is each?

3. The difference between twice one number and a smaller number is 21. The sum of the smaller and twice the larger is 27. Name the numbers.

4. If Mary were twice as old, she would be 19 years older than Ann, and their combined ages would be 41. How old are the girls?

5. If a baker orders three times as much whole-wheat flour as usual, he orders 11 pounds more of it than of white flour. But if he had ordered twice as much whole-wheat, he would have ordered 2 pounds less of it than of white. How much of each does he order?

6. Robert buys five magazines at the same price and a sixth at a higher price, paying 25¢ more for all five than for the sixth. If he buys only 3 of the cheaper magazines, they will cost, in all, a nickel less than the more expensive one. Find each price.

7. For $15.50, a grocer buys two grades of eggs, at 53¢ and 59¢ per dozen. If the cheaper eggs cost $2.52 more than the more costly, how many dozen of each does he buy?

8. Two styles of men's hats cost $40 and $52 a dozen. If an order totals $224 and the cheaper line costs $16 more than the costlier line, how many dozen hats of each style are ordered?

9. One catcher's mitt and 8 fielder's gloves cost $41.60. One catcher's mitt and 12 fielder's gloves cost $59. What is the cost of each?

10. After a sale a store had taken in $10.50 more on its $5.95 dresses than on its $3.95 dresses. The total sale was $287. How many dresses were sold at each price?

10–4 Multiplication in the Addition and Subtraction Method

Sometimes adding or subtracting the members will not eliminate either variable because the coefficients of corresponding terms do not have the same absolute value. You then can use the multiplication property of equality to make the coefficient of a variable in one equation equal to the corresponding coefficient in the other.

EXAMPLE 1. Solve. $3x + 4y = 45$
$6x - 2y = 30$

Solution:

1. Multiply both members of the first equation by 2. $6x + 8y = 90$
$6x - 2y = 30$
2. Subtract the equations, and solve for y. $10y = 60$
$y = 6$
3. Find the value of x by substituting 6 for y in one of the given equations. $3x + 4(6) = 45$
$3x = 21$
$x = 7$
4. *Check:* Substitute in original equations. (This is left to you.)

EXAMPLE 2. Solve $2x + 5y = 28$ and $3x + 2y = 31$.

Solution:

$$\begin{aligned} -2(2x + 5y) &= -2(28) &\rightarrow\quad -4x - 10y &= -56 \\ 5(3x + 2y) &= 5(31) &\rightarrow\quad 15x + 10y &= 155 \\ & & 11x &= 99 \\ & & x &= 9 \end{aligned}$$

For alternate procedure see 10-4 T.M. pg. 34.

Check: $2 \cdot 9 + 5 \cdot 2 \stackrel{?}{=} 28$ $\quad 3 \cdot 9 + 2 \cdot 2 \stackrel{?}{=} 31$ $\quad 3(9) + 2y = 31$
$28 = 28\ \checkmark$ $\quad 31 = 31\ \checkmark$ $\quad 2y = 4$
$y = 2$

∴ The solution set is $\{(9, 2)\}$, Answer. Ask better students to write an explanation of why the given system is equivalent to the system $x = 9$, $y = 2$.

ORAL EXERCISES

First equation is referred to as ①, second as ②.

By what number would you multiply one equation in each pair to eliminate one variable? Give the transformed equation.

1. $w + z = 10$ 2 × ①
$2w + 3z = 23$ or 3 × ①
2. $r + t = 7$ 4 × ①
$4r - 5t = 1$ or 5 × ①
3. $3m + 2n = 8$ 2 × ①
$6m - n = 11$ or 2 × ②
4. $m - 5n = 16$
$2m + 3n = 6$ 2 × ①
5. $5x - 7y = -5$
$2x - y = 7$ 7 × ②
6. $7x + 10y = 0$
$3x + y = 23$ 10 × ②
7. $7w - 2z = 3$ 4 × ①
$3w + 8z = -43$
8. $4p - 9q = 10$ 2 × ①
$8p - 15q = 26$
9. $2u - 5t = 6$ 3 × ①
$6u + 3t = -4$

By what number can you multiply each equation in the pair to eliminate a variable?

10. $6x - 5y = -18$
$4x + 3y = 7$
x: 2 × ①, 3 × ②. y: 3 × ①, 5 × ②
11. $12A + 5B = 31$
$8A + 7B = 50$
A: 2 × ①, 3 × ②. B: 7 × ①, 5 × ②
12. $2r - 9s = 0$
$3r - 6s = -5$
r: 3 × ①, 2 × ②. s: 2 × ①, 3 × ②

13. $9s - 14t = 8$
$6s - 12t = 0$

s: 2×①, 3×②. t: 6×①, 7×②

14. $4u - 7v = 8$
$5u + 9v = 81$

u: 5×①, 4×②. v: 9×①, 7×②

15. $5x + 4y = 22$
$3x + 6y = 24$

x: 3×①, 5×②. y: 3×①, 2×②

WRITTEN EXERCISES

Solve each pair of equations algebraically.

Ⓐ

1. $x - y = 73$
$2x - 7y = 21$

2. $m - n = 52$
$3m - 8n = 6$

3. $\frac{2m}{3} + 2n = 1$
$\frac{4m}{9} - \frac{n}{3} = \frac{1}{9}$

4. $\frac{c}{5} + \frac{d}{2} = -1$
$\frac{d}{4} - \frac{c}{3} = -\frac{1}{2}$

5. $2s - 3t = 1$
$3s - 4t = 7$

6. $2a + 5b = 18$
$3a + 4b = 27$

7. $c - 2d = 5$
$4d - 5c = -22$

8. $\frac{x}{4} - \frac{y}{6} = 0$
$\frac{3x}{8} + \frac{5y}{12} = -4$

9. $9q - 8r = 1$
$6q + 12r = 5$

10. $4c - d = -10$
$3c + 5d = 4$

11. $5p - 3w = 6$
$7w + p = -52$

12. $\frac{x}{2} + \frac{y}{3} = 2$
$\frac{x}{3} + \frac{y}{9} = 1$

Ⓑ

13. $\frac{2}{x} + \frac{1}{y} = 3$
$\frac{3}{x} - \frac{2}{y} = 8$

Hint: Rewrite: $2\left(\frac{1}{x}\right) + \frac{1}{y} = 3$
$3\left(\frac{1}{x}\right) - 2\left(\frac{1}{y}\right) = 8$

Let $a = \frac{1}{x}$ and $b = \frac{1}{y}$. Solve for a and b.

Then $x = \frac{1}{a}$ and $y = \frac{1}{b}$.

14. $\frac{3}{2x} - \frac{2}{y} = -7$
$\frac{9}{2x} + \frac{5}{y} = 12$

15. $\frac{4}{m} + \frac{3}{2n} = 0$
$\frac{2}{3m} + \frac{13}{4n} = 1$

16. $\frac{3}{2m} - \frac{3}{4n} = 1$
$\frac{4}{3m} - \frac{11}{9n} = 2$

17. $\frac{x + 2y}{3} - \frac{2x - y}{2} = -3$
$\frac{3x - 2y}{14} - \frac{3x + 2y}{4} = \frac{3}{4}$

18. $\frac{3x - 2y}{6} + \frac{9x + y}{2} = -1\frac{2}{3}$
$\frac{2x + 3y}{2} - \frac{7x - 4y}{3} = \frac{4}{9}$

Solve for x and y.

19. $2ax - 3by = 13ab$
$5ax + 2by = 4ab$

20. $3rx + 2sy = -19rs$
$4rx - 7sy = 23rs$

PROBLEMS

A few problems may be used for oral work to give practice in forming pairs of equations.

Solve, using two equations in two variables.

A **1.** Mr. Bates's rent and savings each month total \$100. If he saves \$5 more monthly, his savings will be half his rent. What is his rent?

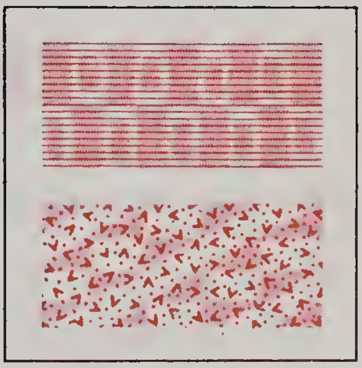
2. A dealer has 30 cars and trucks. When 2 more cars are delivered, he has 3 times as many cars as trucks. How many of each has he?

3. The ages of Eleanor's uncle and aunt total 68 years, and if her uncle's age were doubled the difference between their ages would be 40 years. How old is Eleanor's aunt?

4. The town's share of a \$10,000 legacy was \$4,000 less than three times a charity's share. Find the town's share.

5. A garden 45 meters square is in the center of a rectangular park 150 meters longer than it is wide. The perimeter of the park is 20 meters less than 4 times that of the garden. Find the park's dimensions.

6. A frame 30 inches square encloses two matching rectangular prints, each 4 inches longer than twice its width. The perimeter of the frame is 16 inches less than the combined perimeters of the prints. Find the dimensions of each print.



7. When 3 pads and 2 pencils cost 34¢, and 4 pads and 5 pencils cost 57¢, what is the price of a pad?

8. Alice buys 2 pounds of butter and 3 dozen eggs for \$2.90. She then buys an extra pound of butter and 2 dozen eggs, for \$1.71. Find the unit price of butter and eggs.

9. Mr. Kasner buys 3 golf balls and a soft drink for \$2.10. Later he buys one more golf ball and soft drinks for himself and his three friends for \$1.25. What is the cost of each ball and each drink?

10. The team buys 7 bats and 5 balls for \$16.95. Later, they buy 3 bats and 6 balls for \$13.05. What is the price of each item?

11. A man wishes to gain \$33 each year, so he invests \$600, part at 4% and the rest at 6%. How much does he invest at each rate?

12. Mr. Russo needs \$113. He invests a sum at 6% and another, \$500 more than the first, at 5%. Find the two sums.

10–5 The Substitution Method

You can solve either of a pair of equations for one variable in terms of the other, and use the substitution principle to obtain a third equation with only one variable. This method is sometimes easier to use than the addition and subtraction method.

See T.M. pg. 34 (3) for the logic of this method.

EXAMPLE. **Solve $x + 4y = 3$ and $2x - 3y = 17$.**

Solution:

1. **Solve for x in the first equation.**

$$\begin{aligned} x + 4y &= 3 \\ x &= 3 - 4y \end{aligned}$$

2. **Substitute this expression for x in the other equation.**

$$\begin{aligned} 2x - 3y &= 17 \\ 2(3 - 4y) - 3y &= 17 \end{aligned}$$

3. **Solve for y.**

$$\begin{aligned} 6 - 8y - 3y &= 17 \\ -11y &= 11 \\ y &= -1 \end{aligned}$$

Solving for x and checking are left to you.

The solution set is $\{(7, -1)\}$, Answer.

The check is essential. See T.M. pg. 34 (4).

WRITTEN EXERCISES

Solve each pair of equations by substitution.

A

1. $x + 2y = 5$
$x = 3y$

2. $y = 5x$
$2x + y = 7$

3. $2r - s = 7$
$r - s = 1$

4. $3r - s = 20$
$r - s = 2$

5. $2r + 3s = 19$
$r - s = 12$

6. $3r + 2s = 5$
$r - s = 5$

7. $2m - 3n = 5$
$3m - n = 18$

8. $3m - 2n = 11$
$2m - n = 8$

9. $3x + 2y = 33$
$y - 2x = -8$

10. $2x + 3y = 22$
$y - 2x = -6$

11. $2r - 3s = 0$
$3r + 3s = 33$

12. $3r - 4s = 0$
$2r + s = 33$

B

13. $w + z = 9$
$w - \frac{z}{3} = 5$

14. $\frac{1}{2}(r - c) = 4$
$\frac{1}{3}(r + c) = 4$

15. $s - t = 8$
$\frac{s}{3} - \frac{t}{2} = 1$

16. $x - y = 2$
$\frac{1}{2}(x + y) = 3$

17. $w + z = 9$
$w - \frac{z}{4} = 4$

18. $\frac{1}{3}(r - c) = 2$
$\frac{1}{4}(r + c) = 2$

PROBLEMS

Solve, using two equations in two variables and the substitution method.

A

1. Half the sum of two numbers is $-\frac{1}{2}$. Half their difference is $\frac{3}{2}$. Find the two numbers.
2. Two men traveled toward each other from points 300 miles apart. When they met, Mr. Wright had traveled 12 miles less than twice as far as Mr. Black. How far had each traveled?
3. Frank and James walked in opposite directions. When they were 9 blocks apart, Frank had gone 1.5 blocks less than twice the distance James had gone. Find the distance traveled by each.
4. The senior play cost students 35¢ and adults 60¢. Five hundred tickets were sold for \$254.50. How many adults attended?
5. Mr. Simpson invested \$3000, some at $3\frac{1}{2}\%$, the rest at 6% per year. The return from the $3\frac{1}{2}\%$ investment exceeded that from the 6% investment by \$10. How much was invested at each rate?
6. How fast does each operate if two sorters process 1200 cards in one minute, or 1200 cards in two minutes when one breaks down after operating one-half minute?

B

7. The average of two numbers is $\frac{5}{48}$. One-fourth of their difference is $\frac{1}{96}$. Find the two numbers.
8. One sum at 6% and another at 4% yield \$57. Were the investments interchanged, their income would increase by \$6. Name the sums.
9. A 6% investment over some years yields an income \$150 in excess of itself. If invested at 5% for half the time, the income is \$375 less than the investment. How large is the investment?
10. A grocer blends teas worth 66¢ and 48¢ a pound. If he interchanges the amounts, he saves \$6 in a blend of 100 pounds. Find the ratio of the weight of the two teas in the original blend.

10–6 Graphs of Pairs of Linear Inequalities (Optional)

See the second transparency.

Not only can graphs be used to solve systems of equations; they can also be employed in determining the solution set of simultaneous inequalities such as:

$$y > 3x - 2$$
$$x + y < 6$$

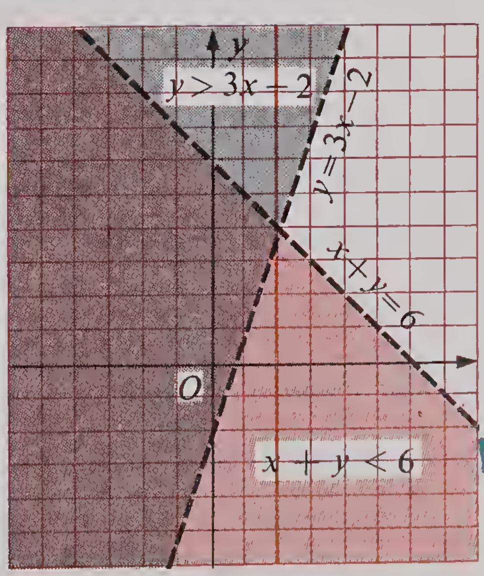


Figure 10–3

First draw the graphs of $y = 3x - 2$ and $x + y = 6$ (Figure 10–3) on the same axes. Just as the solution set of $y > 3x - 2$

Since the lines y = 3x -2 and x + y =6 are not included, they are shown as dashed lines.

consists of all points in the half-plane above the graph of $y = 3x - 2$, the solution set of $x + y < 6$ consists of all points in the half-plane below the graph of $x + y = 6$. The *intersection* (common points) of the two half-planes (black–red) represents the intersection of the two solution sets and contains all, and only those, points satisfying both inequalities. Some points in the common solution set are $(1, 4)$, $(-1, 0)$, and $(-2, -1)$.

WRITTEN EXERCISES

Graph each pair of inequalities, indicating their solution set with cross-hatching.

A

1. $y < 2x$, $x > 1$

2. $y > 2x$, $x > 1$

3. $y \geq 3x$, $y \leq 3$

4. $y \geq x$, $y \geq -1$

5. $y < x - 1$, $y > 1 - x$

6. $y < 2x + 1$, $y > x - 2$

7. $y \geq 3x - 6$, $y \leq 2x + 4$

8. $y \leq 3x - 3$, $y \geq 2x + 2$

B

9. $x \geq 1$, $x \leq 3$

10. $y - 3 \leq 0$, $y + 1 > 0$

11. $x + 2y \geq 2$, $x + 2y \leq 6$

12. $2x - y \leq 2$, $2x - y \geq 0$

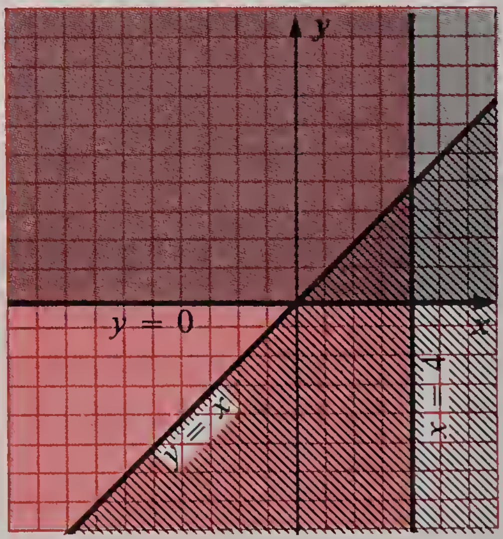
13. $x + 2y \geq 4$, $2x + 4y \leq 8$

14. $y \geq 3x - 6$, $2y \leq 6x - 12$

Graph each inequality in each simultaneous system. Show the solution set of each system as points in a three-way shaded region.

SAMPLE 1. $y \leq x, y \geq 0, x \leq 4$

Solution:



a. The solution set of $y \leq x$ consists of points on $y = x$ and in the diagonally shaded region below it.

b. The solution set of $y \geq 0$ consists of points on the x-axis and in the entire region above it.

c. The solution set of $x \leq 4$ consists of points on the line $x = 4$ and in the entire region to the left of it.

d. The intersection of these three sets is the three-way shaded region (small triangle), including points on its boundaries as well as in its interior.

Ⓒ **15.** $y \geq x$
$x \geq 0$
$y \leq 4$

16. $x + y < 0$
$x > -2$
$y > -2$

17. $x + 3y \leq 6$
$x \geq 0$
$y \geq 0$

18. $y < x - 1$
$y < 3 - x$
$y > -1$

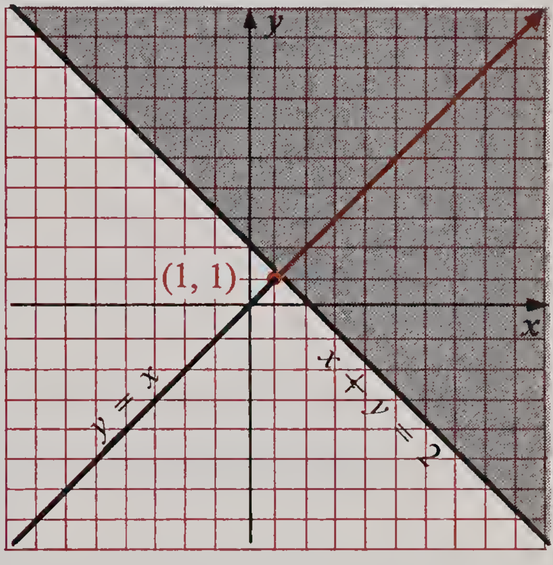
19. $y > 2x - 1$
$x + y > 2$
$x > 0$

20. $y \leq 2x - 3$
$x > 0$
$y \geq 0$

Solve each pair graphically, and check by solving algebraically.

SAMPLE 2. $y = x$
$x + y \geq 2$

Solution: **a.** The heavy red line, including (1, 1), is the graph of the solution set, **Answer**.



b. Substitute x for y in

$x + y \geq 2$: $x + x \geq 2$

Solve for x: $2x \geq 2$

$x \geq 1$

Since $y = x$: $y \geq 1$

$\therefore\ x \geq 1$ and $y \geq 1$

21. $x + y = 0$
$y < x + 2$

22. $x - 2y = 2$
$x + 2y \geq 6$

23. $y = 2x$
$x < 0$

24. $2x + 3y + 6 = 0$
$y \geq 0$

ADDITIONAL PROBLEMS

10–7 Digit Problems

See 10-7 T.M. pg. 35.

The digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 not only differ in value among themselves, but each one represents different values in different positions within a numeral ($76 = 7 \cdot 10 + 6 \cdot 1$; $67 = 6 \cdot 10 + 7 \cdot 1$).

All two-digit *decimal numerals* have the same form: $76 = 7 \cdot 10 + 6$, $67 = 6 \cdot 10 + 7$. In general, $10t + u$, where t is the tens digit and u is the ones (units) digit, and $t \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $u \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. If you wish to represent a number with the same digits in reverse order, you write $10u + t$. In either case, the sum of the digits is represented by $t + u$.

EXAMPLE. **The sum of the digits in a two-digit numeral is 10. When the digits are interchanged, the number designated is 18 more than the original number. Find the original number.**

Solution:

1 Let t = the tens digit in the original numeral
and u = the units digit in the original numeral.

Then $10t + u$ = the original number
and $10u + t$ = the new number.

2 The sum of the digits is 10. $t + u = 10$

The new number is 18 more than the original. $\}$ $10u + t = 18 + 10t + u$

$$(10u + t) - (10t + u) = 18$$
$$9u - 9t = 18$$

3

$$\begin{array}{rcll} u + t &=& 10 & \\ u - t &=& 2 & \longleftarrow \quad u - t = 2 \\ \hline 2u &=& 12 & \longrightarrow 6 - t = 2 \\ u &=& 6 & \qquad\quad t = 4 \end{array}$$

4 Is the sum of the digits 10? $4 + 6 = 10$ ✓

Is 64 eighteen more than 46? $64 - 46 = 18$ ✓

$\therefore$ The number is 46, Answer.

PROBLEMS

A

1. The sum of the digits of a two-digit numeral is 13. If 27 is added to the number, the result is the number with its digits interchanged. Find the original number.
2. The sum of the digits of a two-digit numeral is 12. If the order of the digits is reversed, the result names a number exceeding the original by 36. Find the original number.
3. The sum of the digits of a two-digit numeral is 9. The value of the number is 12 times the tens digit. Find the number.
4. The sum of the digits of a two-digit numeral is 12. The number with its digits interchanged is 13 times the original units digit. Find the original number.
5. The units digit of a two-digit numeral exceeds twice the tens digit by 1. The sum of the digits is 7. Find the number.
6. The units digit of a two-digit numeral is twice the tens digit. The sum of the digits is 12. Find the number.

7. The sum of the digits of a two-digit numeral is 8. The number with the digits interchanged is 7 times the tens digit of the original number. Find the original number.

8. The sum of the digits of a two-digit numeral is 6. The number with the digits in reverse order is 12 times the original units digit. Find the original number.

9. A teller mistakenly reversed the two digits in the face amount of a check, overpaying $9. If the sum of the digits was 9, determine the amount for which the check was drawn.

10. Find a number less than 100 whose tens digit exceeds twice its units digit by 1 and whose digits in reverse order give a number 4 more than 3 times their sum.

Ⓑ 11. Find a three-digit number whose tens digit is 3 times its hundreds digit and twice its units digit, and whose digits total 11.

12. A three-digit number is 198 more than itself reversed. The hundreds digit is 3 times the tens digit, and the sum of the digits is 19. Find the original number.

13. If a two-digit number is divided by its tens digit, the quotient is 11 and the remainder is 4. If the number with its digits interchanged is divided by its original units digit, the quotient is 10 and the remainder is 5. Find the original number.

14. Show that the difference between a three-digit number and the number with the order of the digits reversed is always divisible by 99.

Problems 15–18 refer to two-place decimal fractions between 0 and 1.

Ⓒ 15. The sum of the digits of a two-place fraction is 9. When its digits are reversed, the new fraction exceeds the original by .09. Find the original fraction.

16. When the digits of a two-place fraction are reversed, the new fraction is $\frac{4}{7}$ the original fraction. If the sum of the digits is 9, find the original fraction.

17. The sum of the digits of a two-place fraction is 13. The fraction with its digits reversed is .04 less than twice itself. Find the original fraction.

18. The tenths digit of a two-place fraction exceeds twice the hundredths digit by 1. If the digits are reversed, the original is .01 less than twice the new fraction. Find the original fraction.

10–8 Motion Problems

You can solve some motion problems conveniently by using two equations with two variables.

EXAMPLE. **With a tail wind, a plane flew 180 miles in half an hour. With no change in the wind, the return trip took 40 minutes. Find the speed of the wind and the plane's rate in still air.**

Solution: See 10-8 T.M. pg. 35.

1 **Let s = the rate, in m.p.h., of the plane in still air**

and v = the speed, in m.p.h., of the wind.

2

	r	$\cdot\ t\ =$	d
With tail wind	$s + v$	$\frac{1}{2}$	180
Against head wind	$s - v$	$\frac{2}{3}$	180

→ $\frac{1}{2}(s + v) = 180$

→ $\frac{2}{3}(s - v) = 180$

Note the change of units.

3 **Solve the equations.**

4 **Check the roots.**

The rate of the plane is 315 miles per hour.
The speed of the wind is 45 miles per hour. } **Answer.**

PROBLEMS

Ⓐ **1.** A motorboat covers 6 miles in 45 minutes. The return trip takes $1\frac{1}{2}$ hours. Find the boat's speed in still water.

2. A cyclist rode 1 mile in 3 minutes with the wind, and returned in 4 minutes against the wind. Find his speed without a wind.

3. Mr. Hanson flew 450 miles against the wind in $1\frac{1}{2}$ hours. The return trip took $1\frac{1}{4}$ hours, with no wind change. What was the speed of the wind?

4. It required $1\frac{1}{2}$ hours for a 540-mile plane trip and $1\frac{4}{5}$ hours for the return, in unchanged wind. What would have been the speed of the plane without wind?

5. Larry took 36 minutes to row 3 miles. When he returned, he took 90 minutes. What was the river's current?

6. A canoeist paddles 6 miles downstream in 40 minutes and returns in 3 hours. At the same rate, how fast does he go in still water?

Ⓑ **7.** A man rows 4 miles upstream and back in $2\frac{2}{3}$ hours. He rows 1 mile against the current in the time he rows 3 miles with it. At what rate does he row? What was his average rate of travel?

8. A cyclist takes $3\frac{1}{2}$ hours on a 30-mile round trip. On the return trip against the wind he did 2 miles in the time that he did 5 miles on the trip out. Find his average rate. What was the wind's speed?

9. A round-trip flight of 1105 miles takes $7\frac{1}{2}$ hours. The part of the flight with the wind takes one hour less than the other half of the trip. Find the speed of the plane in still air and the speed of the wind.

10. A steamer sails a distance up a river in the time it sails twice that distance downstream. If the speed of the steamer is s and that of the current is c, find the relationship between s and c.

10–9 Age Problems

You can simplify the solution of age problems by using two variables and by organizing the facts in chart form.

EXAMPLE. **Four years ago, Polly was $\frac{2}{3}$ as old as Paul. Four years from now, she will be $\frac{4}{5}$ as old as he. How old are both now?**

Solution:

Time	Polly	Paul	
4 years ago	$x - 4$	$y - 4$	→ $x - 4 = \frac{2}{3}(y - 4)$
This year	x	y	
4 years hence	$x + 4$	$y + 4$	→ $x + 4 = \frac{4}{5}(y + 4)$

Solve these equations and check.

$$\left.\begin{array}{l}\textbf{Polly is 12 years old.}\\ \textbf{Paul is 16 years old.}\end{array}\right\}\ \textbf{Answer.}$$

PROBLEMS

A 1. A man is 5 times as old as his son. In five years he will be 3 times as old as his son will be. How old is the son now?

2. Ruth's father is 7 times as old as Ruth is. One year ago he was 9 times as old as Ruth was. Find Ruth's present age.

3. Three years ago, Joe's age was 1 year more than twice Jill's. Six years from now, his will be 10 years more than half her age. How old is Joe?

4. Five years ago, Jerry was $\frac{2}{3}$ as old as Jeff. Ten years from now, he will be $\frac{5}{6}$ as old as Jeff. How old is each now?

5. Janet is $\frac{4}{5}$ as old as Phil. Four years ago she was $\frac{3}{4}$ as old as he. How old is each?

6. A man said, "My son is twice as old as my daughter. My wife is 3 times as old as the combined ages of both, and I am as old as my wife and son together. My mother, who is as old as all of us together, is 69." How old is the son?

B **7.** Ray said, "If I were $\frac{3}{5}$ as old as I am and Joyce were $\frac{3}{4}$ as old as she is, we would be 3 years older together than I am alone. But if I were $\frac{2}{5}$ as old as I am and Joyce were $\frac{1}{2}$ as old as she is, together we would be 3 years younger than I am alone." How old is Ray?

8. Mary is twice as old as Jane was when Mary was as old as Jane is now. Find the relationship between Mary's present age (M) and Jane's (J).

9. Mary is twice as old as Jane was when Mary was as old as Jane is now. In 3 years Mary will be 3 times as old as Jane was 4 years ago. Find their present ages.

10. Bob is twice as old as his brother will be when Bob is 8 times as old as his brother is now. Find the relationship between Bob's present age (B) and his brother's (b).

10–10 Problems about Fractions

Among the problems you can solve by using two variables are those about fractions, like this one:

EXAMPLE. **A fraction has a value of $\frac{4}{5}$. If the numerator is increased by 10 and the denominator is decreased by 13, the resulting fraction is equal to twice the reciprocal of the original fraction. Find the original fraction.**

Solution:

1 **Let $\frac{n}{d}$ = the original fraction.**

2 $$\frac{n}{d} = \frac{4}{5} \qquad \text{and} \qquad \frac{n+10}{d-13} = 2\cdot\frac{5}{4} \quad \text{or} \quad \frac{n+10}{d-13} = \frac{5}{2}$$

3 $$\begin{array}{rcl@{\qquad\qquad}rcl}
5d\cdot\frac{n}{d} &=& 5d\cdot\frac{4}{5} & 2(d-13)\cdot\frac{n+10}{d-13} &=& 2(d-13)\cdot\frac{5}{2} \\
5n &=& 4d & 2(n+10) &=& 5(d-13) \\
n &=& \frac{4d}{5} & 2n+20 &=& 5d-65 \\
 & & & 2\left(\frac{4d}{5}\right)+20 &=& 5d-65 \\
n &=& \frac{4(25)}{5} & 8d+100 &=& 25d-325 \\
 & & & -17d &=& -425 \\
n &=& 20 & d &=& 25
\end{array}$$

4 Check is left to you.

$\therefore$ **The original fraction is $\frac{20}{25}$, Answer.**

PROBLEMS

Using two variables, find the original fraction.

Ⓐ **1.** The denominator is 3 more than the numerator. If each is increased by 1, the value of the resulting fraction is $\frac{2}{3}$.

2. The denominator is 5 more than the numerator. If 1 is added to each, the value of the resulting fraction is $\frac{1}{2}$.

3. The denominator exceeds the numerator by 3. If 1 is subtracted from the numerator, and the denominator is unchanged, the resulting fraction has a value of $\frac{1}{2}$.

4. The denominator exceeds the numerator by 5. If 1 is subtracted from the numerator, a fraction is obtained whose value is $\frac{1}{3}$.

5. A fraction has a value of $\frac{3}{4}$. When 7 is added to its numerator, the resulting fraction equals the reciprocal of the original fraction.

6. A fraction's value is $\frac{2}{3}$. When its numerator is increased by 10, the new fraction equals the reciprocal of the value of the original fraction.

Ⓑ **7.** The two digits in the numerator of a fraction whose value is $\frac{4}{7}$ are reversed in its denominator. The reciprocal of the fraction is the value of the fraction obtained when 11 is added to the original numerator and 22 is subtracted from the original denominator.

8. The numerator equals the sum of the two digits in the denominator. The value of the fraction is $\frac{1}{7}$. When both numerator and denominator are increased by 3, the resulting fraction has a value of $\frac{1}{4}$.

9. The two digits in the numerator of a fraction are reversed in its denominator. If 11 is added to the numerator and 7 to the denominator, the value of the resulting fraction is $\frac{1}{2}$. The fraction whose numerator is the sum and whose denominator is the difference of the units and tens digits equals 2.

10. The numerator is a three-digit number whose hundreds digit is 2. The denominator is the number with the digits reversed. If 16 is added to the denominator, the value of the fraction is $\frac{1}{2}$. If 111 is subtracted from the numerator, the resulting fraction equals $\frac{1}{4}$.

List all possible members of each solution set.

Ⓒ **11.** The numerator is a two-digit number and the denominator is that number with the digits reversed. The value of the fraction is $\frac{4}{7}$.

12. The numerator of a fraction whose value is $\frac{34}{67}$ is a three-digit number whose tens digit is 0. The denominator contains the same digits in reverse order.

THE HUMAN
EQUATION

Burned Books and Buried Scholars

You have heard of the Great Wall of China. Have you heard of the emperor who used it as a concentration camp for scholars? His name was Shih Huang Ti, and he came to the throne in 221 B.C.

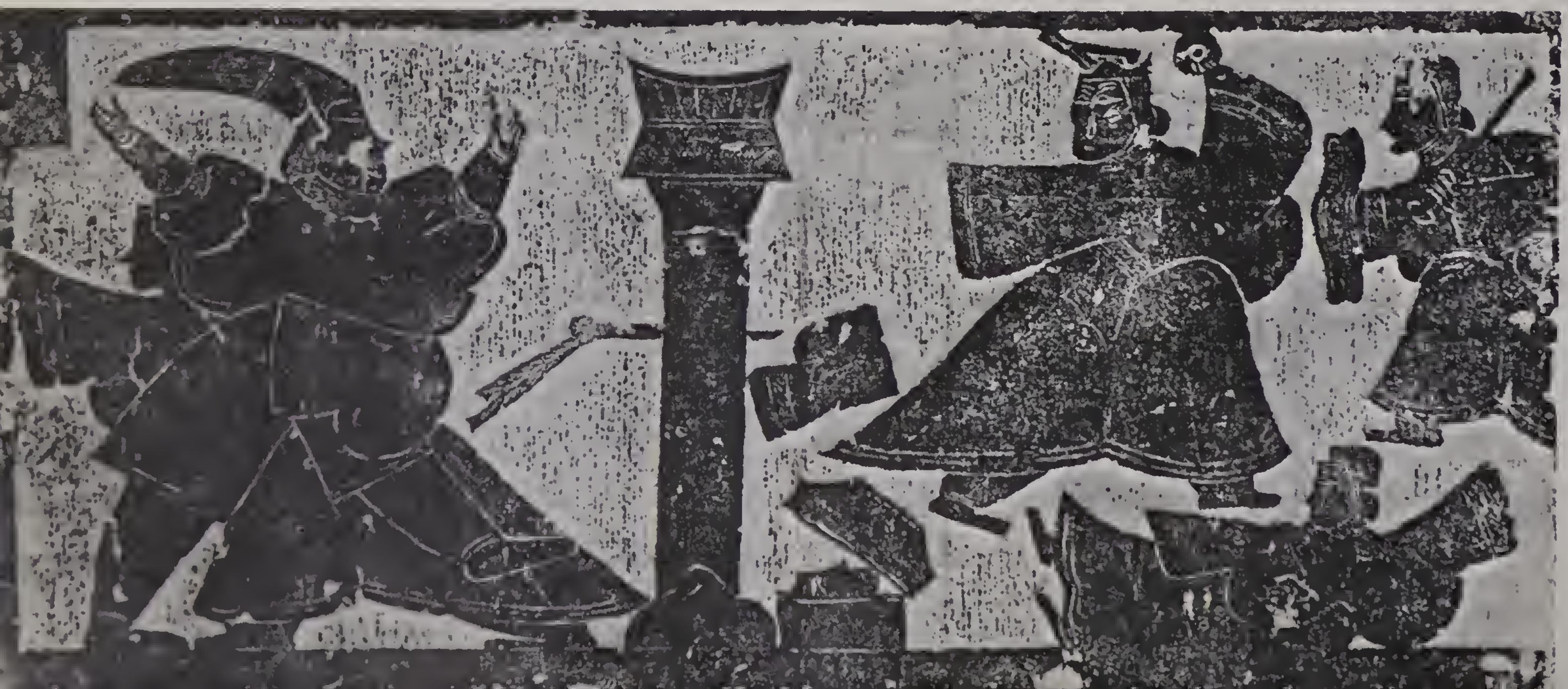
Shih Huang Ti had delusions of grandeur. He was determined to be remembered as the greatest of all emperors. Among other things, he wished to be famous as the ruler in whose reign knowledge increased beyond measure.

He took a strange way to get this last wish. He ordered all books on certain topics — including mathematics and related subjects — to be burned! Apparently he reasoned this way: "If in years to come, there is no mathematics book in all China that was written before my reign, but many books written during my reign, then people will think that mathematics began with me."

Shih Huang Ti knew that scholars did not willingly burn books; so he fixed a penalty for failure to obey his order: branding and four years hard labor on the Great Wall. Even so, 460 scholars banded together to defy the emperor. But Shih Huang Ti was more powerful than they: he had them buried alive.

So the books were burned. And the emperor called for new books. Of course, new books were written. Mathematicians who were neither slaving on the Wall nor buried in the sod worked feverishly to record their knowledge for the use of future generations. One of the books they rewrote is called *Arithmetic in Nine Sections*. There is reason to believe that this book originally was written before 1000 B.C. It is more algebra than arithmetic; it includes many topics that you will study this year and some that are more advanced than a first course in algebra.

This impression from a wall carving shows an attempt to assassinate Shih Huang Ti, the emperor who tried to advance knowledge by destroying books.



Chapter Summary

Inventory of Structure and Method

1. The graphs of **consistent** and **independent equations** intersect in one point, which can be determined by graphic or algebraic methods. To solve such a **pair of simultaneous linear equations,** graph them and find the point of intersection of the lines.
2. When the coefficients of one variable have the same absolute value, use the **addition** or **subtraction property of equality** to eliminate that variable; then solve for the other variable. When the coefficients of both variables have different absolute values in the two equations, use the **multiplication property of equality** before adding or subtracting.
3. A pair of simultaneous linear equations in the same variables can be solved by applying the **substitution principle.**
4. To solve a pair of linear inequalities, graph both on one set of axes; the intersection of their regions contains all points which satisfy both.
5. By using two variables to form two equations, you can solve **digit** problems, **motion** problems, **age** problems, and problems about **fractions.**

Vocabulary and Spelling

intersection (*p. 367*)
simultaneous equations (*p. 368*)
inconsistent equations (*p. 368*)
consistent equations (*p. 368*)
dependent equations (*p. 368*)
independent equations (*p. 368*)
addition and subtraction method (*p. 370*)
substitution method (*p. 378*)
decimal numeral (*p. 381*)

Chapter Test

10–1 **1.** Solve $3x + y = 10$ and $x + 2y = 0$ graphically.

10–2 Use addition or subtraction to solve each pair of equations.

2. $x + y = 37$
$x - y = 8$

3. $5r - 4w = 14$
$5r + 2w = 8$

4. $2s + 3t = 122$
$2s + t = 78$

10–3 **5.** Marie scored 40 on a test, receiving 3 points for each right answer and losing 1 point for each wrong answer. Had 4 points been awarded for each correct answer and 2 points been deducted for each incorrect answer, Marie would have scored 50. How many questions were in the test?

Solve by using multiplication with addition or subtraction.

10-4 **6.** $x - y = 26$; $3x - 8y = 3$ **7.** $8w + 5z = 31$; $3w - 2z = 0$

10-5 Solve by substitution.

8. $3m - 2n = 1$; $m - n = 2$ **10.** $r + 2s = 30$; $2r - s = -45$

9. $2r + s = 3$; $3r - 2s = 57$

10-6 **11.** Solve $y < 2x + 3$ and $x + y > 3$ graphically.

10-7 **12.** The sum of the two digits of a number is 15. The digits are reversed when 9 is added to the number. Find the number.

10-8 **13.** Walter's boat goes 48 miles upstream in 4 hours and returns in 3 hours. How fast is the current?

10-9 **14.** Henry is 20 years older than George. In 5 years George will be half as old as Henry is then. Use two variables to find their present ages.

10-10 **15.** If 3 is added to the numerator and denominator of a fraction with a value of $\frac{3}{4}$, the resulting fraction equals $\frac{4}{5}$. Use two variables to find the original fraction.

Chapter Review

10-1 The Graphic Method *Pages 367–370*

1. In solving a pair of linear equations having the same variables, both graphs are drawn on the _?_ _?_.

2. The point at which the two lines cross represents the _?_ _?_ which satisfies both equations.

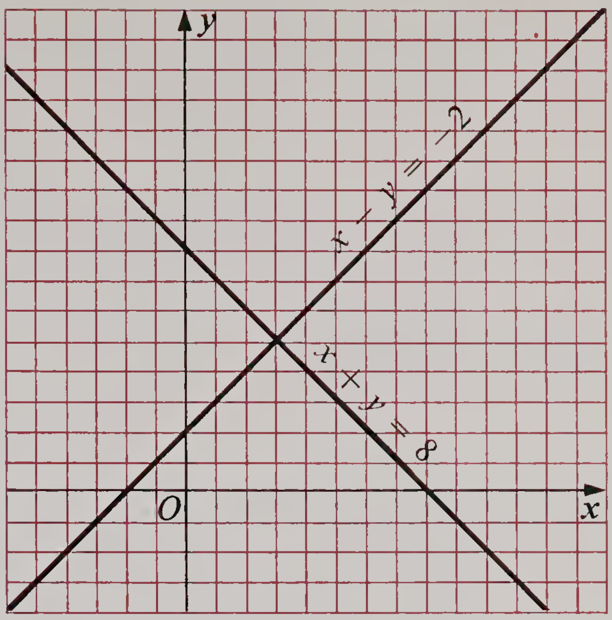
3. The coordinates of the point of intersection of a pair of linear equations are a _?_ of any pair of linear equations whose graphs pass through this point.

Questions 4–6 refer to the graphs of $x + y = 8$ *and* $x - y = -2$.

4. Give the coordinates of a point which satisfies $x + y = 8$ but not $x - y = -2$.

5. Name a number pair satisfying $x - y = -2$, but not $x + y = 8$.

6. The point __?__ satisfies both equations.



10-2 The Addition and Subtraction Method *Pages 370–372*

Solve each pair of equations.

7. $2x + y = 11$
$2x - y = 9$

8. $3r - t = 10$
$2r - t = 40$

9. $5m - 3n = 11$
$5m + 3n = 29$

10. If the variables in a pair of equations are replaced by numbers for which both equations are true, the difference of the left members __?__ the difference of the right members.

10-3 Problems with Two Variables *Pages 372–374*

Translate each statement into a pair of equations.

11. One number is 5 less than twice another, and their sum is 100.

12. The length of a rectangle is 5 times its width. The perimeter is 72 feet.

Solve, using two variables and two equations.

13. A 7-foot board is cut in two. Three times the longer part would be 9 feet more than five times the shorter. Find both lengths.

10-4 Multiplication in the Addition and Subtraction Method *Pages 374–377*

Solve by using multiplication with addition or subtraction.

14. $x + 3y = 26$
$3x + 2y = 29$

15. $7r - 10s = 1$
$56r - 40s = 28$

16. $9w + 5z = 33$
$6w - 7z = -9$

10–5 The Substitution Method *Pages 378–379*

Solve by substitution.

17. $2u + 3v = 7$
$u - v = 11$

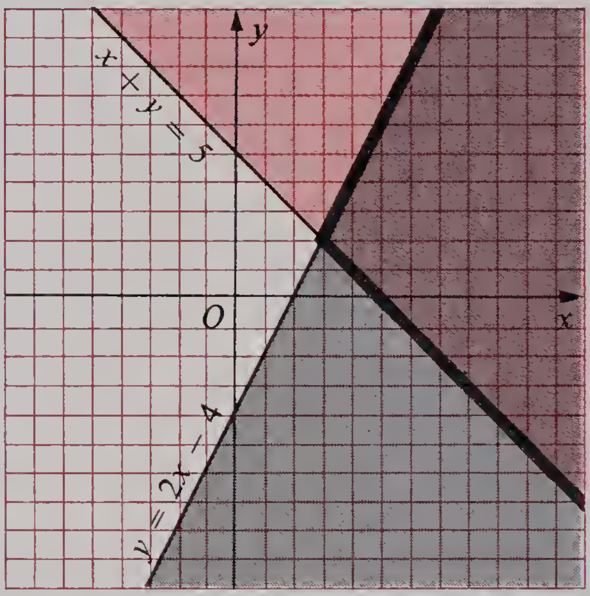
18. $3w - 2z = 38$
$2w - z = 18$

19. $6p + 3q = -3$
$p + 2q = 10$

10–6 Graphs of Pairs of Linear Inequalities (Optional) *Pages 379–381*

20. The graph of a linear inequality in two variables is represented by a _?_.

Questions 21–22 refer to the graphs of $x + y \geq 5$ *and* $y \leq 2x - 4$.



21. Give the coordinates of a point which satisfies $x + y \geq 5$ but not the other inequality.

22. Name a number pair satisfying $y \leq 2x - 4$ but not $x + y \geq 5$.

10–7 Digit Problems *Pages 381–383*

23. The value represented by a digit depends on its _?_ in a number.

24. In 31, the tens digit is _?_; the units digit is _?_.

25. The tens digit of a two-digit numeral exceeds the units digit by 2. The sum of the digits is 16. What is the number?

26. The sum of the two digits of a numeral is 12. With its digits reversed, the number is 36 less than it was. Find the original number.

10–8 Motion Problems *Pages 383–385*

27. If you row downstream, the rate is _?_ than in still water.

28. An airplane cruises at 250 m.p.h. It flies in a head wind of 45 m.p.h. at _?_ m.p.h.

29. Norman cycled 15 miles in $1\frac{1}{4}$ hours into the wind. With the wind behind him, he made the trip in $\frac{3}{4}$ hour. What would Norman's speed be without wind?

30. A steamer goes 91 miles up a river in $6\frac{1}{2}$ hours. The return trip is scheduled for $3\frac{1}{2}$ hours. What is the rate in still water?

10-9 Age Problems *Pages 385–386*

Solve, using two variables and two equations.

31. Allen is $\frac{4}{5}$ as old as Bertrand. In 4 years he will be $\frac{5}{6}$ as old as Bertrand. How old is each boy now?

32. Three years ago, June was $\frac{2}{3}$ Jack's age. In two years, June will be $\frac{3}{4}$ his age. How old are they?

33. Mr. Granger is 10 years more than twice as old as Tim. Last year he was 4 years less than 3 times as old as Tim. How old is each?

10-10 Problems about Fractions *Pages 386–387*

34. If you add 1 to its denominator, a certain fraction becomes equal to $\frac{5}{8}$. If you subtract 4 from its numerator, that fraction becomes equal to $\frac{2}{5}$. Find the fraction.

35. A fraction's value is $\frac{2}{5}$. If its numerator loses 1 and its denominator gains 5, the reciprocal of the result equals 5. Find the fraction.

Extra for Experts

Diophantine Equations

Can you find a solution of $x + 2y = 3$ in the set of directed numbers? Of course you can; (0, 1.5) is a solution, as is $(-1, 2)$. In fact, the number of solutions is unlimited. Can you see, however, that if the replacement set for x and for y is {positive integers}, the only solution is (1, 1) since $y = \frac{3 - x}{2}$?

Because a system of one or more equations with more variables than equations may have an infinite solution set, the equations are called *indeterminate*. When the replacement sets of the variables are restricted to subsets of {integers}, the equations are called *Diophantine* (di′o-**fan**-tin) equations, after the Greek algebraist Diophantus. The solutions must be so restricted in many practical situations:

EXAMPLE. How many quarters and how many dimes together make up \$1.60?

Solution: Let d = the number of dimes and q = the number of quarters, where d and $q \in$ {nonnegative integers}.

$$10d + 25q = 160 \rightarrow 2d + 5q = 32 \rightarrow q = \frac{2(16 - d)}{5}$$

For q to be a nonnegative integer, $(16 - d)$ must represent a nonnegative multiple of 5:

$16 - d = 5t$, where $t \in$ {nonnegative integers}.

Substituting $5t$ in the expression for q,

$$q = \frac{2 \cdot 5t}{5} = 2t.$$

Thus, d and q are expressed in terms of t by the equations,

$d = 16 - 5t$ and $q = 2t$, where $t \in$ {nonnegative integers}.

Specific values for t give corresponding pairs (d, q):

t	0	1	2	3
d	16	11	6	1
q	0	2	4	6

Additional choices of t result in negative values of d, so that the chart gives all the acceptable pairs, (d, q). Check these values in the problem.

Questions

1. A 55-yard cloth is cut into pieces $3\frac{1}{2}$ and $4\frac{1}{2}$ yards long, without remnants. How many pieces of each length are there?
2. If a baker arranges his almost 6 dozen loaves of bread in groups of 4, 1 is left over; in groups of 5, 4 are left over. How many loaves has he?
3. Find the smallest positive integer which when divided by 2, 3, and 7 leaves remainders of 1, 2, and 6, respectively.
4. This "Chinese Problem of a Hundred Fowl," dates at least to the sixth century: If a rooster is worth 5 yuan, a hen is worth 3 yuan, and 3 chicks are worth 1 yuan, how many of each, 100 in all, would be worth 100 yuan? Assume that at least 5 roosters are required.

5. If at least one coin of each kind is used, how can you change a dollar into 15 coins, each less than a quarter in value?

6. Solve in positive integers: $\begin{cases} 4a - 11b + 12c = 22 \\ a + 5b - 4c = 17 \end{cases}$

7. How many combinations of three-cent and five-cent stamps can you buy with exactly fifty cents, getting at least one of each kind?

8. A compound of carbon, hydrogen, and oxygen has a molecular weight of 46. Using atomic weights of 12 for carbon, 1 for hydrogen, and 16 for oxygen, what are the possible formulas for the compound in the form $C_xH_yO_z$?

Just for Fun

Make a Friend Repeat Himself

You can make a person repeat himself, whether or not he wishes to do so.

You say	If he selects 28	If he selects 63
Multiply any two-digit number by 15.	15 × 28 = 420	15 × 63 = 945
Now multiply by 7.	7 × 420 = 2940	7 × 945 = 6615
Subtract 4 times the original number from the product. Why are you repeating yourself?	4 × 28 = 112 2940 − 112 = 2828	4 × 63 = 252 6615 − 252 = 6363
Let's start over: Multiply the original number by 13.	13 × 28 = 364	13 × 63 = 819
Now multiply by 8.	8 × 364 = 2912	8 × 819 = 6552
Subtract three times the original number — You're repeating yourself again!	3 × 28 = 84 2912 − 84 = 2828	3 × 63 = 189 6552 − 189 = 6363

The person repeats himself because he is multiplying by 101. Any two-digit number multiplied by 101 gives a four-digit number containing the same digits in the same order.

CHAPTER 1

The Real Numbers

The ancient Greeks thought that all lengths should be measurable in terms of integers or common fractions. However, they found several examples in which they were not. You already know about one of these.

If the diameter of a circle is one foot, what is the circumference? Is it $3\frac{1}{7}$, 3.14, or 3.1416? No matter what common fraction you use, it will not measure the circumference exactly. Likewise, the diagonal of a one-foot square can be given only as a decimal approximation.

The architect who designed the public auditorium of Pittsburgh (top picture) used an approximation of π. A space navigator needs even greater accuracy in his approximation of π, so he takes advantage of a computer's talents (bottom picture).

Many of the problems of mathematics and science cannot be solved with only common fractions. You need a different kind of number about which you will learn in this chapter.

THE SYSTEM OF RATIONAL NUMBERS

11–1 The Nature of Rational Numbers

Pupils should now reread the first transparency.

You define a **number system** by giving a set of numbers and telling how to add and multiply the members. Thus, the numbers you first met in arithmetic, $\{1, 2, 3, 4, \ldots\}$, form the **system of positive integers**, which is closed under addition and multiplication.

The next numbers you met were elements of the set of common fractions. The system of positive fractions and integers is closed under addition, multiplication, and division. When you extended your idea of number to include zero and the negative integers and fractions, you had a system of numbers closed under subtraction, too.

The resulting set of positive and negative integers and fractions, together with zero, is the set of *rational numbers.* A **rational number** is any number which can be expressed as the ratio of two integers. Because the system of rational numbers is closed under addition, subtraction, multiplication, and division (except by zero), these are called the **rational operations**

Emphasize this point. See T.M. pg. 35 (1).

Stress that the set of integers is a subset of the set of rational numbers.

A rational number can be expressed in an unlimited number of ways:

$$0 = \frac{0}{7} = \frac{0}{-5} \qquad 6 = \frac{6}{1} = \frac{-18}{-3} \qquad -\frac{3}{4} = \frac{-3}{4} = \frac{6}{-8}$$

$$\frac{5}{17} = \frac{-5}{-17} = \frac{20}{68} \qquad 1.8 = \frac{18}{10} = \frac{27}{15} \qquad 23\% = \frac{23}{100} = \frac{115}{500}$$

Furthermore, you always can tell which of two rational numbers is the greater, by writing them with the same positive denominator and comparing their numerators.

$-\frac{1}{2} > -\frac{3}{2}$ **because** $-1 > -3$; $\frac{3}{2} > \frac{5}{4}$ **because** $\frac{6}{4} > \frac{5}{4}$ **since** $6 > 5$

This test can be developed in another form:

Let a and b be integers and c and d be positive integers; if one of the following statements is true, the others are true.

$$\frac{a}{c} > \frac{b}{d} \qquad \frac{a}{c}(cd) > \frac{b}{d}(cd) \qquad ad > bc$$

$\frac{5}{6} > \frac{3}{4}$ because $5(4) > 3(6)$ $\qquad -\frac{1}{2} > -\frac{3}{2}$ because $-1(2) > -3(2)$

In other words, $\frac{a}{c} > \frac{b}{d}$ if, and only if, $ad > bc$.

For each integer there is a next larger one. This is not true for the set of rational numbers; it possesses the ***property of density***:

Between every pair of different rational numbers there is another rational number.

EXAMPLE. **Find a rational number between $\frac{3}{4}$ and $\frac{5}{6}$.**

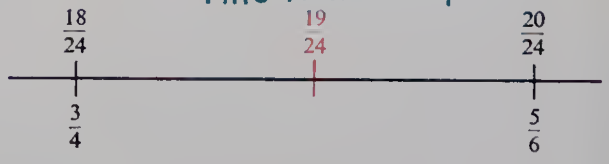
Solution: See Written Exercise 23, page 400.

1. **Find the difference of the numbers.** $\frac{5}{6} - \frac{3}{4} = \frac{10}{12} - \frac{9}{12} = \frac{1}{12}$
2. **Add half this differenceto the smaller.** $\frac{3}{4} + \frac{1}{2}\left(\frac{1}{12}\right) = \boxed{\frac{19}{24}}$
3. **Check. Is** $\frac{3}{4} < \frac{19}{24} < \frac{5}{6}$**?**

This is the midpoint.

$3(24) \overset{?}{<} 19(4)$ **and** $19(6) \overset{?}{<} 5(24)$

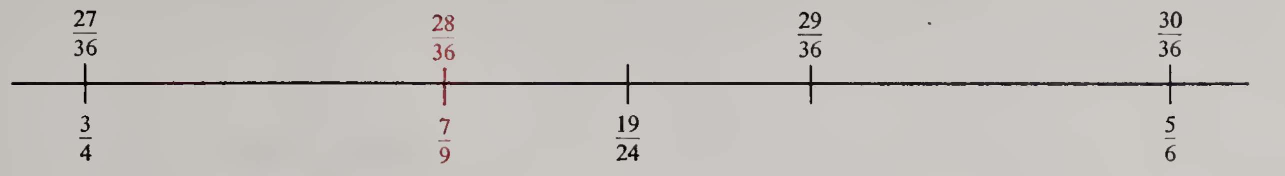
$72 < 76$ ✓ $\qquad 114 < 120$ ✓



$\therefore$ **A rational number between (it is halfway between) $\frac{3}{4}$ and $\frac{5}{6}$ is $\frac{19}{24}$, Answer.**

Another rational number between $\frac{3}{4}$ and $\frac{5}{6}$ is $\frac{7}{9}$:

$\frac{3}{4} + \frac{1}{3}(\frac{1}{12}) = \frac{27}{36} + \frac{1}{36} = \frac{28}{36} = \frac{7}{9}$. **Check that** $\frac{3}{4} < \frac{7}{9} < \frac{5}{6}$.



Do you see that the number of rational numbers between $\frac{3}{4}$ and $\frac{5}{6}$ is unlimited? The property of density implies that between every pair of rational numbers there is an infinite set of rational numbers.

ORAL EXERCISES

Express each number as a quotient of integers.

SAMPLE. 4.3 *What you say:* 43 divided by 10.

1. 2.2 $22 \div 10$ **5.** .07 $7 \div 100$ **9.** -7 $-7 \div 1$ **13.** $(-3)(-\frac{4}{3})$ $4 \div 1$
2. 32.5 $325 \div 10$ **6.** .005 $5 \div 1000$ **10.** -2 $-2 \div 1$ **14.** $\frac{1}{7} \div 7$ $1 \div 49$
3. $-4\frac{1}{2}$ $-9 \div 2$ **7.** 12% $12 \div 100$ **11.** $\frac{2}{3} + (-\frac{5}{3})$ $-1 \div 1$ **15.** $1\frac{1}{4} + \frac{3}{4}$ $2 \div 1$
4. $-3\frac{1}{5}$ $-16 \div 5$ **8.** 15% $15 \div 100$ **12.** $\frac{4}{7} - \frac{6}{7}$ $-2 \div 7$ **16.** $0 \cdot (-\frac{7}{2})$ $0 \div 2$ or any other integer

State which number is the greater.

17. [✓ $\frac{4}{3}$], $\frac{2}{3}$ **19.** -5, [✓ $\frac{1}{3}$] **21.** [✓ $-\frac{4}{7}$], $-\frac{9}{7}$ **23.** [✓ 5], $\frac{34}{7}$
18. [✓ $\frac{17}{6}$], $\frac{13}{6}$ **20.** [✓ $\frac{1}{2}$], -3 **22.** $-\frac{5}{2}$, [✓ $-\frac{3}{2}$] **24.** 9, [✓ $\frac{55}{6}$]

If $a > 0$, and $x \in \{1, 2, 3, 4\}$, taking each value in succession, which fractions grow larger and which, smaller? Larger = L; smaller = S

25. $\frac{x}{1}$ L **27.** $\frac{a+1}{x+a}$ S **29.** $\frac{a}{2x-1}$ L **31.** $\frac{3a}{3x-2}$ S
26. $\frac{1}{x}$ S **28.** $\frac{x+3}{a}$ L **30.** $\frac{a}{5-x}$ L **32.** $\frac{2a}{9-2x}$ L

WRITTEN EXERCISES

Replace the ? by $=$, $<$, or $>$ to make a true statement.

A

1. $\frac{8}{15}$? $\frac{7}{16}$ **3.** $\frac{19}{12}$? $\frac{56}{35}$ **5.** $\frac{200}{105}$? $\frac{240}{126}$ **7.** $-17\frac{1}{3}$? $-\frac{400}{12}$
2. $\frac{-22}{45}$? $-\frac{12}{25}$ **4.** $-\frac{46}{49}$? $-\frac{53}{56}$ **6.** $\frac{138}{162}$? $\frac{184}{216}$ **8.** $\frac{321}{15}$? $21\frac{1}{5}$

Arrange the members of each set in increasing order.

9. $\{\frac{1}{2}, -\frac{4}{9}, \frac{16}{31}\}$
10. $\{\frac{2}{3}, -\frac{3}{7}, \frac{9}{14}\}$
11. $\{\frac{25}{20}, \frac{24}{21}, \frac{4}{3}\}$
12. $\{\frac{22}{7}, \frac{10}{3}, 3, \frac{157}{50}\}$
13. $\{-\frac{11}{4}, -\frac{87}{32}, -\frac{16}{6}\}$
14. $\{-\frac{3}{2}, -\frac{99}{70}, -\frac{7}{5}, -\frac{17}{12}\}$

Find the number halfway between these numbers.

15. $\frac{11}{12}, 1$
16. $\frac{9}{14}, \frac{5}{7}$
17. $\frac{1}{100}, \frac{1}{10}$
18. $-\frac{1}{100}, -\frac{1}{1000}$
19. $-6\frac{1}{7}, -7\frac{2}{5}$
20. $4\frac{2}{3}, 5\frac{1}{8}$

B **21.** Find the number one-third of the way from $\frac{7}{8}$ to $1\frac{1}{4}$.

22. Find the number one-fifth of the way from $-\frac{3}{2}$ to $-\frac{1}{4}$.

23. Show that the number halfway between a and b is $\dfrac{a+b}{2}$.

24. What number is one-third of the way from p to q (a) if $p < q$, and (b) if $p > q$?

Explain why each statement is true.

25. Three is the smallest integer greater than 2.

26. There is no smallest rational number greater than 2.

In each case, tell whether the statement is true with the blanks filled as indicated.

27. If m and n are different $\underline{\ (x)\ }$, there are as many $\underline{\ (y)\ }$ between m and n as you please.

- **a.** (x) rational numbers; (y) rational numbers
- **b.** (x) integers; (y) integers
- **c.** (x) rational numbers; (y) integers
- **d.** (x) integers; (y) rational numbers

In Exercises 28–31 *a*, *b*, *c*, and *d* are nonzero integers. Explain why these expressions represent rational numbers.

28. $\dfrac{a}{c} + \dfrac{b}{d}$
29. $\dfrac{a}{c} - \dfrac{b}{d}$
30. $\dfrac{a}{c} \cdot \dfrac{b}{d}$
31. $\dfrac{a}{c} \div \dfrac{b}{d}$

(Closure properties of the set of rational numbers)

11–2 Decimal Forms of Rational Numbers

In arithmetic you learned to change common fractions to decimals and decimals to common fractions. To change a common fraction to a decimal, you carry out the indicated division.

$$\frac{3}{16} = 3 \div 16 \qquad \frac{1}{12} = 1 \div 12 \qquad \frac{7}{11} = 7 \div 11$$

$$\begin{array}{r} .1875 \\ 16\overline{)3.0000} \\ \underline{1\,6} \\ 1\,40 \\ \underline{1\,28} \\ 120 \\ \underline{112} \\ 80 \\ \underline{80} \\ 0 \end{array} \qquad \begin{array}{r} .08333 \\ 12\overline{)1.00000} \\ \underline{96} \\ 40 \\ \underline{36} \\ 40 \\ \underline{36} \\ 40 \\ \underline{36} \\ 4 \end{array} \qquad \begin{array}{r} .6363 \\ 11\overline{)7.0000} \\ \underline{6\,6} \\ 40 \\ \underline{33} \\ 70 \\ \underline{66} \\ 40 \\ \underline{33} \\ 7 \end{array}$$

A decimal with a finite number of places, like .1875, above, is called *terminating*, *ending*, or *finite*. Such a decimal represents a rational number; for example, .1875 equals $\frac{1875}{10000}$ or $\frac{3}{16}$.

In the division of 1 by 12, however, you never have a remainder of 0, but the remainder 4 repeats step after step, and 3 repeats in the quotient. A decimal which continues indefinitely is called *nonterminating* or *unending*. Moreover, the decimal for $\frac{1}{12}$ is *repeating* or *periodic*, because the same digit (or block of digits) repeats unendingly. You may write

$$\frac{1}{12} = .08333\ldots \qquad \text{or} \qquad \frac{1}{12} = .08\overline{3},$$

where the dots and the bar indicate *continue unendingly*.

When 7 is divided by 11, the successive remainders are 4, 7, 4, 7, . . . , and the quotient is a repeating decimal.

$$\frac{7}{11} = .636363\ldots, \qquad \text{or} \qquad \frac{7}{11} = .\overline{63}.$$

When you divide an integer by 11, the remainder at each step is from {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Within no more than ten steps after only zeros are left in the dividend, either the remainder is 0 and the division terminates, or a sequence of other remainders repeats unendingly. This sort of reasoning leads to the following result.

The decimal form of any rational number $\frac{r}{s}$ either terminates or eventually repeats in a block of fewer than s digits.

Conversely, you can show that this statement is true.

All terminating decimals and all repeating decimals represent rational numbers which can be written in the form $\frac{r}{s}$.

The preceding conversion of .1875 to $\frac{3}{16}$ shows that a terminating decimal can be written as a common fraction. The following examples show that repeating decimals can also be written as common fractions.

Assume that $.3\overline{24}$ represents a number. See T.M. pg. 35 (2).

EXAMPLE 1. **Write .324 as a common fraction.**

Solution:

$$\begin{aligned} 100N &= 32.424\overline{24} \\ N &= .324\overline{24} \\ \hline 99N &= 32.10000 \\ N &= \frac{32.1}{99} = \frac{107}{330} \end{aligned}$$

EXAMPLE 2. **Write $.\overline{125}$ as a common fraction.**

Solution:

$$\begin{aligned} 1000N &= 125.\overline{125} \\ N &= .\overline{125} \\ \hline 999N &= 125.000 \\ N &= \frac{125}{999} \end{aligned}$$

If the number of digits in the block of repeating digits is p, multiply the given number N by 10^p, producing a number with the same repeating block as the given number, so that subtracting the given number from it yields a terminating decimal.

It often is convenient to break off a lengthy decimal, leaving an approximation of the number represented. You may write

$$\tfrac{1}{12} \doteq .08333 \quad \text{or} \quad \tfrac{1}{12} \doteq .083 \quad \text{or} \quad \tfrac{1}{12} \doteq .08,$$

where $\doteq$ is read *equals approximately*, using this rule:

To round a decimal, add 1 to the last digit kept if the first digit dropped is 5 or more; otherwise, leave the digits unchanged.

For example: $.325 \doteq .33$, $.324 \doteq .32$

ORAL EXERCISES

Approximate each number to the nearest tenth.

1. 3.42 3.4
2. 5.61 5.6
3. −1.36 −1.4
4. −2.745 −2.7
5. $.37\overline{2}$ .4
6. $.83\overline{6}$ .8
7. $-74.3\overline{5}$ −74.4
8. −68.55 . . . −68.6
9. $.04\overline{7}$ 0.0
10. $-.09\overline{4}$ −.1
11. $\frac{1}{4}$ .3
12. $\frac{3}{4}$ .8

Give a second form for each number.

13. $\frac{4}{5}$ .8
14. 8 $\frac{8}{1}$
15. .78 $\frac{39}{50}$
16. 1.35 $\frac{27}{20}$
17. 1.099 . . . 1.1
18. $23.\overline{9}$ 24.0
19. $-\frac{56}{27}$ $-2.\overline{074}$
20. $-\frac{42}{21}$ −2
21. $4.33\overline{3}$ $\frac{13}{3}$
22. $16.\overline{6}$ $\frac{50}{3}$
23. −.0082 $-\frac{41}{5000}$
24. −7.024 $-\frac{878}{125}$

WRITTEN EXERCISES

Write as terminating or repeating decimals.

A **1.** $\frac{2}{25}$
2. $\frac{3}{50}$
3. $\frac{7}{32}$
4. $\frac{9}{8}$
5. $-\frac{10}{11}$
6. $-\frac{5}{3}$
7. $\frac{-2}{7}$
8. $\frac{-17}{80}$
9. $\frac{81}{40}$
10. $\frac{52}{70}$
11. $-\frac{4}{15}$
12. $-\frac{8}{21}$

Write as common fractions.

13. $.666$
14. $.4$
15. $.33\overline{7}$
16. $.3\overline{37}$
17. $.27272727\ldots$
18. $.303303303\ldots$
19. $-2.\overline{567}$
20. $-1.20\overline{2}$

Find the difference of these numbers, and give a number between them.

B **21.** $.27$ and $.\overline{27}$
22. $.33$ and $.\overline{33}$
23. $.9$ and $.\overline{9}$
24. $.16$ and $.\overline{16}$
25. $.0833$ and $\frac{1}{12}$
26. $.0909$ and $\frac{1}{11}$

Compare the decimal forms of the members of each set of numbers.

C **27.** $\{\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \ldots, \frac{8}{9}\}$
28. $\{\frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \ldots, \frac{10}{11}\}$
29. $\{\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \ldots, \frac{6}{7}\}$
30. $\{\frac{1}{13}, \frac{2}{13}, \frac{3}{13}, \ldots, \frac{12}{13}\}$

IRRATIONAL NUMBERS

11–3 Roots of Numbers

The **power of a number** is the product of factors each equal to that number: $5^2 = 5(5)$; $5^3 = 5(5)(5)$; and $5^n = 5(5)\ldots(5)$, taking 5 as a factor n times. This operation of *raising to a power* is called *involution.*

Just as addition and multiplication have inverse operations, so has raising to a power. Its inverse operation is called *extracting a root* (*evolution*). For any positive integer n, a number x is an **n^{th} root** of the number a if it satisfies $x^n = a$. That is, since $3^4 = 81$, 3 is a *fourth root* of 81.

To indicate the n^{th} root of a, use the expression $\sqrt[n]{a}$, which is called a *radical* (in Latin *radix* means root). The symbol $\surd$ indicates that a root is to be extracted; n is the *root index*, signifying the root to be taken; the bar, usually incorporated in the radical symbol, covers the *radicand* (rad′i-**cand**), the expression whose root is to be extracted. With no root index, $\surd$ indicates square root: $\sqrt{81} = 9$; $\sqrt[3]{125} = 5$; $\sqrt[4]{81} = 3$.

When you square a positive or a negative number, you get a positive result. That is, $5^2 = 25$, and $(-5)^2 = 25$. Thus, every positive number has two square roots, one positive and the other negative. Zero, however, has only one square root, zero. You use the expression $\sqrt{25}$ to indicate the positive root 5 (the *principal root*), $-\sqrt{25}$ to indicate the negative root -5, and $\pm\sqrt{25}$ (read positive and negative square root of 25) to represent both roots. Thus, $\sqrt{\frac{4}{9}} = \frac{2}{3}$, $-\sqrt{\frac{4}{9}} = -\frac{2}{3}$, and $\pm\sqrt{\frac{4}{9}} = \pm\frac{2}{3}$. Since the square of every directed number is either positive or zero, negative numbers do not have square roots in the set of directed numbers.

See T.M. pg. 36 (3).

One method of finding the square root of a large number is to determine its factors, and then to express it as a product of powers, and take the square roots of the powers.

Be sure that pupils see why negative numbers do not have square roots in the system of directed numbers. See T.M. pg. 36 (4).

EXAMPLE 1. **Evaluate $\sqrt{2025}$.**

Solution:

$$2025 = 5(405) = 5(5)(81) = 5^2(9^2)$$
$$\sqrt{2025} = \sqrt{5^2(9^2)} = 5(9) = 45$$

Check: $45(45) = 2025$ ✓ $\sqrt{2025} = 45$, **Answer.**

This method of solution is based on the ***product property of square roots.***

If $a \geq 0$ and $b \geq 0$, then $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

To prove this property, we show that $\sqrt{a} \cdot \sqrt{b}$ is a nonnegative number and that its square is ab. Notice that only principal roots are used.

The class should supply reasons for each step. See T.M. pg. 36 (5).

To show that $\sqrt{a} \cdot \sqrt{b} \geq 0$:

If $a \geq 0$, and $b \geq 0$,

then $\sqrt{a} \geq 0$, $\sqrt{b} \geq 0$,

and $\sqrt{a} \cdot \sqrt{b} \geq 0$.

To show that $(\sqrt{a} \cdot \sqrt{b})^2 = ab$:

$$(\sqrt{a} \cdot \sqrt{b})^2 = (\sqrt{a} \cdot \sqrt{b})(\sqrt{a} \cdot \sqrt{b})$$
$$= \underbrace{(\sqrt{a} \cdot \sqrt{a})}_{}\underbrace{(\sqrt{b} \cdot \sqrt{b})}_{}$$
$$= a \cdot b$$
$$\therefore (\sqrt{a} \cdot \sqrt{b})^2 = ab$$

Emphasize that $\sqrt{a}$ means the number such that $\sqrt{a} \cdot \sqrt{a} = a$.

Can you guess the ***quotient property of square roots***?

If $a \geq 0$ and $b > 0$, then $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

EXAMPLE 2. Evaluate $\sqrt{\frac{3600}{2401}}$.

Solution:

$$3600 = (36)(100) = 6^2(10^2)$$

$$2401 = 7(343) = 7(7)(49) = 7^2(7^2) = 7^4$$

$$\sqrt{\frac{3600}{2401}} = \sqrt{\frac{6^2(10^2)}{7^4}} = \frac{6(10)}{7^2} = \frac{60}{49}$$

Check: $\frac{60}{49} \cdot \frac{60}{49} = \frac{3600}{2401}\checkmark \qquad \sqrt{\frac{3600}{2401}} = \frac{60}{49}$, **Answer.**

ORAL EXERCISES

Give the principal square root.

1. 144 12
2. 121 11
3. $\frac{81}{64}$ $\frac{9}{8}$
4. $\frac{4}{49}$ $\frac{2}{7}$
5. $100z^2$ $10|z|$
6. $144w^4$ $12w^2$
7. $81a^2b^2$ $9|ab|$
8. $36x^2y^2$ $6|xy|$
*9. $.25r^6t^4$ $.5|r^3|t^2$
10. $.04k^4t^2$ $.2k^2|t|$
11. $\frac{4d^4}{g^8}$ $\frac{2d^2}{g^4}$
*12. $\frac{9p^2}{q^{10}}$ $\frac{3|p|}{|q^5|}$

Simplify each expression.

13. $\sqrt{1}$ 1
14. $\sqrt{81}$ 9
15. $\sqrt{\frac{16}{49}}$ $\frac{4}{7}$
16. $\sqrt{\frac{1}{9}}$ $\frac{1}{3}$
17. $\sqrt{(-2)^2}$ 2
18. $\sqrt{(-3)^2}$ 3
19. $-\sqrt{36x^4}$ $-6x^2$
20. $-\sqrt{64y^8}$ $-8y^4$
21. $-\sqrt{9m^2n^4}$ $-3|m|n^2$
22. $-\sqrt{49m^2n^6}$ $-7|m|n^3$
23. $\pm\sqrt{\frac{s^2}{121}}$ $\pm\frac{s}{11}$
24. $(\sqrt{81})^2$ 81
25. $(\sqrt{49})^2$ 49
26. $\sqrt{4^2 + 3^2}$ 5
27. $-\sqrt{13^2 - 12^2}$ -5
28. $\pm\sqrt{10^2 - 8^2}$ ± 6
29. $\pm\sqrt{5^2 - 4^2}$ ± 3
30. $\pm\sqrt{\frac{t^4}{100}}$ $\pm\frac{t^2}{10}$

WRITTEN EXERCISES

Evaluate each expression.

A

1. $\sqrt{484}$
2. $\sqrt{441}$
3. $\sqrt{1296}$
4. $\sqrt{2304}$
5. $-\sqrt{1156}$
6. $-\sqrt{784}$
7. $-\sqrt{\frac{1089}{25}}$
8. $-\sqrt{\frac{484}{9}}$
9. $\pm\sqrt{\frac{1}{324}}$
10. $\pm\sqrt{\frac{1}{1521}}$
11. $\pm\sqrt{\frac{16}{1225}}$
12. $\pm\sqrt{\frac{49}{1296}}$

*Since $|r^3| = |r|^3$ and $|q^5| = |q|^5$, the following answers are also acceptable: Ex. 9, $.5|r|^3t^2$; Ex. 12, $\frac{3|p|}{|q|^5}$. The latter forms are especially convenient when one substitutes values for the variables.

Solve. Use {rational numbers} as the replacement set of the variable.

SAMPLE. $y^2 = 9$

Solution:

$$y^2 = 9$$
$$\therefore y = \pm\sqrt{9} = \pm 3$$

Check:

$$(3)^2 \overset{?}{=} 9 \qquad (-3)^2 \overset{?}{=} 9$$
$$9 = 9 \checkmark \qquad 9 = 9 \checkmark$$

$\{3, -3\}$, **Answer.**

13. $x^2 = 36$

14. $y^2 = 49$

15. $4a^2 - 1 = 0$

16. $9b^2 - 4 = 0$

17. $3t^2 - 27 = 0$

18. $5k^2 - 125 = 0$

B **19.** If $r^2 + s^2 = 12$ and $rs = 2$, find the positive value of $r + s$.

20. If $r - s = 5$ and $rs = 3$, find the value of $r^2 + s^2$.

21. If $(x - y)^2 = 5$ and $xy = 1$, find the negative value of $x + y$.

22. If $x + y = 8$ and $x^2 + y^2 = 40$, find the positive value of $x - y$.

23. Find the fallacy in the following argument.

Let a be any rational number.

$$a^2 = (-a)^2$$
$$\therefore \sqrt{a^2} = \sqrt{(-a)^2}$$
$$\text{But } \sqrt{a^2} = a \quad \text{and} \quad \sqrt{(-a)^2} = -a$$
$$\therefore \quad a = -a$$

24. Find the fallacy in this proof that every rational number is 1.

Let x be any rational number.

$$x - 1 = -(1 - x)$$
$$\therefore (x - 1)^2 = (1 - x)^2$$
$$\therefore \quad x - 1 = 1 - x$$
$$\therefore \quad 2x = 2$$
$$\therefore \quad x = 1$$

C **25.** Solve the equation $\sqrt{x^2} - x = 18$.

26. Solve the equation $y - \sqrt{y^2} = 50$.

27. Prove the quotient property of square roots.

28. Prove that $\sqrt{a^3} = (\sqrt{a})^3$, if $a \geq 0$.

29. Prove that if $a > b > 0$, then $a^2 > b^2$. (*Hint:* Show that $a^2 > ab$ and that $ab > b^2$.) Use this result to explain why a positive number cannot have two different positive square roots.

Able students should do Exercises 29 and 30.

30. Prove that if $b < a < 0$, then $a^2 < b^2$. Use this result to explain why a positive number cannot have two different negative square roots.

11–4 Properties of Irrational Numbers

Rational numbers like 25, 36, and $\frac{49}{16}$ which are squares of rational numbers are called ***perfect squares***. However, not every positive rational number is a perfect square.

Do those rational numbers which are not perfect squares have rational square roots? Consider some positive integer n. Assume that its square root is a fraction $\frac{a}{b}$ in lowest terms. That is, $\sqrt{n} = \frac{a}{b}$, where a, b, and n are positive integers, and a and b have no common factors.

If $\sqrt{n} = \frac{a}{b}$, then $n = \frac{a^2}{b^2}$. Since a^2 has the same prime factors as a, and b^2 has the same prime factors as b, if a and b have no factors in common, neither do a^2 and b^2, and $\frac{a^2}{b^2}$ is in lowest terms. *If a fraction in lowest terms is equal to an integer, the denominator of the fraction must be 1.* Thus, since n is an integer, and $\frac{a^2}{b^2}$ is in lowest terms, $b^2 = 1$, and $b = 1$, which means that $\frac{a}{b} = \frac{a}{1} = a$. Therefore, if the square root of a positive integer is a rational number, $\frac{a}{b}$, the root is in fact an integer, $\frac{a}{1}$. Thus, only integers which are squares of integers can have rational square roots.

See T.M. pg. 36 (6).

As there are positive integers which are not squares of integers, if symbols like $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ are to have any meaning, your concept of numbers must be extended to include *irrational numbers*, numbers which cannot be expressed in the form $\frac{r}{s}$, where r and s are integers. Together, the rational and irrational numbers form the *real number system*, which has all the properties you have studied, including the property of density. In addition, the real numbers have the ***property of completeness***:

> On the number line, each point corresponds to one real number; and each real number corresponds to one point on the line.

Terminating and repeating decimals represent rational numbers; therefore, the decimals for irrational numbers must neither terminate

Give a few simple examples like $3 < \sqrt{16} < \frac{16}{3}$ and $\frac{49}{8} < \sqrt{49} < 8$.

nor repeat. One method of finding successive digits in the decimals of irrational numbers which are square roots is based on the ***property of pairs of divisors*** of any number:

> If you divide a number by a divisor which is smaller in absolute value than the square root of that number, the quotient will be larger in absolute value than the square root.

Consider 100 and its square root 10. $100 \div 10 = 10$; but $100 \div 25 = 4$, $100 \div 5 = 20$, and $100 \div 50 = 2$. If the divisor is greater than 10, the quotient is less than 10, and vice versa.

EXAMPLE 1. **Approximate $\sqrt{17}$.**

Solution: See T.M. pg. 36 (7).

1. **As your first approximation select the integer whose square is nearest 17.**
2. **Divide 17 by a. Carry the quotient to twice as many digits as are in the divisor.** Do not round the quotient.
3. **From the property of pairs of divisors, you know that $\sqrt{17}$ is between a and $\frac{17}{a}$. Take their average to find a better approximation to $\sqrt{17}$.**
4. **Use this average as your new a. Repeat Steps 2, 3, 4 until you have as close an approximation as you wish. Your a is accurate to as many places as match in a and $17 \div a$.**

Carry the average to the same number of digits as in $\frac{17}{a}$; do not round.

Let a be a number such that

$$a^2 \doteq 17$$

First $a = 4$ (1)

$17 \div 4 \doteq \underline{4.2}$ (2)

$2\overline{)8.2}$ (3)

Second $a = 4.1$ (4)

$17 \div 4.1 \doteq \underline{4.146}$ (2)

$2\overline{)8.246}$ (3)

Third $a = 4.123$ (4)

$17 \div 4.123 \doteq \underline{4.1232112}$ (2)

$2\overline{)8.2462112}$ (3)

Fourth $a = 4.1231056$ (4)

Check:

$$\begin{array}{r} 4.12310565 \\ 4.1231056\overline{)17.00000000} \end{array} \checkmark$$

$\therefore \sqrt{17} \doteq 4.1231056$, Answer.

The approximation found by this method represents a rational number which differs from the true root usually by no more than 2 in the last digit retained. See T.M. pg. 36 (8).

To obtain a good approximation rapidly, start with an approximation even closer than the nearest integer. The tables of squares and square roots (Appendix) help in determining the first approximations for your roots.

The product and quotient properties of square roots, too, may be useful in finding the decimal approximation for roots of numbers less than 1 and numbers greater than 100. Notice how even powers of 10 are used in this example.

EXAMPLE 2. **Evaluate (a) $\sqrt{12996}$ and (b) $\sqrt{.012996}$.**

Solution:

a. $12996 = 10^4(1.2996)$; $\sqrt{12996} = 100\sqrt{1.2996}$

b. $.012996 = \left(\frac{1}{10^2}\right)(1.2996)$; $\sqrt{.012996} = \frac{1}{10}\sqrt{1.2996}$

Both solutions require the determination of $\sqrt{1.2996}$. To find this:

1. First approx. $= 1.1$
2. $1.2996 \div 1.1 = 1.181$
3. Second approx. $= \frac{2.281}{2}$

1. Second approx. $= 1.140$
2. $1.2996 \div 1.140 = 1.140$

$\therefore \sqrt{1.2996} = 1.14$, the true root.

Substituting this value in the expressions in (a) and (b) gives:

a. $\sqrt{12996} = 100(1.14) = 114$, **Answer.**

b. $\sqrt{.012996} = \frac{1}{10}(1.14) = .114$, **Answer.**

The next example shows how you can use the values in the table of square roots in solving some square root problems.

EXAMPLE 3. **Approximate the square roots of (a) .57 and (b) 570.**

Solution: **Make use of the table of square roots.**

a. $.57 = \frac{57}{100}$

$\sqrt{\frac{57}{100}} \doteq \frac{7.550}{10}$

$\therefore \sqrt{.57} \doteq .7550$,

b. $570 = 57(10)$

$\sqrt{57(10)} \doteq 7.550(3.162)$

$\sqrt{570} \doteq 23.87$, **Answer.**

ORAL EXERCISES

R= rational; I = irrational

Identify each expression as rational or irrational. Give the numbers represented by the rational ones.

SAMPLE. $\sqrt[3]{\frac{27}{8}}$ *What you say:* Rational; $\frac{3}{2}$

1. $\sqrt{4}$ R; 2
2. $-\sqrt{25}$ R; -5
3. $\sqrt{\frac{25}{16}}$ R; $\frac{5}{4}$
4. $3\sqrt{18}$ I
5. $\sqrt[3]{8}$ R; 2
6. $\sqrt{\frac{24}{6}}$ R; 2
7. $3\sqrt{81}$ R; 27
8. $\sqrt{.49}$ R; .7
9. $\sqrt[3]{\frac{9}{27}}$ I
10. $\sqrt{1.44}$ R; 1.2
11. $-\sqrt{.64}$ R; -.8
12. $\pm\sqrt{\frac{2}{18}}$ R; $\pm\frac{1}{3}$

13. $\sqrt{49} - \sqrt{81}$ R; -2
14. $\frac{\sqrt{13}}{\sqrt{13}}$ R; 1
15. $\sqrt{18} - \sqrt{18}$ R; 0
16. $-\sqrt{35} + \sqrt{35}$ R; 0
17. $\sqrt{121} - \sqrt{100}$ R; 1
18. $\frac{\sqrt{28}}{\sqrt{7}}$ R; 2

Name the integer closest to the square root of each number.

19. 30 5
20. 59 8
21. 11.2 3
22. 27.28 5
23. 3.61 2
24. 1.69 1
25. 14.4 4
26. 42.398 7

Express as the product of a number between 1 and 100 and a power of 100.

27. 800 $8 \cdot 100$
28. 1200 $12 \cdot 100$
29. 625.2 $(6.252) \cdot 100$
30. 397.4 $(3.974) \cdot 100$
31. 7734 $77.34 \cdot 100$
32. 8296 $82.96 \cdot 100$
33. 2500 $25 \cdot 100$
34. 3600 $36 \cdot 100$

Express as the quotient of a number between 1 and 100 and a power of 100.

35. .32 $32 \div 100$
36. .68 $68 \div 100$
37. .1391 $13.91 \div 100$
38. .467 $46.7 \div 100$
39. .5 $50 \div 100$
40. .7 $70 \div 100$
41. .000428 $4.28 \div (1$
42. .000331 $3.31 \div (10$

To compensate for every two places the decimal is moved to the left (right), multiply (divide) by 100.

WRITTEN EXERCISES

In Exercises 5-10, 15 ar 16, use product property of roots.

Find the indicated square roots.

A

1. $\sqrt{3.61}$
2. $\sqrt{4.41}$
3. $\sqrt{10.24}$
4. $\sqrt{30.25}$
5. $\sqrt{1764}$
6. $\sqrt{5329}$
7. $-\sqrt{316.84}$
8. $-\sqrt{408.04}$
9. $\sqrt{24{,}336}$
10. $\sqrt{110{,}889}$

Find each square root to the nearest hundredth.

11. $\sqrt{2}$
12. $\sqrt{3}$
13. $\sqrt{50.3}$
14. $\sqrt{10.7}$
15. $-\sqrt{356}$
16. $-\sqrt{279}$
17. $\sqrt{27}$
18. $\sqrt{35}$

Find both roots to the nearest tenth.

B

19. $x^2 = 960$
20. $y^2 = 405$
21. $t^2 - 3.2 = 0$
22. $u^2 - 12.6 = 0$
23. $800 = 5z^2$
24. $1000 = 8w^2$

25. Find $\sqrt{21}$ by taking $a = 4$ and by taking $a = 5$.

26. Find $\sqrt{90}$ by taking $a = 9$ and by taking $a = 10$.

Solve to the nearest tenth.

Ⓒ **27.** $.9p^2 = 1.062$

28. $(x + 1)^2 + (x - 1)^2 = 55$

29. $11y^2 - 36 = 0$

30. $9z^2 - 11 = 0$

31. If a is rational and $\sqrt{b}$ is irrational, prove that $a + \sqrt{b}$ is irrational.

32. Prove that $\sqrt[3]{n}$ may be irrational by an argument like that for $\sqrt{n}$.

PROBLEMS

If you wish, have pupils use Table 3, for square roots.

Find each answer to the nearest tenth, unless otherwise directed.

Ⓐ **1.** Find the side of a square whose area is 75 square inches.

2. The area of a square is 46 square feet. How long is one side?

3. The area of a circle is $A \doteq 3.14r^2$. Find the radius, correct to hundredths, of a circle whose area is 87.92 square centimeters.

4. The area of a circle is $A \doteq .7854d^2$. Find the diameter, correct to hundredths, of a circle whose area is 15.708 square centimeters.

5. Find the side of a square whose area is $\frac{35}{9}$ square meters.

6. The area of a square is $\frac{63}{16}$ square meters. Find its side.

7. A rectangle whose area is 225 square meters has a length three times its width. Find the length and width of this rectangle.

8. The length of a rectangle is twice its width. Its area is 1814 square centimeters. Find the dimensions of the rectangle.

11–5 Geometric Interpretation of Square Roots

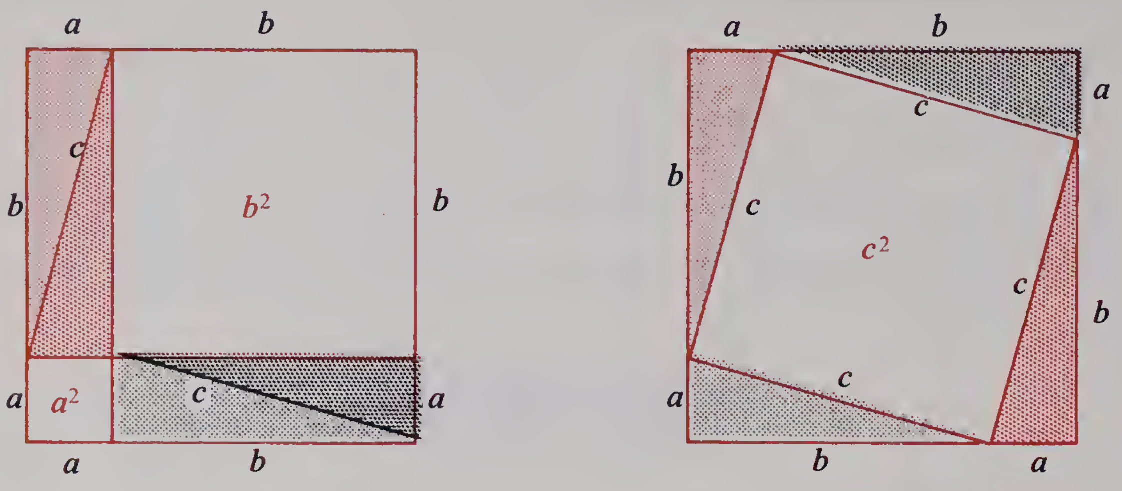
How can you locate irrational square roots on the number line without using approximations? Pythagoras proved the existence of distances which could not be measured by rational numbers in the ***Pythagorean theorem*** (pith-ag′uh-re-an **the**-uh-rem). (A *theorem* is a statement which can be proved.)

See T.M. pg. 37 (9).

In any right triangle the square of the length of the hypotenuse equals the sum of the squares of the lengths of the other two sides.

The **hypotenuse** of a right triangle is the longest side and is opposite the right angle.

Figure 11–1 illustrates the Pythagorean theorem: $c^2 = a^2 + b^2$, where c is the length of the hypotenuse, and a and b are the lengths of the other two sides.

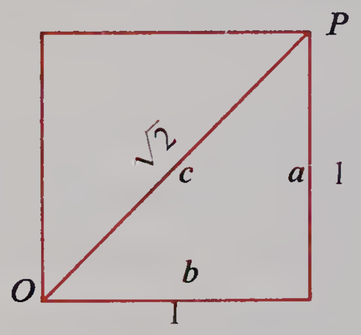

a
b
c
b
b²
b
a
a²
c
a
a
b
a
b
a
c
b
c
c²
c
b
a
c
b
a

· *Figure 11–1* ·

To find a length equal to $\sqrt{2}$, draw a square whose sides are 1 (Figure 11–2). The diagonal OP divides it into two right triangles in which $a = 1$ and $b = 1$.

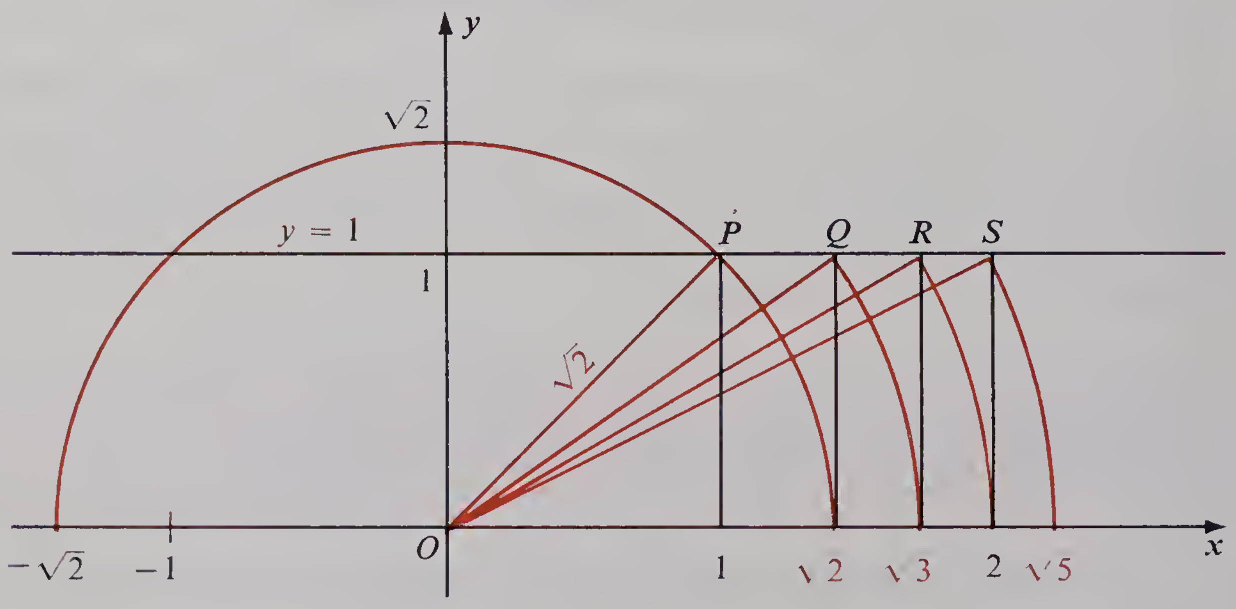
Use this fact: if $r^2 = s^2$, and neither r nor s is negative, then $r = s$.

$$
\begin{aligned}
c^2 &= a^2 + b^2\\
c^2 &= 1^2 + 1^2\\
c^2 &= 1 + 1\\
c^2 &= 2\\
c &= \sqrt{2}
\end{aligned}
$$



· *Figure 11–2* ·

Figure 11–3 combines this square with the number axes. The semicircle has the origin as its center and $\sqrt{2}$ as its radius.


y
√2
y = 1
P
Q
R
S
1
√2
−√2
−1
O
1
√2
√3
2
√5
x

· *Figure 11–3* ·

Points Q, R, and S are on the line $y = 1$ at distances of $\sqrt{3}$, $\sqrt{4}$, and $\sqrt{5}$ from the origin. Each can be found by using the previously constructed square root and drawing perpendiculars to form new right triangles. If OQ, OR, and OS represent lengths, then:

$$\begin{array}{lll}
(OQ)^2 = (\sqrt{2})^2 + 1^2 & (OR)^2 = (\sqrt{3})^2 + 1^2 & (OS)^2 = (\sqrt{4})^2 + 1^2 \\
(OQ)^2 = 2 + 1 & (OR)^2 = 3 + 1 & (OS)^2 = 4 + 1 \\
(OQ)^2 = 3 & (OR)^2 = 4 & (OS)^2 = 5 \\
(OQ) = \sqrt{3} & (OR) = \sqrt{4} = 2 & (OS) = \sqrt{5}
\end{array}$$

EXAMPLE. **Is a triangle whose sides are 3, 4, and 5 a right triangle?**

Solution:

$$c^2 = a^2 + b^2$$
$$5^2 \overset{?}{=} 3^2 + 4^2$$
$$25 = 25 \checkmark$$

Watch for this error: $\sqrt{3^2 + 4^2} = 3 + 4$. See T.M. pg. 37 (10).

A 3-4-5 triangle is a right triangle, Answer.

WRITTEN EXERCISES

Determine whether or not each is a right triangle.

Ⓐ **1.** The three sides are 6, 8, and 10 inches.

2. The sides are 9 feet, 12 feet, and 15 feet.

3. The sides are 48, 52, and 20 centimeters.

4. The sides are 45, 51, and 24 centimeters.

In each right triangle find the missing dimension to the nearest hundredth.

5. $a = 15$ inches; $b = 20$ inches

6. $a = 32$ meters; $b = 24$ meters

7. $a = 6$ miles; $b = 1\frac{3}{4}$ miles

8. $a = 2\frac{6}{7}$ yards; $b = 3$ yards

9. $b = 9$ meters; $c = 41$ meters

10. $a = 10$ feet; $c = 25$ feet

PROBLEMS

If desired, have pupils use Table 3, for square roots.

Make a sketch for each problem. Work to 2 decimal places.

Ⓐ **1.** If the bottom of a 17-foot ladder is 8 feet from a wall, how high on the wall does it reach?

2. A wire from the top of a telephone pole to a point on the ground 16 feet from the pole is thirty-four feet long. How high is the pole?

3. A rectangular lot is 80 feet long and 50 feet wide. How long is a straight line from one corner to the corner diagonally opposite?

4. Two sides of a triangular bracket are 16 inches long. How long is the third side?

B 5. A right triangle has sides whose lengths in inches are expressed by consecutive even integers. Find the length of each side.

6. The sides of one right triangle, when expressed in feet, are given by consecutive integers. Find the length of each side.

7. The sides of one right triangle have a ratio of 3:4. The hypotenuse is 35 feet in length. Find the length of each side.

8. A man drives 30 miles north, then 5 west, then south to a point 15 miles northwest of his starting point. How far south does he drive?

C 9. A one-way tunnel entrance is a rectangle 10′ w. × 14′ h. surmounted by a semicircular arch. How high a truck, 7 ft. wide, can enter?

10. Find the length of the diagonal of a box 24″ × 18″ × 15″ deep.

Have a student write these square-root properties on the board.

RADICAL EXPRESSIONS

11–6 Multiplication, Division, and Simplification of Radicals

The product and quotient properties of square roots (page 404) together with the commutative and associative properties enable you to multiply, divide, and simplify square-root radicals quickly.

Recall: $\frac{a}{b} = \frac{a}{b} \cdot \frac{c}{c} = \frac{ac}{ac}$ since $\frac{c}{c} = 1.$

$$\sqrt{7} \cdot \sqrt{3} = \sqrt{21} \qquad \sqrt{3} \cdot \sqrt{27} = \sqrt{81} = 9$$

$$(2\sqrt{5})(4\sqrt{2}) = (2 \cdot 4)(\sqrt{5} \cdot \sqrt{2}) = 8\sqrt{10}$$

$$\frac{\sqrt{10}}{\sqrt{2}} = \sqrt{\frac{10}{2}} = \sqrt{5} \qquad \sqrt{\frac{11}{75}} = \sqrt{\frac{11}{75} \cdot \frac{3}{3}} = \sqrt{\frac{33}{225}} = \frac{\sqrt{33}}{15}$$

An expression having a square-root *radical* is in simplest form when

1. no integral radicand has a square factor other than 1,
2. no fractions are under a radical sign, and
3. no radicals are in a denominator.

$$\sqrt{20} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5} \quad \text{and} \quad 3\sqrt{96} = 3\sqrt{16 \cdot 6} = 12\sqrt{6}$$

$$\sqrt{\frac{5}{6}} = \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{5}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{30}}{6}; \qquad \frac{1}{\sqrt{2}} = \frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\frac{2\sqrt{3}}{3\sqrt{8}} = \frac{2\sqrt{3}}{3\sqrt{8}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{6}}{3\sqrt{16}} = \frac{2\sqrt{6}}{3 \cdot 4} = \frac{\sqrt{6}}{6}$$

The process of changing the form of a fraction with an irrational denominator to an equal fraction with a rational denominator is called **rationalizing the denominator**. Rationalizing the denominator of a radical expression helps in approximating its value. It is easier to approximate $\frac{1}{\sqrt{2}}$ as $\frac{1.414}{2}$ than as $\frac{1}{1.414}$, given $\sqrt{2} \doteq 1.414$.

ORAL EXERCISES

Express in simplest form.

1. $\sqrt{5}\cdot\sqrt{2}$ $\sqrt{10}$
2. $\sqrt{5}\cdot\sqrt{3}$ $\sqrt{15}$
3. $\frac{\sqrt{28}}{\sqrt{7}}$ 2
4. $\frac{\sqrt{27}}{\sqrt{3}}$ 3
5. $\sqrt{5}\cdot 8\sqrt{3}$ $8\sqrt{15}$
6. $4\sqrt{3}\cdot\sqrt{7}$ $4\sqrt{21}$
7. $\frac{\sqrt{50}}{\sqrt{2}}$ 5
8. $\frac{\sqrt{80}}{\sqrt{5}}$ 4
9. $3\sqrt{2}\cdot 2\sqrt{8}$ 24
10. $4\sqrt{2}\cdot 7\sqrt{3}$ $28\sqrt{6}$
11. $\frac{\sqrt{15}}{\sqrt{3}}$ $\sqrt{5}$
12. $\frac{\sqrt{21}}{\sqrt{3}}$ $\sqrt{7}$
13. $(3\sqrt{10})^2$ 90
14. $(2\sqrt{3})^2$ 12
15. $\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}}$ $\frac{1}{2}$
16. $\sqrt{\frac{1}{3}}\cdot\sqrt{\frac{1}{3}}$ $\frac{1}{3}$

17. $\sqrt{32}$ $4\sqrt{2}$
18. $\sqrt{500}$ $10\sqrt{5}$
19. $\sqrt{\frac{2}{3}}$ $\frac{1}{3}\sqrt{6}$
20. $\frac{1}{\sqrt{3}}$ $\frac{1}{3}\sqrt{3}$
21. $\sqrt{27}$ $3\sqrt{3}$
22. $\sqrt{12}$ $2\sqrt{3}$
23. $\sqrt{\frac{3}{2}}$ $\frac{1}{2}\sqrt{6}$
24. $\frac{2}{\sqrt{14}}$ $\frac{\sqrt{14}}{7}$
25. $\sqrt{72}$ $6\sqrt{2}$
26. $\sqrt{24}$ $2\sqrt{6}$
27. $\sqrt{\frac{1}{8}}$ $\frac{1}{4}\sqrt{2}$
28. $\sqrt{\frac{1}{x}}$ $\frac{1}{x}\sqrt{x}$
29. $\sqrt{45}$ $3\sqrt{5}$
30. $\sqrt{48}$ $4\sqrt{3}$
31. $\sqrt{\frac{7}{18}}$ $\frac{1}{6}\sqrt{14}$
32. $\sqrt{\frac{1}{x^3}}$ $\frac{1}{x^2}\sqrt{x}$
33. $\sqrt{28}$ $2\sqrt{7}$
34. $\sqrt{80}$ $4\sqrt{5}$
35. $\sqrt{\frac{3}{40}}$ $\frac{1}{20}\sqrt{30}$
36. $\frac{5}{\sqrt{a}}$ $\frac{5}{a}\sqrt{a}$

Give two factors, one of which is the largest possible perfect square.

SAMPLE. 75 *What you say:* 25 times 3

37. 300 $100\cdot 3$
38. 50 $25\cdot 2$
39. 250 $25\cdot 10$
40. 90 $9\cdot 10$
41. 160 $16\cdot 10$
42. 360 $36\cdot 10$
43. 162 $81\cdot 2$
44. 128 $64\cdot 2$
45. 125 $25\cdot 5$
46. 44 $4\cdot 11$

WRITTEN EXERCISES

Express in simplest form. (Assume that all given radicands are nonnegative real numbers.)

A
1. $3\sqrt{17}\cdot 8\sqrt{17}$
2. $5\sqrt{26}\cdot 13\sqrt{26}$
3. $\sqrt{3}\cdot\sqrt{2}\cdot\sqrt{6}$
4. $2\sqrt{3}\cdot\sqrt{7}\cdot\sqrt{2}$
5. $\sqrt{\frac{3}{5}}\cdot\sqrt{\frac{5}{3}}$
6. $2\sqrt{\frac{3}{5}}\cdot 3\sqrt{\frac{5}{4}}$
7. $\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{3}{5}}$
8. $\frac{2}{3}\sqrt{\frac{5}{7}}\cdot\frac{1}{5}\sqrt{\frac{7}{20}}$
9. $\sqrt{8\frac{2}{5}}\cdot\sqrt{4\frac{2}{7}}$
10. $\sqrt{7\frac{1}{2}}\cdot\sqrt{3\frac{1}{3}}$
11. $(-5\sqrt{a})(2\sqrt{a})$
12. $(3\sqrt{b})(-4\sqrt{b})$

13. $\dfrac{15\sqrt{10}}{5\sqrt{2}}$

14. $\dfrac{3\sqrt{22}}{6\sqrt{2}}$

15. $\sqrt{162}$

16. $2\sqrt{18}$

17. $4\sqrt{98}$

18. $\dfrac{\sqrt{75}}{\sqrt{3}}$

19. $\dfrac{\sqrt{2}}{\sqrt{98}}$

20. $\frac{3}{4}\sqrt{12}$

21. $6\sqrt{\frac{27}{4}}$

22. $6\sqrt{\frac{20}{9}}$

23. $\dfrac{21\sqrt{5}}{7\sqrt{25}}$

24. $\dfrac{\sqrt{a^5}}{\sqrt{a}}$

25. $12\sqrt{\frac{27}{2}}$

26. $4\sqrt{\frac{5}{8}}$

27. $6\sqrt{\frac{11}{12}}$

28. $\dfrac{\sqrt{3y}}{\sqrt{y}}$

29. $\dfrac{\sqrt{60}}{3\sqrt{6}}$

30. $\sqrt{72c^2}$

31. $\sqrt{40d^5}$

32. $3\sqrt{90k^3}$

B

33. $(-5\sqrt{ab^2})(-3\sqrt{a})$

34. $(-7\sqrt{c^2d})(2\sqrt{d})$

35. $\sqrt{x}(\sqrt{x} - 3)$

36. $\sqrt{y}(2 - \sqrt{y})$

37. $(\sqrt{6})(-2\sqrt{3})(\sqrt{8})$

38. $(-5\sqrt{14})(-\sqrt{7})(\sqrt{8})$

39. $(\sqrt{21y})(\sqrt{y})(\sqrt{7})$

40. $(\sqrt{x})(\sqrt{3x})(\sqrt{24})$

41. $(2a\sqrt{5a})^2$

42. $(3b\sqrt{2b})^2$

43. $2\sqrt{6}(3\sqrt{2} + \sqrt{3})$

44. $5\sqrt{10}(2\sqrt{5} - \sqrt{15})$

45. $-6\sqrt{1\frac{1}{6}}$

46. $5\sqrt{1\frac{2}{5}}$

47. $-5\sqrt{2.4}$

48. $6.47\sqrt{.35}$

49. $\dfrac{2\sqrt{7} + \sqrt{14}}{\sqrt{7}}$

50. $\dfrac{50\sqrt{34}}{5\sqrt{17} - 3\sqrt{17}}$

51. $\dfrac{5\sqrt{2} + \sqrt{8}}{\sqrt{2}}$

52. $\dfrac{\sqrt{3} - \sqrt{12}}{\sqrt{3}}$

53. $\dfrac{7\sqrt{a^3} + \sqrt{a}}{\sqrt{a}}$

54. $\dfrac{\sqrt{b} - 4\sqrt{b^3}}{\sqrt{b}}$

PROBLEMS

Find answers to the nearest tenth, unless otherwise directed.

B

1. One number is three times another. The difference of their squares is $\frac{3}{4}$. Find the numbers.

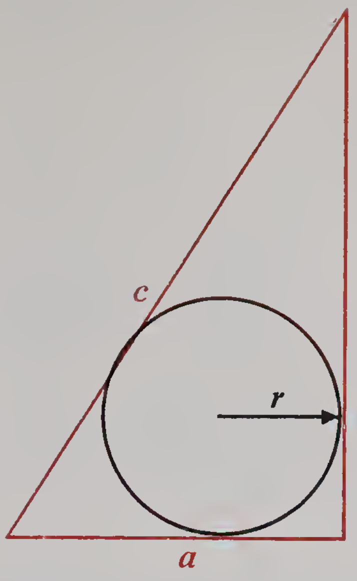
2. Find two numbers in the ratio 4 to 3, whose squares differ by 12.

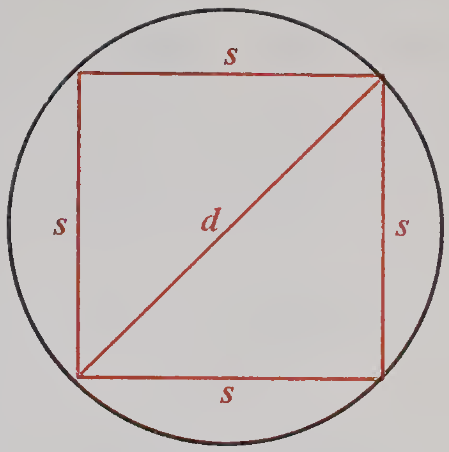
3. The dimensions of a rectangle are in the ratio 3 to 2. Its area is 128 square inches. Find its dimensions.

4. The area of a square is 70 square inches. How long is its diagonal?

5. Find the area of a triangle whose sides are 30, 60, and 70 centimeters long, by the rule $A = \sqrt{s(s - a)(s - b)(s - c)}$, where a, b, and c are the lengths of the sides and s is half the perimeter.

6. Find the radius r of the circle inscribed in the right triangle shown, by the rule $r = \frac{1}{2}(a + b - c)$. The shorter sides of the triangle have lengths of .5 meter and .8 meter. (Sketch p. 417 top left)





Ⓒ **7.** A square of side s is inscribed in a circle of diameter d, as shown above. If $d = 4.6$, find s.

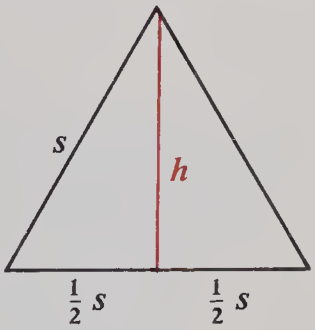
8. A square whose area is 25 square inches is inscribed in a circle of diameter d. Find the radius of the circle.

9. Will a square whose area is 25 square inches fit inside a circle whose area is 77 square inches? Support your answer by calculations.

10. Show that an equilateral triangle, whose side is 4 inches, is smaller in area than a square inscribed in a circle whose diameter is 4 inches.

11. The altitude h divides the equilateral triangle of side s into two identical right triangles, as shown. Find h, if $s = 2$.

12. Find the length of the side s of an equilateral triangle of altitude h, if $h = 6$.



11–7 Addition and Subtraction of Radicals

Because $6\sqrt{2}$ and $5\sqrt{2}$ have the common factor $\sqrt{2}$, you can simplify their sum by using the distributive property.

$$6\sqrt{2} + 5\sqrt{2} = (6 + 5)\sqrt{2} = 11\sqrt{2}$$

The analogy with 6x + 5x is helpful.

The sum or difference of square-root radicals having the same radicand is the sum or difference of the coefficients of the radicals, multiplied by the common radical. On the other hand, the addition or subtraction of radicals having unlike radicands can only be indicated.

$$5\sqrt{6} - 8\sqrt{5} + 2\sqrt{6} + 6\sqrt{5} = 7\sqrt{6} - 2\sqrt{5}$$

Compare with 7x -2y.

By reducing each radical to simplest form, you sometimes can combine terms in a sum of radicals.

EXAMPLE. Simplify $-2\sqrt{48} + 7\sqrt{3} + 2\sqrt{147}$.

Solution:

$$\begin{aligned}-2\sqrt{48} + 7\sqrt{3} + 2\sqrt{147} &= -2\sqrt{16 \cdot 3} + 7\sqrt{3} + 2\sqrt{49 \cdot 3}\\ &= -2(4\sqrt{3}) + 7\sqrt{3} + 2(7\sqrt{3})\\ &= -8\sqrt{3} + 7\sqrt{3} + 14\sqrt{3}\\ &= 13\sqrt{3}, \text{ Answer.}\end{aligned}$$

To add or subtract square-root radicals:

1. Express each radical in simplest form.
2. By the distributive property, combine radicals with like radicands.
3. Indicate the sum or difference of radicals with unlike radicands.

ORAL EXERCISES

From each group select the radicals having the same radicand.

1. [✓ $\sqrt{2}$], [✓ $3\sqrt{2}$], $2\sqrt{3}$
2. [✓ $\sqrt{10}$], [✓ $2\sqrt{10}$], $\sqrt{20}$
3. $\sqrt{5}$, [✓ $\sqrt{17}$], [✓ $5\sqrt{17}$]
4. [✓ $\sqrt{11}$], $11\sqrt{3}$, [✓ $3\sqrt{11}$]
5. $6\sqrt{5}$, [✓ $5\sqrt{7}$], [✓ $3\sqrt{7}$], $5\sqrt{6}$
6. $2\sqrt{10}$, [✓ $7\sqrt{6}$], $10\sqrt{2}$, [✓ $2\sqrt{6}$]

Combine these radicals.

7. $2\sqrt{2} + 3\sqrt{2}$ $5\sqrt{2}$
8. $5\sqrt{3} - 2\sqrt{3}$ $3\sqrt{3}$
9. $4\sqrt{5} + \sqrt{5}$ $5\sqrt{5}$
10. $\sqrt{7} + 8\sqrt{7}$ $9\sqrt{7}$
11. $6\sqrt{11} - 3\sqrt{11} - 4\sqrt{11}$ $-\sqrt{11}$
12. $2\sqrt{13} + 4\sqrt{13} - 9\sqrt{13}$ $-3\sqrt{13}$
13. $6\sqrt{15} - 8\sqrt{15} + \sqrt{15}$ $-\sqrt{15}$
14. $3\sqrt{10} - 9\sqrt{10} + 2\sqrt{10}$ $-4\sqrt{10}$
15. $2\sqrt{a} + \sqrt{a} - 3\sqrt{a}$ 0
16. $\sqrt{c} - 4\sqrt{c} + \sqrt{c}$ $-2\sqrt{c}$

WRITTEN EXERCISES

Combine these radicals. (Assume that all given radicands are nonnegative real numbers.)

A

1. $3\sqrt{5} + 5\sqrt{5} - 2\sqrt{5}$
2. $2\sqrt{11} - 6\sqrt{11} - 5\sqrt{11}$
3. $\sqrt{2} - \sqrt{3} + 5\sqrt{2} + 5\sqrt{3}$
4. $\sqrt{5} + \sqrt{2} - 3\sqrt{5} + 2\sqrt{2}$

5. $\sqrt{2} - \sqrt{8}$
6. $\sqrt{12} + \sqrt{3}$
7. $\sqrt{2} - \sqrt{\frac{1}{2}}$
8. $\sqrt{6} + \sqrt{\frac{2}{3}}$
9. $2\sqrt{32} - 3\sqrt{50}$
10. $2\sqrt{50} - 3\sqrt{18}$

11. $3\sqrt{63} + \frac{1}{4}\sqrt{28}$
12. $2\sqrt{150} - \frac{3}{8}\sqrt{96}$
13. $4\sqrt{\frac{5}{2}} - 10\sqrt{\frac{1}{10}}$
14. $10\sqrt{\frac{3}{5}} + 30\sqrt{\frac{5}{3}}$
15. $\sqrt{3} - 4\sqrt{48} + 3\sqrt{75}$
16. $\sqrt{7} + 4\sqrt{28} - 3\sqrt{63}$

Ⓑ **17.** $\sqrt{3} + 2\sqrt{27} - 6\sqrt{\frac{1}{3}}$
18. $3\sqrt{10} - 4\sqrt{90} + 5\sqrt{\frac{1}{10}}$
19. $12\sqrt{\frac{2}{3}} - 2\sqrt{1\frac{1}{2}} + \sqrt{54}$

20. $10\sqrt{\frac{2}{5}} + \sqrt{\frac{5}{2}} - \frac{1}{4}\sqrt{40}$
21. $\sqrt{\frac{3}{4}} + \sqrt{\frac{4}{3}} - \frac{1}{6}\sqrt{75}$
22. $6\sqrt{\frac{5}{12}} + \sqrt{\frac{12}{5}} - \frac{1}{5}\sqrt{60}$

Ⓒ **23.** $2\sqrt{12x} - \dfrac{2x^2}{3}\sqrt{\dfrac{108}{x}}$
24. $a^5\sqrt{\dfrac{c}{a^3}} + 2\sqrt{ca^7}$

25. $\sqrt{\dfrac{n^2}{64} + \dfrac{n^2}{36}}$
26. $\sqrt{\dfrac{t^2}{9} - \dfrac{t^2}{25}}$

Solve each equation.

27. $a\sqrt{2} - \sqrt{6} = \sqrt{24} - a\sqrt{8}$
28. $\sqrt{90} - b\sqrt{45} = b\sqrt{80} - \sqrt{1210}$
29. $\sqrt{x} + 3\sqrt{x} = 6 - 2\sqrt{x}$
30. $2(1 + \sqrt{y}) = 7\sqrt{y} - 4\sqrt{y}$

11–8 Multiplication of Binomials Containing Radicals

Sometimes in dealing with radicals, you may wish to find a product like $(3 + \sqrt{7})(3 - \sqrt{7})$. Do you see that this indicated product resembles $(a + b)(a - b) = a^2 - b^2$? Two binomials of the form $x + \sqrt{y}$ and $x - \sqrt{y}$ are called *conjugates* of each other. They differ only in the sign between the two terms, and their product is a rational number, as this example shows.

EXAMPLE 1. $(3 + \sqrt{7})(3 - \sqrt{7})$

Pupils should be encouraged to generalize: $(r + \sqrt{s})(r - \sqrt{s}) = r^2 - s$; also, $(\sqrt{t} + v)(\sqrt{t} - v) = t - v^2$, etc.

Solution:

$$\begin{aligned}(3 + \sqrt{7})(3 - \sqrt{7}) &= 3^2 - (\sqrt{7})^2\\ &= 9 - 7\\ &= 2, \text{ Answer.}\end{aligned}$$

Consider the product $(2 + \sqrt{5})(2 + \sqrt{5})$, whose form is $(a + b)(a + b) = a^2 + 2ab + b^2$.

EXAMPLE 2. $(2 + \sqrt{5})(2 + \sqrt{5})$

Solution: $(2 + \sqrt{5})(2 + \sqrt{5}) = 2^2 + 2(2\sqrt{5}) + (\sqrt{5})^2$
$= 4 + 4\sqrt{5} + 5 = 9 + 4\sqrt{5}$, **Answer.**

Binomial multiplication by the conjugate of the denominator can help you *rationalize a denominator.*

EXAMPLE 3. **Rationalize the denominator:** $\dfrac{1}{2\sqrt{3} + 1}$.

Solution: **Multiply the numerator and denominator by $2\sqrt{3} - 1$.**

Stress that you use a multiplier such that the new denominator will not contain a radical.

$$\frac{1}{2\sqrt{3}+1} = \frac{1(2\sqrt{3}-1)}{(2\sqrt{3}+1)(2\sqrt{3}-1)} = \frac{2\sqrt{3}-1}{(2\sqrt{3})^2 - 1^2}$$

$$= \frac{2\sqrt{3}-1}{12-1} = \frac{2\sqrt{3}-1}{11}, \text{ Answer.}$$

WRITTEN EXERCISES

Express in simplest form.

A
1. $(2 + \sqrt{3})(2 - \sqrt{3})$
2. $(4 - \sqrt{5})(4 + \sqrt{5})$
3. $(\sqrt{2} - \sqrt{3})(\sqrt{2} + \sqrt{3})$
4. $(\sqrt{7} + \sqrt{6})(\sqrt{7} - \sqrt{6})$
5. $(2\sqrt{3} - 5)(2\sqrt{3} + 3)$
6. $(5\sqrt{2} - 4)(5\sqrt{2} + 1)$
7. $(1 + \sqrt{7})^2$
8. $(5 - \sqrt{10})^2$
9. $(5\sqrt{2} - 1)^2$
10. $(3\sqrt{7} + 2)^2$
11. $2\sqrt{6}(3\sqrt{2} + \sqrt{3})$
12. $5\sqrt{10}(2\sqrt{5} - \sqrt{15})$

B
13. $(4\sqrt{3} + 1)(2\sqrt{3} - 3)$
14. $(5\sqrt{7} - 2)(\sqrt{7} + 2)$
15. $(2\sqrt{6} - \sqrt{3})(\sqrt{6} + 3\sqrt{3})$
16. $(6\sqrt{15} + \sqrt{5})(2\sqrt{15} - 3\sqrt{5})$

Rationalize the denominator of each fraction.

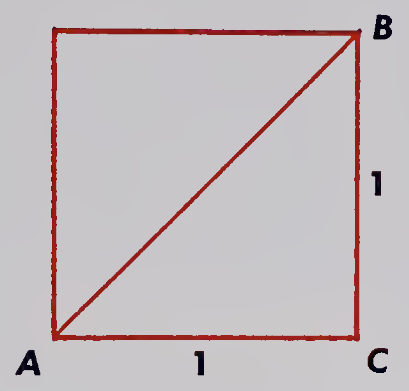
17. $\dfrac{1}{\sqrt{5} - 1}$
18. $\dfrac{1}{\sqrt{7} + 1}$
19. $\dfrac{\sqrt{2}}{\sqrt{2} + 3}$
20. $\dfrac{\sqrt{6}}{5 - \sqrt{6}}$
21. $\dfrac{3 - \sqrt{5}}{2 - \sqrt{5}}$
22. $\dfrac{\sqrt{6} - 1}{3 + \sqrt{6}}$
23. $\dfrac{2}{3\sqrt{2} - 2}$
24. $\dfrac{6}{3 + 2\sqrt{3}}$

(continued on page 421)

The insert opposite helps pupils to understand the significance of an infinite decimal and to see the inadequacy of the system of rational numbers. Together with the first insert, this one provides a picture of the development of the real number system.

The Real Number System

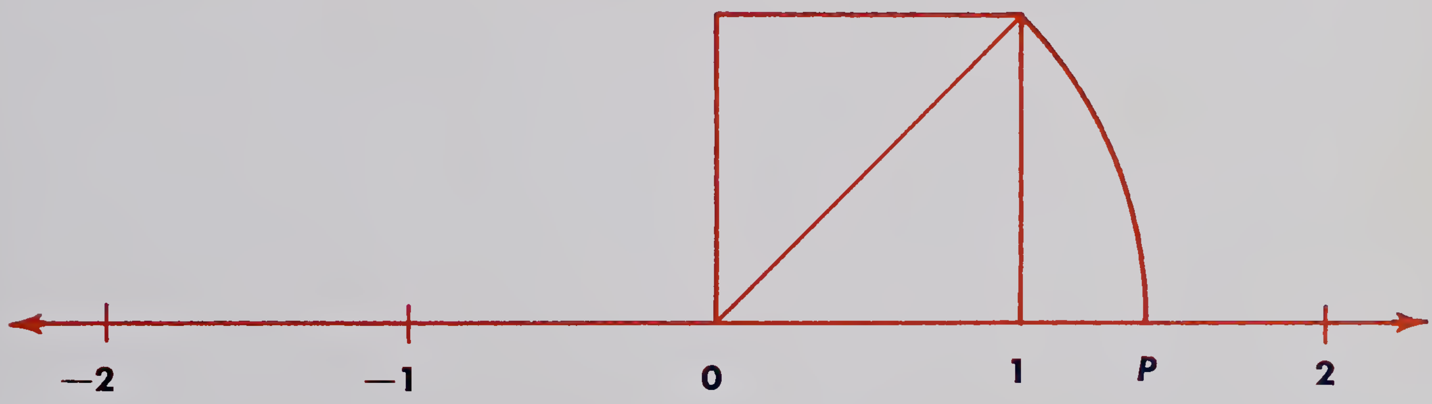
In the system of rational numbers you can add, subtract, multiply, and divide, except by zero. Furthermore, these operations satisfy all the familiar rules of arithmetic. Therefore, since you can use rational numbers for counting and for measuring to any desired degree of accuracy, it might seem that these numbers are entirely adequate for all possible applications.



$\overline{AB} = \sqrt{1^2 + 1^2} = \sqrt{2}$

As a matter of fact, the system of rational numbers is insufficient for many uses in mathematics. Can you solve the equation $x^2 = 2$? You cannot if the replacement set of x is just the set of rational numbers! The Greek mathematician Pythagoras is credited with having made the remarkable discovery that there is no rational number whose square is 2.

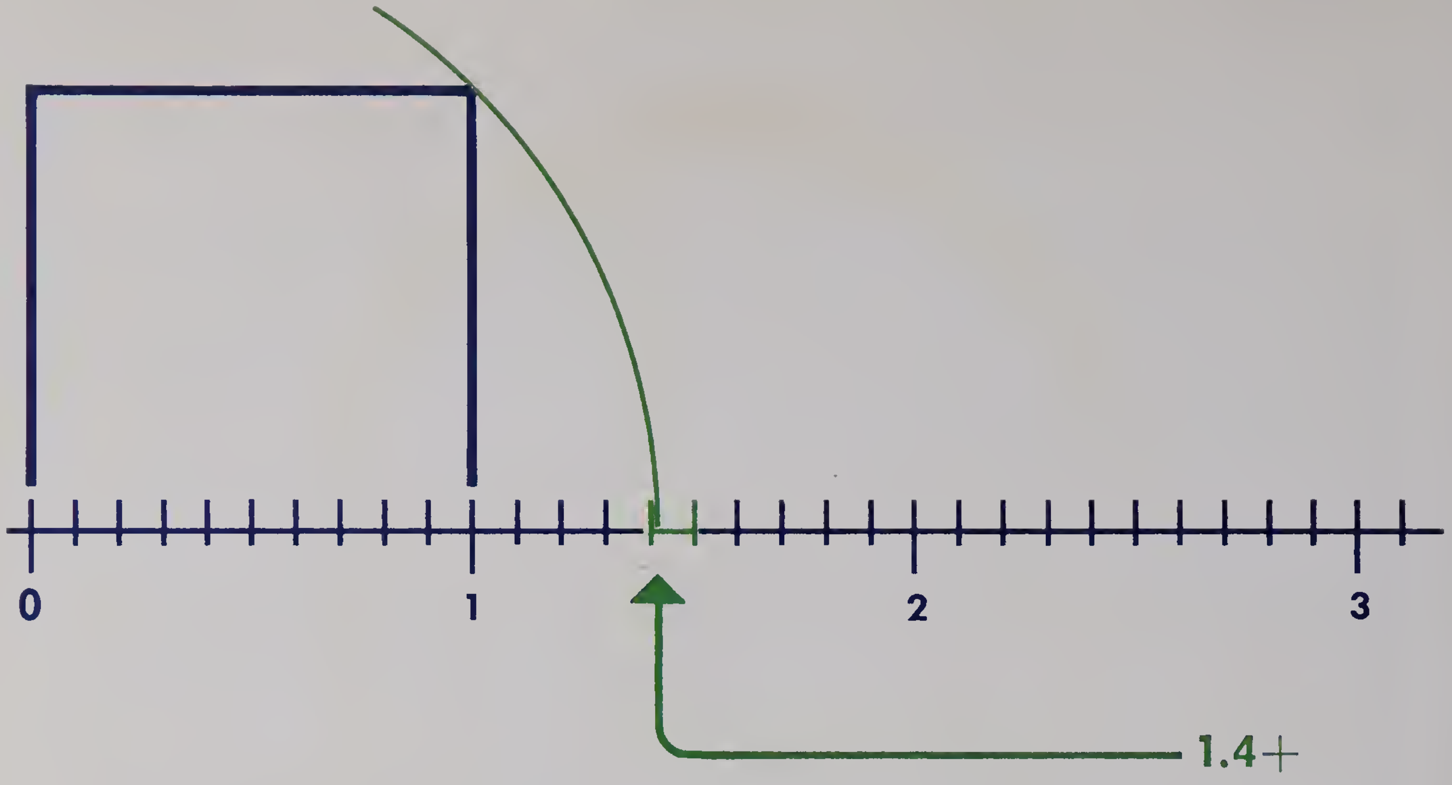
This means that there is no rational number that measures the length of the diagonal of a square having sides one unit in length. By using a scale to measure the diagonal, you may find a rational number such as 1.4 or 1.41 to *approximate* the length, but you will find no rational number that measures the length *exactly*. Thus, on the number line the point P cannot be paired with a rational number.



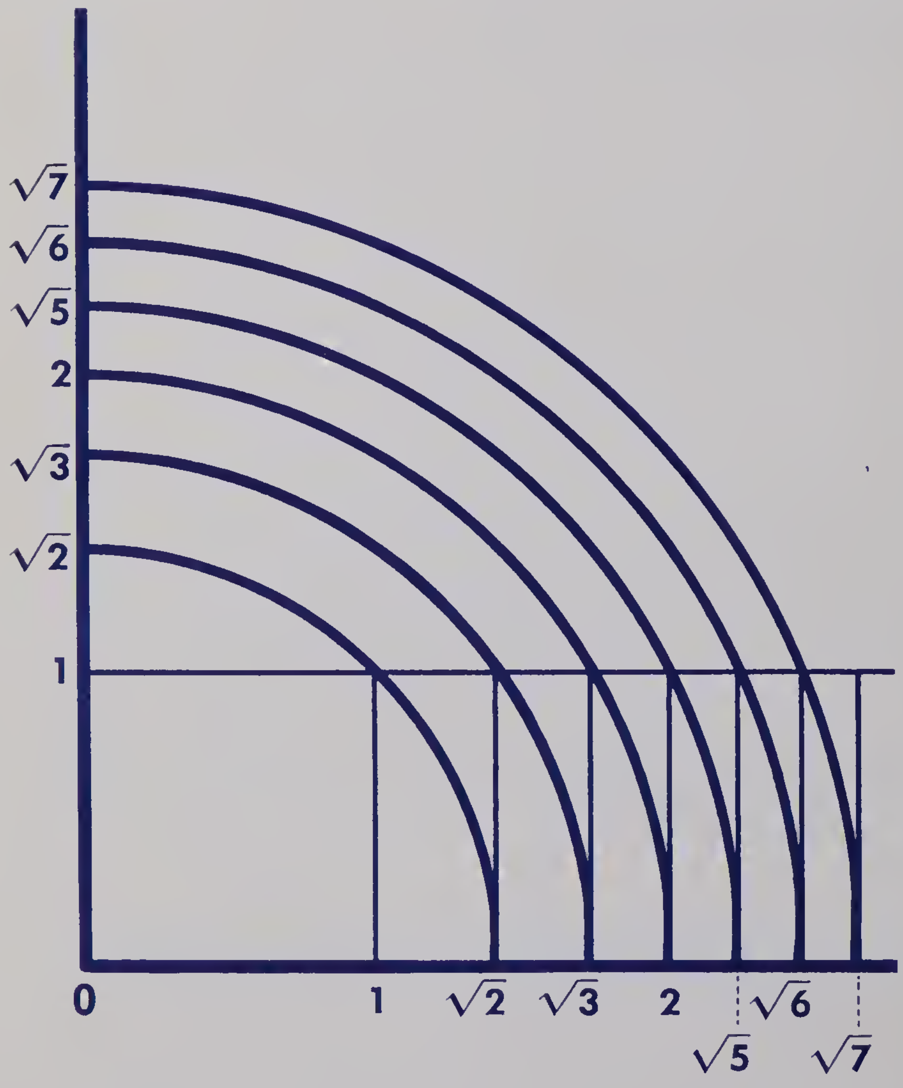
Just as the omission of this point would cause a gap in the line, the absence of a positive number whose square is 2 (symbolized by $\sqrt{2}$) produces a gap in the set of rational numbers. To fill such gaps in the number system, the irrational numbers were invented. A formal way of stating that the set made up of the rational and irrational numbers has no gaps in it is to say that it possesses the property of completeness.

The familiar long division process enables you to express any rational number as a terminating or a repeating decimal; for example, $\frac{5}{4} = 1.25$ and $\frac{2}{15} = .133\ldots = .1\overline{3}$.

Since terminating and repeating decimals always represent rational numbers, the decimal for an irrational number can neither terminate nor repeat. A process for finding successive digits in the decimal representation of $\sqrt{2}$ is illustrated in the sequence of drawings on the upper halves of the following pages. The drawings on the lower halves of these pages are a separate sequence illustrating methods of finding points on the number line corresponding to some other irrational numbers.



On the number line the point associated with $\sqrt{2}$ lies between 1 and 2 because $1^2 < 2 < 2^2$. By subdividing this unit interval into ten equal parts each of length .1, you can see that $1.4 < \sqrt{2} < 1.5$.



The diagram on the left shows the construction for locating on the axes the points associated with the square roots of consecutive integers. Both of the points marked $\sqrt{2}$ are on the circumference of the circle with center at the origin and radius equal to the diagonal of the unit square.

To locate $\sqrt{3}$ draw the circle whose radius coincides with the diagonal of the rectangle whose horizontal side is $\sqrt{2}$ units long, and whose vertical side is 1 unit long. The successive square roots are located by repeating this process with a rectangle of unit height based on the line segment whose length is given by the preceding square root.

$$\sqrt{2} = \sqrt{1^2 + 1^2}$$

$$\sqrt{3} = \sqrt{(\sqrt{2})^2 + 1^2}$$

$$\sqrt{4} = \sqrt{(\sqrt{3})^2 + 1^2}$$

. . .

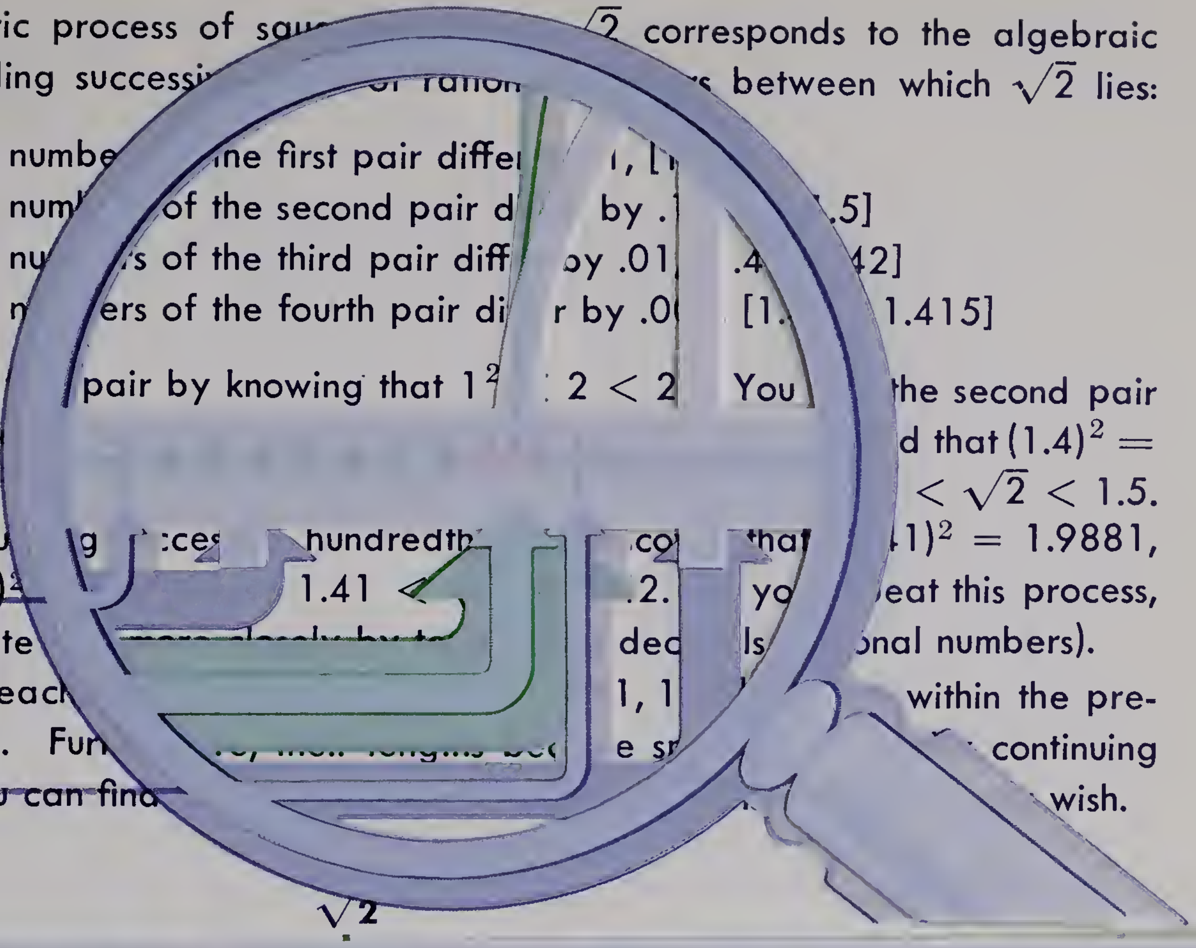
The geometric process of squ $\sqrt{2}$ corresponds to the algebraic process of finding successi ration s between which $\sqrt{2}$ lies:

The numbe ne first pair differ 1, [1
The num of the second pair d by .1 .5]
The nu s of the third pair diff by .01 .4 42]
The n ers of the fourth pair di r by .00 [1. 1.415]

You find the pair by knowing that 1^2 $2 < 2$ You the second pair by computing s d that $(1.4)^2 =$ 1.96 and (1.5) $< \sqrt{2} < 1.5$.

Next, by squ g cces hundredth co that $1)^2 = 1.9881$, and th $2)^2$ 1.41 .2. yo eat this process, you approximate dec ls onal numbers).

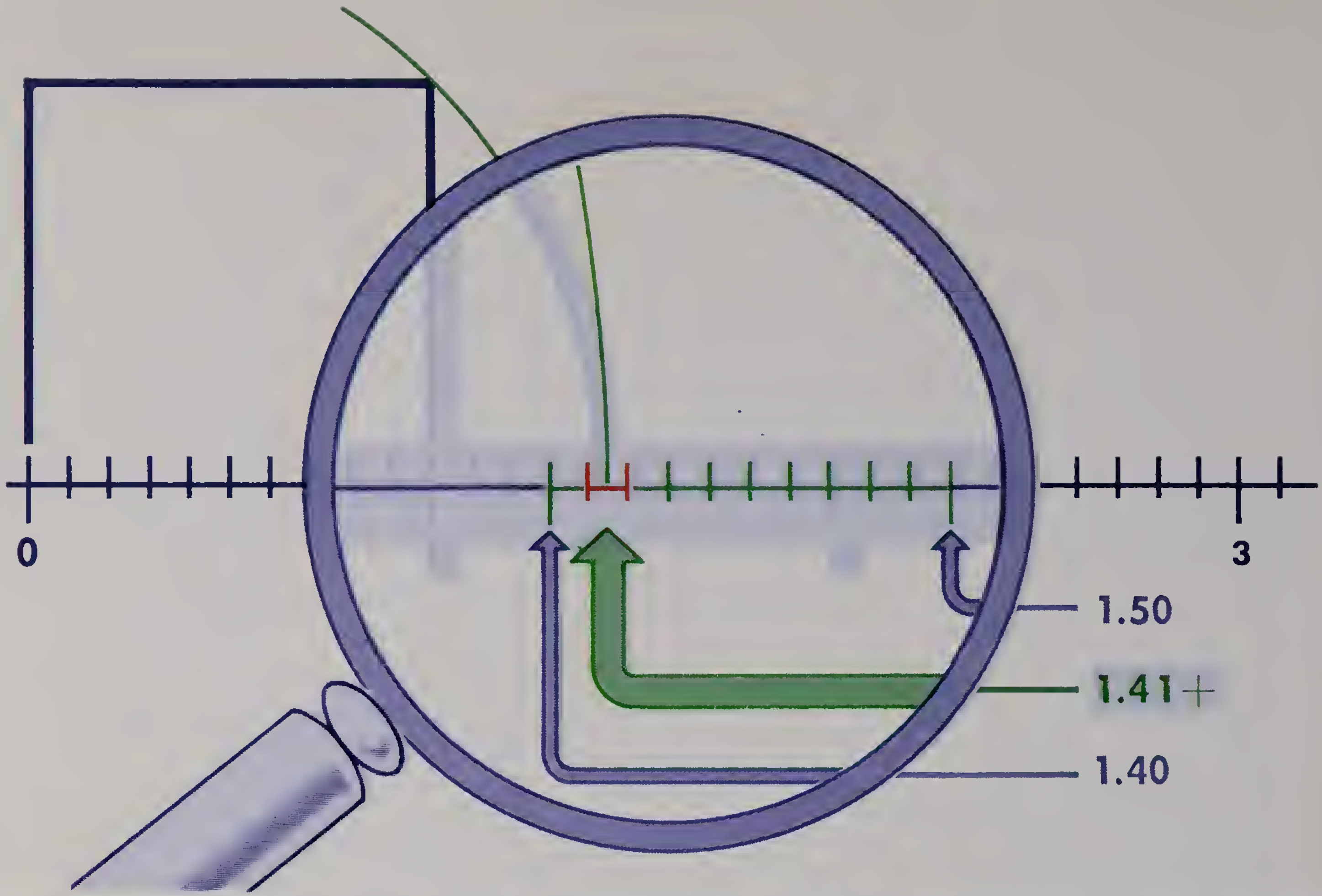
N each 1, 1 within the preceding interval. Fur e sr continuing the pr ou can find wish.



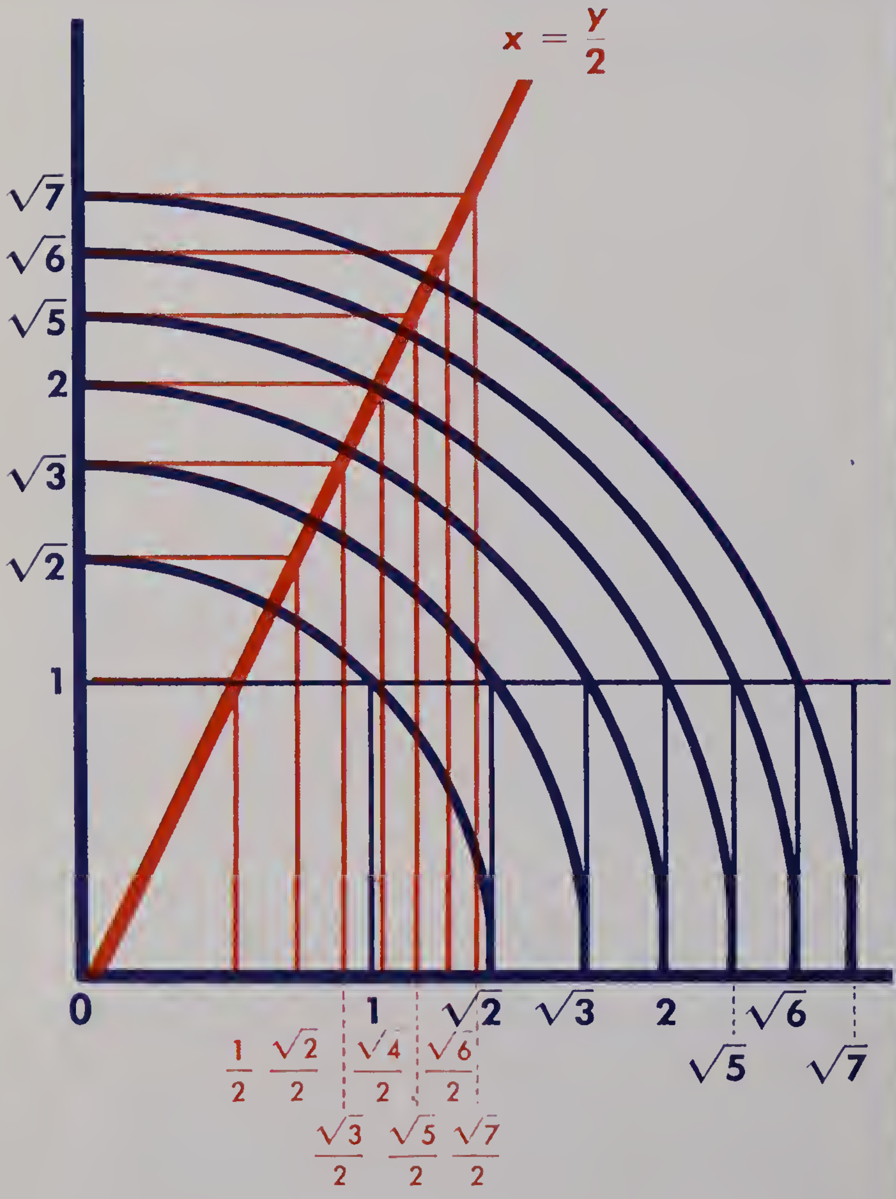
A succession of intervals of this type is called a *sequence of nested intervals.*
ates that there is one, and only one, number be-
quence of nested intervals. This means that a
ys closes on exactly one number.
ervals represent nonterminating, nonrepeating
rational numbers always terminate or repeat,
complete. By extending the number system to
nonterminating, nonrepeating decimals (the ir-
eal number system. It is the property of complete-
pair every real number with exactly one point
each point on the line exactly one real number.
ervals: [1.4, 1.5], [1.49, 1.5], [1.499, 1.5], . . . ;
or $\frac{3}{2}$? This means that $\frac{3}{2}$ has *two* decimal repre-
nd a nonterminating, repeating one, 1.4999 . . .
imal can be expressed as a repeating decimal
ample, $1 = 0.99\overline{9}$ and $1.32 = 1.31\overline{9}$.
nterminating decimals are based on the laws of
, for example, to add .333 . . . and .666 . . .

.333	.3333	.33333
.666	.6666	.66666
.999	.9999	.99999

digits in the addends, you obtain as the sum
vith $\frac{1}{3} + \frac{2}{3} = 1$.


0
3
1.50
1.41+
1.40

To pinch in more closely on $\sqrt{2}$, subdivide the interval from 1.4 to 1.5 into ten parts, each of length .01. The drawing has been enlarged, as if by a magnifying glass, to enable you to pick out more readily the interval of length .01 containing $\sqrt{2}$. Do you see that $1.41 < \sqrt{2} < 1.42$?


$x = \frac{y}{2}$
$\sqrt{7}$
$\sqrt{6}$
$\sqrt{5}$
2
$\sqrt{3}$
$\sqrt{2}$
1
0
1
$\sqrt{2}$
$\sqrt{3}$
2
$\sqrt{6}$
$\sqrt{5}$
$\sqrt{7}$
$\frac{1}{2}$
$\frac{\sqrt{2}}{2}$
$\frac{\sqrt{4}}{2}$
$\frac{\sqrt{6}}{2}$
$\frac{\sqrt{3}}{2}$
$\frac{\sqrt{5}}{2}$
$\frac{\sqrt{7}}{2}$

To locate halves of square roots of integers, draw the line whose equation is $x = \frac{y}{2}$. Then, from the graphs of the roots of integers shown on the y-axis draw horizontal lines to meet the line $x = \frac{y}{2}$. The x-coordinates of these points will be half their y-coordinates. Thus, by marking these coordinates on the x-axis you will have filled in the points corresponding to the halves of the square roots of the integers.

To find the point corresponding to $\frac{\sqrt{3}}{2}$, start at the point $(0, \sqrt{3})$ and extend the horizontal line $y = \sqrt{3}$ to the right to meet the line $x = \frac{y}{2}$. By setting $y = \sqrt{3}$ in this latter equation, you see that the x-coordinate of this point is $\frac{\sqrt{3}}{2}$.

The geometric process of sq
√2 corresponds to the algebraic
process of finding successi
between which √2 lies:

The numbe ne first pair diffe y 1, [1,
The num of the second pair d er by .1, [1. .5]
The nu rs of the third pair diff by .01, [1.4 42]
The n ers of the fourth pair di r by .001, [1. 1.415]

You find the pair by knowing that 1^2 2 < 2^2. You the second pair
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1.96 and (1.5) $< \sqrt{2} < 1.5$.

Next, by squ g cce hundredth sco that 41$)^2 = 1.9881$,
and th 2) 1.41 2. you eat this process,
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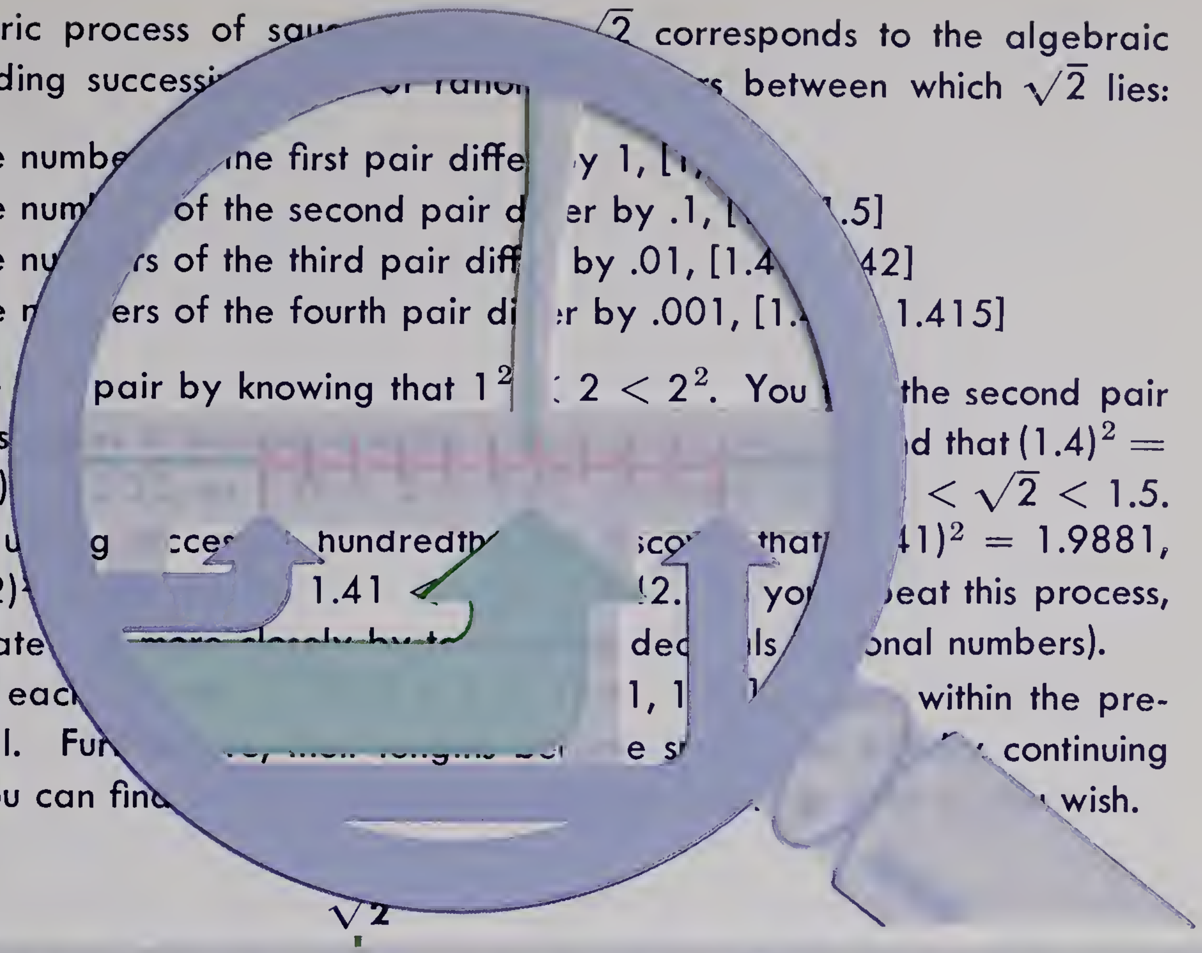
N each 1, 1 within the pre-
ceding interval. Fu e s continuing
the pr ou can find wish.



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rational numbers always terminate or repeat,
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or $\frac{3}{2}$? This means that $\frac{3}{2}$ has two decimal repre-
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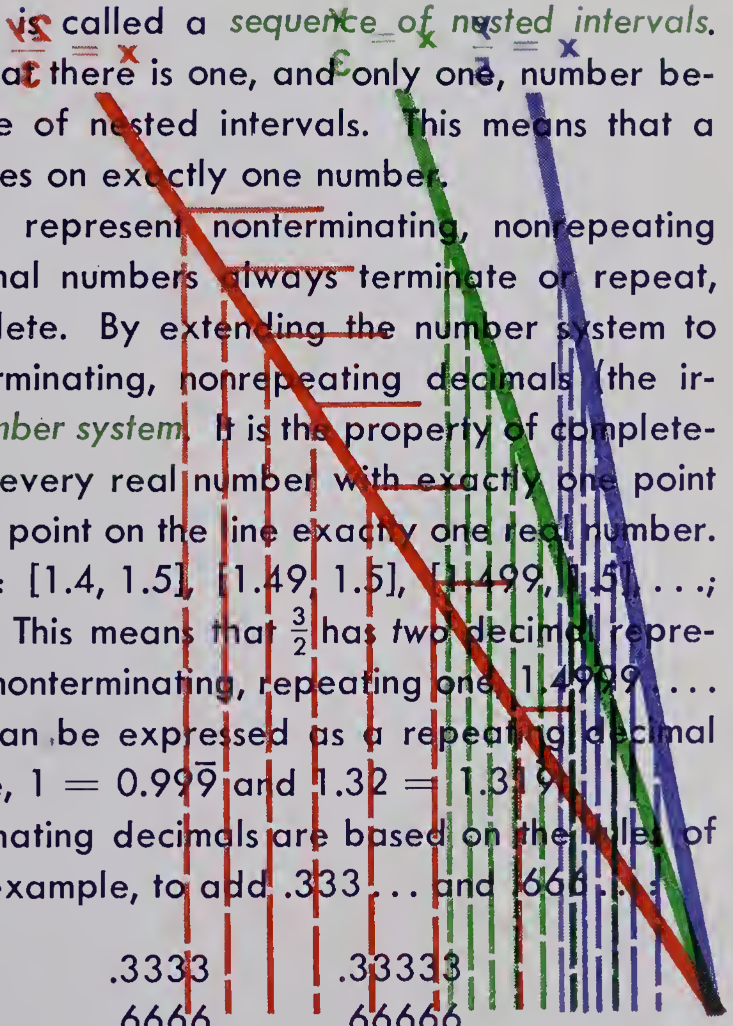
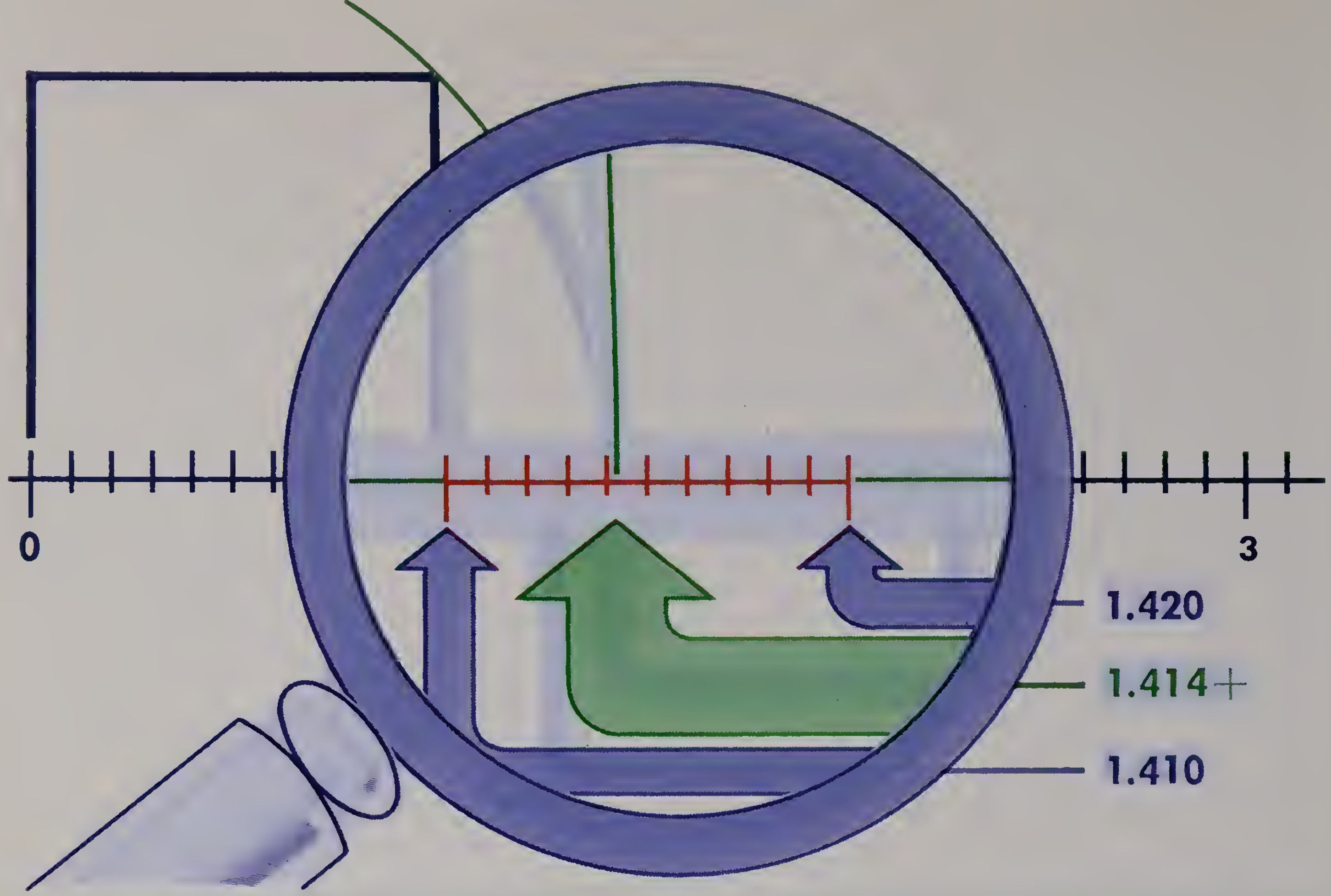
, for example, to add .333 . . . and .666 . . . :

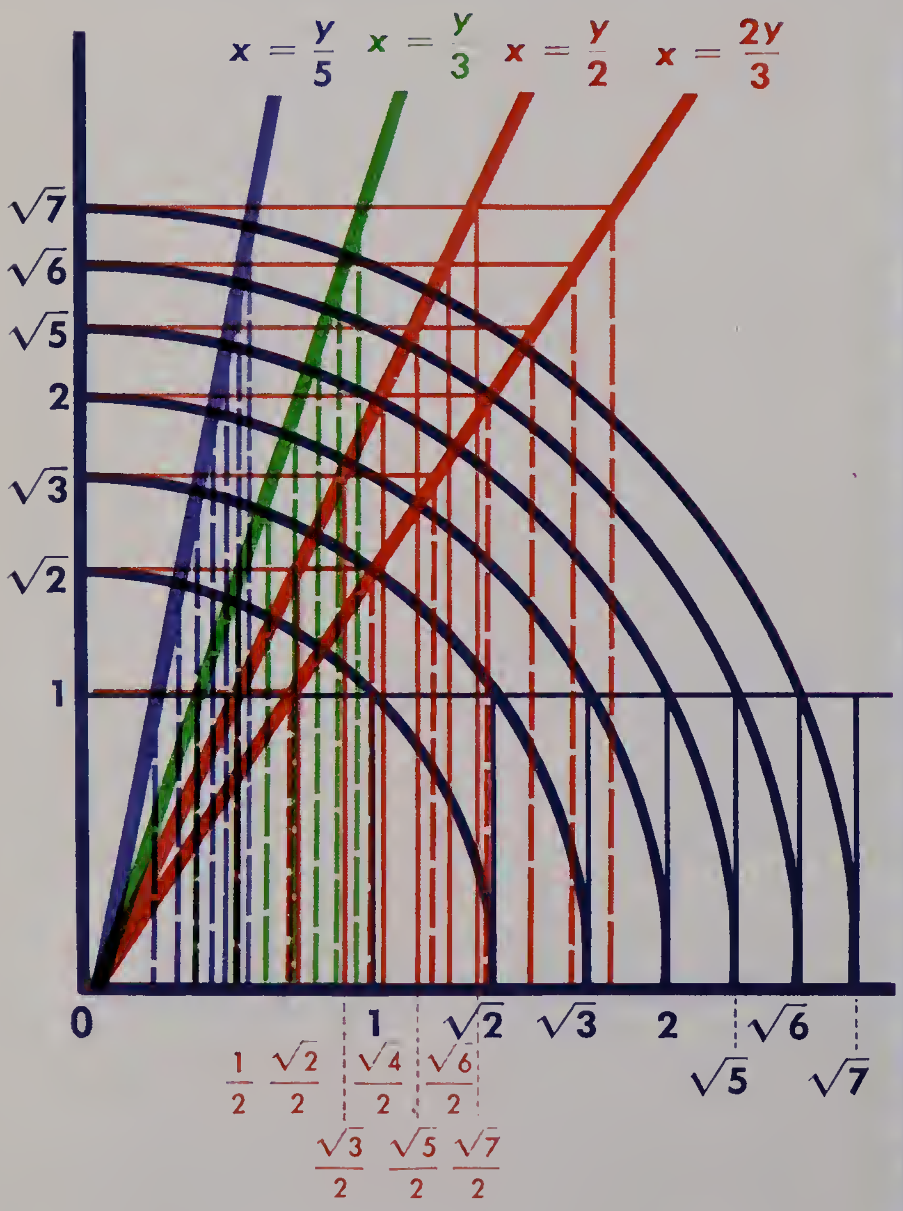


$$\begin{array}{r} .333 \\ \underline{.666} \\ .999 \end{array} \qquad \begin{array}{r} .3333 \\ \underline{.6666} \\ .9999 \end{array} \qquad \begin{array}{r} .33333 \\ \underline{.66666} \\ .99999 \end{array}$$

digits in the addends, you obtain as the sum
with $\frac{1}{3} + \frac{2}{3} = 1$.


0
3
1.420
1.414+
1.410

To close in further on $\sqrt{2}$ subdivide the interval from 1.41 to 1.42 into 10 equal parts, each of length .001. From the enlarged drawing you can see that $1.414 < \sqrt{2} < 1.415$. By repeating this process you can approximate $\sqrt{2}$ to more and more decimal places.


$x = \frac{y}{5}$
$x = \frac{y}{3}$
$x = \frac{y}{2}$
$x = \frac{2y}{3}$
$\sqrt{7}$
$\sqrt{6}$
$\sqrt{5}$
2
$\sqrt{3}$
$\sqrt{2}$
1
0
1
$\sqrt{2}$
$\sqrt{3}$
2
$\sqrt{6}$
$\sqrt{5}$
$\sqrt{7}$
$\frac{1}{2}$
$\frac{\sqrt{2}}{2}$
$\frac{\sqrt{4}}{2}$
$\frac{\sqrt{6}}{2}$
$\frac{\sqrt{3}}{2}$
$\frac{\sqrt{5}}{2}$
$\frac{\sqrt{7}}{2}$

The preceding construction is repeated by using the lines $x = \frac{y}{5}$, $x = \frac{y}{3}$, and $x = \frac{2y}{3}$ to locate these numbers on the x-axis:

$$\frac{1}{5}, \frac{\sqrt{2}}{5}, \ldots \frac{\sqrt{7}}{5}; \quad \frac{1}{3}, \frac{\sqrt{2}}{3}, \ldots \frac{\sqrt{7}}{3}; \quad \frac{2}{3}, \frac{2\sqrt{2}}{3}, \ldots \frac{2\sqrt{7}}{3}.$$

Since $\sqrt{\frac{a}{b}} = \frac{\sqrt{ab}}{b}$ for any positive integers a and b, you can locate the positive square roots of any rational number on the x-axis. This process does not locate all of the irrational numbers. (It does locate all the rational numbers.) Numbers which are not square roots of rational numbers, such as $\sqrt[3]{2}$, $\sqrt[4]{6}$, and multiples of π, cannot be located by this method.

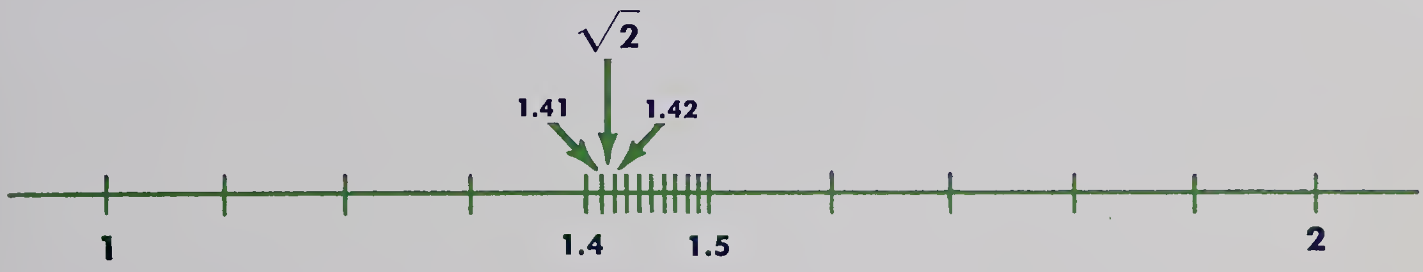
The geometric process of squeezing in on $\sqrt{2}$ corresponds to the algebraic process of finding successive pairs of rational numbers between which $\sqrt{2}$ lies:

The numbers of the first pair differ by 1, [1, 2].
The numbers of the second pair differ by .1, [1.4, 1.5]
The numbers of the third pair differ by .01, [1.41, 1.42]
The numbers of the fourth pair differ by .001, [1.414, 1.415]

You find the first pair by knowing that $1^2 < 2 < 2^2$. You find the second pair by computing squares of successive tenths: $(1.1)^2$, $(1.2)^2$, until you find that $(1.4)^2 = 1.96$ and $(1.5)^2 = 2.25$; as $1.96 < 2 < 2.25$, you know that $1.4 < \sqrt{2} < 1.5$.

Next, by squaring successive hundredths you discover that $(1.41)^2 = 1.9881$, and that $(1.42)^2 = 2.0164$, so $1.41 < \sqrt{2} < 1.42$. As you repeat this process, you approximate $\sqrt{2}$ more closely by terminating decimals (rational numbers).

Notice that each interval [1, 2], [1.4, 1.5], [1.41, 1.42], . . . lies within the preceding interval. Furthermore, their lengths become smaller, so that by continuing the process you can find in the succession an interval that is as short as you wish.



A succession of intervals of this type is called a *sequence of nested intervals.* The **property of completeness** states that there is one, and only one, number belonging to every interval in a sequence of nested intervals. This means that a sequence of nested intervals always closes on exactly one number.

Many sequences of nested intervals represent nonterminating, nonrepeating decimals. Since the decimals for rational numbers always terminate or repeat, the rational number system is not complete. By extending the number system to include numbers represented by nonterminating, nonrepeating decimals (the irrational numbers), you obtain the *real number system.* It is the property of completeness of this set that permits you to pair every real number with exactly one point on the number line and to assign to each point on the line exactly one real number.

Consider this set of nested intervals: [1.4, 1.5], [1.49, 1.5], [1.499, 1.5], . . .; do you see that it closes in on 1.5 or $\frac{3}{2}$? This means that $\frac{3}{2}$ has *two* decimal representations: a terminating one, 1.5 and a nonterminating, repeating one, 1.4999 . . . . In general, every terminating decimal can be expressed as a repeating decimal whose repeating block is $\overline{9}$; for example, $1 = 0.99\overline{9}$ and $1.32 = 1.31\overline{9}$.

The rules for operating with nonterminating decimals are based on the rules of operation for terminating decimals, for example, to add .333 . . . and .666 . . .:

$$\begin{array}{r} .3 \\ \underline{.6} \\ .9 \end{array} \qquad \begin{array}{r} .33 \\ \underline{.66} \\ .99 \end{array} \qquad \begin{array}{r} .333 \\ \underline{.666} \\ .999 \end{array} \qquad \begin{array}{r} .3333 \\ \underline{.6666} \\ .9999 \end{array} \qquad \begin{array}{r} .33333 \\ \underline{.66666} \\ .99999 \end{array}$$

You can see that by taking more digits in the addends, you obtain as the sum .9999999 . . . , or 1. This checks with $\frac{1}{3} + \frac{2}{3} = 1$.

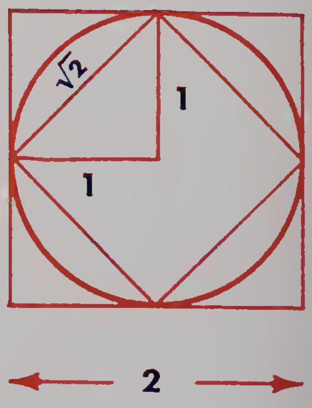
To multiply these same numbers, proceed as follows:

.3	.33	.333	.3333	.33333
.6	.66	.666	.6666	.66666
.18	198	1998	19998	199998
	198	1998	19998	199998
	.2178	1998	19998	199998
		.221778	19998	199998
			.22217778	199998
				.2222177778

Although it was not apparent in the first few products, you can see that the product begins .222. By taking more digits in the factors you can determine additional places in the product. As a matter of fact, $\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9} = .222\overline{2}$.

You can tell which of two real numbers is the greater by comparing their decimals; for example, $.97423 < .97425$. It is also easy to insert between any two real numbers another real number; for instance, between the real numbers just named is the number .974233. Because there is a real number between any two real numbers, the set of real numbers is said to have the **property of density.**

There are more irrational numbers than there are rational ones. The sequence of drawings on the lower halves of the preceding pages shows you how to locate on the number line the square root of any positive rational number. Unless a rational number is the square of a rational number, its square root is an irrational number, so that many of the points located correspond to irrational numbers. Moreover, there are irrational numbers which cannot be expressed as square roots, cube roots, or as roots at all. The measure of the circumference of a circle whose diameter is 1 unit in length is given by the irrational number $\pi = 3.14159265\ldots$, which is an unending, nonrepeating decimal which cannot be found by taking roots of rational numbers. However, it can be found by the method of nested intervals, as was first demonstrated by Archimedes.



Archimedes used a circle of radius 1 whose area is measured by π. The area of this circle is between the areas of two squares, one with its vertices on the circle and the other with its sides just touching the circle, which are called *inscribed* and *circumscribed* squares. The areas of these two squares are given by $(\sqrt{2})^2$ and 2^2, respectively, and, since the area of the circle is between these two numbers, $2 < \pi < 4$. By increasing the number of sides of the inscribed and circumscribed regular polygon, you obtain smaller and smaller intervals containing π. Using regular octagons, for example, you find that $2.8 < \pi < 3.3$. Archimedes used polygons of 96 sides and found that $3\frac{10}{71} < \pi < 3\frac{1}{7}$. By taking polygons with enough sides, you can approximate π as closely as you wish.

The system of real numbers can be built up step by step from the natural numbers used for counting, and it can be shown to satisfy all the familiar rules of arithmetic while also possessing the properties of density and completeness.

11–9 Radical Equations

An equation having a variable in a radicand is a **radical equation**. The simplest kind of radical equation is one like $\sqrt{x} = 3$, which you solve by squaring each of its members.

$$\sqrt{x} = 3 \qquad (\sqrt{x})^2 = 3^2 \qquad x = 9$$

Check: $\sqrt{9} = 3$ ✓ $\therefore$ **The solution set is $\{9\}$, Answer.**

To solve radical equations which have several terms in each member but only one radical term, you first isolate the radical term in one member. Then you can square each member and solve the resulting equation.

EXAMPLE **Solve $3 = 2\sqrt{x} + x$.**

Solution:

1. Isolate the radical term in one member of the equation.	$3 = 2\sqrt{x} + x$ $3 - x = 2\sqrt{x}$
2. Square both members.	$(3 - x)^2 = (2\sqrt{x})^2$ $9 - 6x + x^2 = 4x$
3. Solve the resulting equation.	$x^2 - 10x + 9 = 0$ $(x - 1)(x - 9) = 0$ $x - 1 = 0$ | $x - 9 = 0$ $x = 1$ | $x = 9$
4. Check. Substitute, and then take the principal root of the number in the radicand.	$3 \overset{?}{=} 2\sqrt{1} + 1$ | $3 \overset{?}{=} 2\sqrt{9} + 9$ $3 \overset{?}{=} 2 \cdot 1 + 1$ | $3 \overset{?}{=} 2 \cdot 3 + 9$ $3 = 2 + 1$ ✓ | $3 \overset{?}{=} 6 + 9$ No **The solution set is $\{1\}$, Answer.**

Can you explain why the "squared" equation in Step 2 may not be equivalent to the given equation? Notice that

if $a = b$, then $a^2 = b^2$; but if $a^2 = b^2$, it need not be true that $a = b$.

$5^2 = (-5)^2$, but $5 \neq -5$.

See 11-9 T.M. pg. 37.

WRITTEN EXERCISES

Solve each radical equation.

Ⓐ

1. $\sqrt{2y} = 2$

2. $\sqrt{3x} = \frac{1}{3}$

3. $\sqrt{5x} = \frac{5}{2}$

4. $\sqrt{7z} = \frac{7}{4}$

5. $\sqrt{s} + 5 = 7$

6. $\sqrt{t} - \frac{1}{2} = 2$

7. $\sqrt{\frac{r}{2}} = 1$

8. $\sqrt{\frac{v}{3}} = 2$

9. $\sqrt{x + 10} = 3$

10. $\sqrt{x - 4} = 9$

11. $\sqrt{2y - 3} = 3$

12. $\sqrt{3y + 4} = 1$

13. $4\sqrt{5m} = 20$

14. $\frac{1}{2}\sqrt{10m} = \frac{5}{2}$

15. $\sqrt{p} + 5 = 3$

16. $\sqrt{p} + 2 = 1$

17. $\sqrt{4y - 3} + 7 = 10$

18. $\sqrt{5y - 1} - 8 = -1$

19. $\sqrt{\frac{9x}{2}} - 2 = 7$

20. $\sqrt{\frac{4x}{3}} - 2 = 2$

Ⓑ

21. $\sqrt{\frac{3x - 5}{4}} = 2$

22. $\sqrt{\frac{5y + 1}{6}} = 1$

23. $\sqrt{x} = 2\sqrt{5}$

24. $3\sqrt{k} = 4\sqrt{3}$

25. $4\sqrt{5t^2 + 5} = 20$

26. $2\sqrt{3s^2 - 12} = 12$

27. Solve for E: $v = \sqrt{\frac{2E}{m}}$

28. Solve for h: $r = \sqrt{\frac{7v}{22h}}$

29. $\sqrt{a^2 + 9} = a + 3$

30. $\sqrt{b^2 - 16} = b - 4$

31. $\sqrt{n} = -n$

32. $\sqrt{n} = -\frac{n}{2}$

Ⓒ

33. $\sqrt{x^2 + 4} - 1 = x$

34. $\sqrt{x^2 - 4} - 2 = x$

35. $\sqrt{2k - 4} = k - 2$

36. $\sqrt{3r - 9} = r - 3$

37. $\sqrt{19 - y} = y - 7$

38. $\sqrt{y + 3} = y - 9$

39. $\sqrt{x + 2} + 3 = 0$

40. $1 + \sqrt{x + 4} = 0$

PROBLEMS

Ⓐ

1. Three times the square root of a number is 30. Find the number.

2. One-third the square root of a number is 5. What is the number?

3. The square root of 3 less than twice a number equals 3. Find the number.

4. When 9 is added to 5 times a certain number, the square root of the result is 8. Find the number.

5. The perimeter of a square in terms of its area is $p = 4\sqrt{A}$. Solve for A in terms of p, and find the area of a square whose perimeter is 32 inches.

6. The distance, in miles, from a turret h feet above ground to the horizon is $d = \sqrt{\dfrac{3h}{2}}$. Solve this rule for h in terms of d, and determine the turret's height if the lookout sees 9 miles to the horizon.

7. Solve this form of the Pythagorean theorem for the positive value of a: $c = \sqrt{a^2 + b^2}$. Then find a when $c = 17$ and $b = 15$.

8. Solve the rule $s = \frac{1}{4}\sqrt{f}$ (the number of seconds s a body falls a distance of f feet from a position of rest) for f to find the approximate height of the Washington Monument, given that an object reaches the ground almost 6 seconds after being dropped from its top.

B **9.** The rule $d = \sqrt{\dfrac{H}{1.6n}}$ gives the diameter d of each of n cylinders of an automobile with H horsepower. What horsepower is developed in a six-cylinder engine whose cylinders have a three-inch diameter?

10. A moving car hits a wall with the force with which it would strike the ground in falling from a certain height. The relationship of the car's rate r, in miles per hour, to the height h, in feet, is: $r = \sqrt{\dfrac{h}{.0336}}$. How far (to the nearest foot) would a car fall and strike the ground with the force with which it would hit a wall at 60 miles per hour?

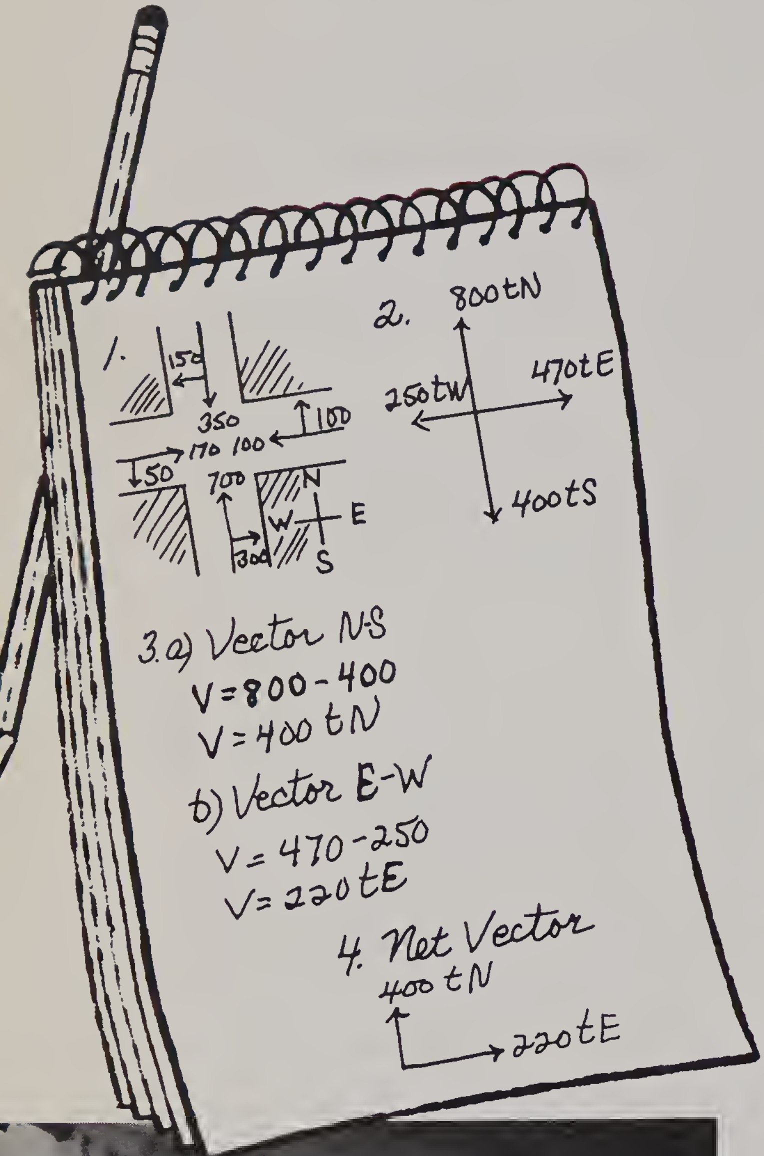
11. The velocity of sound, in meters, at t degrees is $V = 333\sqrt{14.0037t}$. At what temperature will sound travel 11,988 meters per second?

12. The amount a to which \$1 grows when invested for 2 years at $r\%$ per year is $a = \left(1 + \dfrac{r}{100}\right)^2$. At what rate will \$1 become \$1.21?

13. The geometric average of two positive numbers is the positive square root of their product. Find a pair of consecutive positive integers whose geometric average is $6\sqrt{2}$.

14. The illumination i in foot-candles on a surface f feet from a light with c candle power is: $i = \dfrac{c}{f^2}$. How far away can a lamp of 54 candle power light a screen as well as a lamp of 6 candle power at 10 feet can?

Civic Planners and Mathematics



One of the earliest planned cities, a housing project for men working on the Pyramids, was built in Egypt around 3000 B.C. Since that time, city planning has been carried on for a variety of reasons, and in varying degrees of detail. Some cities, like Pompeii, Washington, D.C., and Brazilia, were planned from their inception; others, like Paris and Rome, were re-planned after poverty and congestion had made them almost uninhabitable.

The photograph shows two modern-day civic planners marking suggested improvements on an acetate overlay. These planners must bring into harmony the large populations, intricate transportation networks, and vast industries of today's urban areas. With the aid of modern techniques of statistical and mathematical analysis, however, the modern civic planner can effectively cope with many of these dynamic problems.

One type of problem a civic planner solves is shown on the pad. Surveys were made of the traffic flow at an intersection where many accidents occurred. The data obtained were analyzed mathematically to show: (1) the traffic in each direction at the intersection; (2) the total traffic through the intersection; (3a) north traffic minus south traffic; (3b) east minus west traffic; and (4) the direction and amount of the heaviest traffic flow.

The civic planner found the greatest flow of traffic to be north and east. He advised that the intersection be modified to favor traffic in these directions, either by having longer traffic-light intervals, or by providing more lanes. These changes would reduce accidents and allow faster, less congested movement of vehicles through the intersection.

Chapter Summary

Inventory of Structure and Method

1. A **rational number** can be expressed as a fraction in an unlimited number of ways, and as a **terminating decimal** or a **repeating decimal.** Between every pair of rational numbers is an infinite set of rational numbers. To approximate a number in decimal notation, you retain the digits unchanged if the first digit dropped is less than 5; you increase the last digit by one if the first digit dropped is 5 or more.
2. If $a \geq 0$ and $b > 0$, then $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$; and $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.
3. In any right triangle, if c is the hypotenuse, and a and b are the other sides, $a^2 + b^2 = c^2$.
4. Roots of rational numbers are not all rational numbers. **Irrational numbers** are represented by **unending, nonrepeating decimals**. The real number system can be put into one-to-one correspondence with the points of a line. If you divide a number by a divisor which is smaller in absolute value than the square root of the number, the quotient is larger in absolute value than the square root.
5. Square roots having the same radicand can be added or subtracted by applying the distributive property.
6. Squaring both members of an equation produces a new equation which includes the roots of the given equation as a subset of its own roots.

Vocabulary and Spelling

number system (*p. 397*)
rational number (*p. 397*)
rational operations (*p. 397*)
property of density (*p. 398*)
terminating (*p. 401*)
nonterminating (*p. 401*)
repeating, periodic (*p. 401*)
equals approximately (*p. 402*)
power of a number (*p. 403*)
raising to a power (involution) (*p. 403*)
extracting a root (evolution) (*p. 403*)
root (*p. 403*)
radical (*p. 403*)
root index (*p. 403*)
radicand (*p. 403*)
principal root (*p. 404*)
perfect square (*p. 407*)
irrational numbers (*p. 407*)
real number system (*p. 407*)
property of completeness (*p. 407*)
property of pairs of divisors (*p. 408*)

Pythagorean theorem (*p. 411*)
hypotenuse (*p. 411*)
radical in simplest form (*p. 414*)
rationalizing the denominator (*p. 415*)
conjugate (*p. 419*)
radical equation (*p. 421*)

Chapter Test

11-1 Arrange in order of size, with the largest first.

1. $\frac{8}{15}, \frac{7}{16}, \frac{9}{17}$ **2.** $\frac{117}{72}, \frac{79}{48}, \frac{470}{288}$

Find a number halfway between these numbers.

3. $\frac{15}{14}, 1$ **4.** $-\frac{34}{7}, \frac{10}{9}$

Tell whether or not each statement is true.

5. There is no smallest rational number greater than 0.

6. Between two integers is an infinite number of integers.

11-2 Write as terminating or repeating decimals.

7. $\frac{5}{7}$ **8.** $-\frac{13}{15}$ **9.** $12\frac{3}{8}$

Write as common fractions.

10. $.7272\ldots$ **11.** $.\overline{8}$ **12.** $2.\overline{345}$

11-3 Evaluate each expression, using the product and quotient properties.

13. $\sqrt{900}$ **14.** $-\sqrt{\frac{1}{576}}$

Find each solution set.

15. $x^2 = 400$ **16.** $3y^2 - 363 = 0$

11-4 Compute the indicated square root to the nearest hundredth.

17. $\sqrt{4.72}$ **18.** $-\sqrt{8.2314}$

11-5 Find the third side of right triangle *ABC* if its hypotenuse is *AB*.

19. $BC = 20, AB = 25$ **20.** $AC = 12, BC = 16$

11-6 Find these products and quotients in simplest form.

21. $4\sqrt{13} \cdot 2\sqrt{13}$ **23.** $3\sqrt{7} \cdot 2\sqrt{12} \cdot \sqrt{21}$

22. $\dfrac{2\sqrt{49}}{3\sqrt{40}}$ **24.** $\dfrac{2\sqrt{a^3b^7}}{\sqrt{8a^5b^5}}$

11-7 Combine these radicals.

25. $3\sqrt{5} + 2\sqrt{5} - \sqrt{5}$ **27.** $3\sqrt{27} - \sqrt{\frac{1}{3}}$

26. $10\sqrt{\frac{1}{50}} - 6\sqrt{\frac{1}{2}}$ **28.** $\frac{1}{2}\sqrt{\frac{3}{8}} + \frac{3}{4}\sqrt{\frac{1}{6}}$

11-8 Express in simplest form.

29. $(3 - \sqrt{2})(3 + \sqrt{2})$ **31.** $(4 + \sqrt{5})^2$

30. $\dfrac{1}{3\sqrt{3} + 2}$ **32.** $\dfrac{\sqrt{6} + 1}{3 - \sqrt{6}}$

11-9 Find the solution set.

33. $\sqrt{3y} + 12y = 0$ **34.** $\sqrt{3z} - z = 2$

Chapter Review

11-1 The Nature of Rational Numbers *Pages 397–400*

1. A number system is a set of numbers with the operations $\underline{\quad ? \quad}$ and $\underline{\quad ? \quad}$ defined.

2. Of the fractions $\frac{11}{12}$ and $\frac{37}{40}$, the greater is $\underline{\quad ? \quad}$.

3. The number halfway between $\frac{7}{12}$ and $\frac{5}{8}$ is $\underline{\quad ? \quad}$.

4. The property of density implies that between any two different rational numbers is an $\underline{\quad ? \quad}$ number of others.

Tell whether or not each statement is true.

5. If a rational operation is performed on a rational number, the result is always a rational number.

6. The set of rational numbers between 1 and 2 is finite.

11-2 Decimal Forms of Rational Numbers *Pages 400–403*

Write as decimals.

7. $2\frac{5}{16}$ **8.** $-\frac{11}{12}$ **9.** $\frac{35}{32}$

Write as common fractions.

10. .636363 . . . **11.** $1.4\overline{6}$ **12.** $.\overline{297}$

Round each number to the nearest tenth.

13. 8.446 **14.** $1.0\overline{6}$ **15.** $\frac{13}{20}$ **16.** 4.97

11-3 Roots of Numbers *Pages 403–407*

17. In $\sqrt[5]{n + 1}$, the root index is $\underline{\quad ? \quad}$, and $n + 1$ is the $\underline{\quad ? \quad}$.

18. Every positive number has $\underline{\quad ? \quad}$ square roots.

19. The expression $\sqrt{9} = \underline{\quad ? \quad}$; but $-\sqrt{9} = \underline{\quad ? \quad}$.

Evaluate each expression.

20. $\sqrt{1024}$ **21.** $\pm\sqrt{\frac{676}{25}}$ **22.** $-\sqrt{\frac{1}{361}}$

Find each solution set.

23. $n^2 - 64 = 0$ **24.** $32s^2 = 8$ **25.** $(r - 5)^2 = 25$

11–4 Properties of Irrational Numbers *Pages 407–411*

26. In the sentences, $x^2 = 5$, $y^2 = 9$, $z^2 = .4$, $u^2 = 4\frac{1}{4}$, $v^3 = 8$, only $\underline{\ ?\ }$ and $\underline{\ ?\ }$ can represent rational numbers.

27. Every point on the number line corresponds to a $\underline{\ ?\ }$ number which may be either $\underline{\ ?\ }$ or $\underline{\ ?\ }$.

28. The decimal form of an irrational number is neither $\underline{\ ?\ }$ nor $\underline{\ ?\ }$.

Compute to the nearest hundredth.

29. $\sqrt{35}$ **30.** $\sqrt{9.641}$ **31.** $-\sqrt{.6803}$

32. The area of a square floor is 250 square feet. How long is it, to the nearest hundredth of a foot?

11–5 Geometric Interpretation of Square Roots *Pages 411–414*

33. The $\underline{\ ?\ }$ of a right triangle is longer than either of the other sides.

The hypotenuse of right triangle ABC is AB. How long is the side not given?

34. $AC = 8$ cm.; $BC = 15$ cm. **35.** $AC = 40''$; $AB = 41''$

Determine whether the following are the sides of a right triangle.

36. 39, 36, 15 **37.** 18, 30, 24

11–6 Multiplication, Division, and Simplification of Radicals *Pages 414–417*

Perform the indicated operations.

38. $2\sqrt{7} \cdot 3\sqrt{7}$ **40.** $5\sqrt{12} \div 2\sqrt{3}$ **42.** $\sqrt{3}(2\sqrt{2} - 3\sqrt{3})$

39. $\dfrac{6\sqrt{81}}{2\sqrt{27}}$ **41.** $\dfrac{\sqrt{4r^2s^3t^4}}{2\sqrt{9r^6st^6}}$ **43.** $\dfrac{\sqrt{8} - 2\sqrt{28}}{2\sqrt{2}}$

44. The expression $\dfrac{\sqrt{3}}{\sqrt{2} - 1}$ is not in simplest form because its $\underline{\ ?\ }$ contains a radical.

45. You $\underline{\ ?\ }$ the denominator to transform a fraction having an irrational denominator into an equivalent fraction having a $\underline{\ ?\ }$ denominator.

Simplify these radicals.

46. $\frac{2}{3}\sqrt{162}$ **47.** $-2\sqrt{\frac{4}{5}}$

48. $\frac{5}{\sqrt{8}}$ **49.** $\frac{6}{5\sqrt{12}}$

11-7 Addition and Subtraction of Radicals *Pages 417–419*

Simplify.

50. $6\sqrt{18} - 2\sqrt{98} + 2\sqrt{9}$ **51.** $3\sqrt{6} + 6\sqrt{\frac{1}{6}} - 8\sqrt{1.5}$

52. To combine similar radicals by addition or subtraction, apply the __?__ property.

53. A rectangle's sides are in the ratio 1 to 3, and its diagonal is 39 inches. Find the width, to the nearest tenth of an inch.

11-8 Multiplication of Binomials Containing Radicals *Pages 419–420*

Express in simplest form.

54. $(2\sqrt{5} + 5)^2$ **56.** $\frac{2}{\sqrt{6} + 2}$

55. $(3\sqrt{2} - 1)(3\sqrt{2} + 1)$ **57.** $\frac{10}{2\sqrt{3} - \sqrt{2}}$

11-9 Radical Equations *Pages 421–423*

58. The result of squaring each member of an equation is not always __?__ to the original equation.

Find the roots of the following equations.

59. $\sqrt{2x} = 6$ **62.** $\sqrt{2s - 1} + 3 = 2$

60. $2\sqrt{3n} = 9$ **63.** $\sqrt{2x - 2} - x = 1$

61. $2 + \sqrt{r} = 5$ **64.** $\sqrt{3y + 13} = y + 3$

Cumulative Review: Chapters I-II

1. Simplify: If $x \neq 0$, $x - \frac{x^2 - y^2}{x} =$ __?__.

2. Simplify: If $x \neq 3$ and $x \neq -3$,

$$\frac{2x + 6}{x^2 - 9} \cdot \frac{x^2 - 6x + 9}{2} \div (x - 3) = \underline{\ ?\ }.$$

3. Expand $(a - b)^2$.

4. Give the roots of $x^2 - 2x = 3$.

5. The lowest common denominator of $\frac{1}{x^2} + \frac{z}{x} + \frac{x}{z^2}$ is __?__.

Complete each statement.

6. A man invests \$10,000, part (x) at 4% and the rest at 5%. The amount (in terms of x) invested at 5% is ___?___.

7. A canoeist who paddles r miles per hour in still water paddles ___?___ miles in 1 hour against a current of c m.p.h.

8. The set of rational numbers is a ___?___ of the set of real numbers.

9. The product of a number and its ___?___ is always 1.

Match an answer in the second column to each condition in the first ($m > 1$).

10. The average of three consecutive odd integers beginning with $m - 2$.

11. The value of x in $\frac{15}{20m} = \frac{3}{x}$.

12. The radius of a circle whose area is $4\pi m^2$.

13. The root x of $mx - 3m^2 = x - 3m$.

14. The value of $5|-m|$.

(a) m
(b) $2m$
(c) $3m$
(d) $4m$
(e) $5m$
(f) $6m$

Solve Exercises 15–18 for each variable.

15. $3 = m^2 + 2m$

16. $\frac{x+1}{5} + \frac{x-1}{2} = 9 - \frac{x}{3}$

17. $y^2 - 3y = 0$

18. $$\begin{aligned} 9x + 4y &= -21 \\ 7x - 3y &= 2 \end{aligned}$$

19. Solve and graph the solution set of $4 - 2(3r - 1) \leq 12$.

20. Solve graphically: $3x - 2y = 12$, $x + y = -1$.

21. Graph the solution set of $3x - 2y > 12$, $x + y < -1$ as a doubly shaded area.

22. If $y = 1 - \frac{2}{t}$, and t is positive and increases, how does y change?

Perform the required operations.

23. Find the rational number midway between $\frac{1}{2}$ and $\frac{2}{3}$.

24. Write $\frac{3}{25}$ as a terminating decimal.

25. Find the rational number represented by $\sqrt[3]{\frac{8}{27}}$.

26. Simplify $\sqrt{5n} \cdot \sqrt{10n^3}$.

27. Simplify $\frac{\sqrt{216}}{2\sqrt{2}}$.

28. Express $(\sqrt{2} + 1)(\sqrt{2} - 1)$ as a rational number.

29. Solve $x + 3\sqrt{x} = 10$.

30. The total area of the six faces of a cube is 540 square inches. How long, to the nearest tenth, is one edge?

31. Find the third side of a right triangle whose hypotenuse is 17 and whose other side is 8.

32. If 3 tons of hard coal and 2 tons of soft coal cost $112, and 2 tons of hard and 6 tons of soft coal cost $168, what does a ton of each cost?

33. In 2 hours John overtakes scouts walking at 3 m.p.h., who started 1 hour ahead of him. How fast does John walk?

34. Had x students voted for Alan instead of for Ruth, he would have had four times as many votes as Ruth. However, if $2x$ students had voted for Ruth instead of for Alan, the vote would have been a tie. How many of all 500 votes went to each?

Extra for Experts

Algebraic Fallacies

Seemingly valid methods of working with algebraic statements may lead to illogical results because they conceal violations of the properties of numbers. Test your ability to detect improper procedures by trying to find the errors in the following arguments.

Questions

Justify the valid steps in each series of transformations, and explain the error in each invalid step.

1. To "prove" that 1 = 0, let a and b be two integers such that

$$a = b + 1.$$

Then,

$$\begin{aligned} (a - b)a &= (a - b)(b + 1) \\ a^2 - ab &= ab + a - b^2 - b \\ a^2 - ab - a &= ab + a - a - b^2 - b \\ a(a - b - 1) &= b(a - b - 1) \\ a &= b \\ b + 1 &= b \\ \therefore 1 &= 0 \end{aligned}$$

(*Hint:* Substitute $a = b + 1$ in $a - b - 1$.)

2. To "prove" that 1 = 8, let $y = 3$; then,

$$\frac{y+9}{y-1} - 6 = \frac{15-5y}{y-8}$$

$$\frac{15-5y}{y-1} = \frac{15-5y}{y-8}$$

$$\frac{1}{y-1} = \frac{1}{y-8}$$

$$y - 1 = y - 8$$

$$-1 = -8$$

$$\therefore 1 = 8$$

(*Hint:* What is the value of $15 - 5y$ if $y = 3$?)

3. To "prove" that $0 > 3$, let a be any number such that

$$a > 3.$$

Then,

$$3a > 3(3)$$

$$3a - a^2 > 9 - a^2$$

$$a(3-a) > (3-a)(3+a)$$

$$a > 3 + a$$

$$\therefore 0 > 3$$

4. To "prove" that $1 > 1$, let

$$c > 1$$

and

$$c = d.$$

Then,

$$c - 1 = d - 1.$$

$$c - 1 = -1(1-d)$$

$$(c-1)^2 = (-1)^2(1-d)^2$$

$$(c-1)^2 = (1-d)^2$$

$$c - 1 = 1 - d$$

$$\therefore c - 1 = 1 - c$$

$$2c = 2$$

$$c = 1$$

$$\therefore 1 > 1$$

(*Hint:* $5^2 = 25$ and $(-5)^2 = 25$, but $5 \neq -5$.)

Having examined these fallacies, you should be aware of the disastrous consequences of dividing by zero, of multiplying an inequality by a negative number without changing the order of the inequality, and of assuming that numbers whose squares are equal must equal each other.

THE HUMAN EQUATION

A Man Called Blockhead

Some words have more than one meaning, foreign words as well as English words. In medieval Italian, *bigollo* meant both "traveler" and "blockhead." When, about 1200, Leonardo of Pisa signed himself *Leonardo Bigollo*, he may have meant that he was a great traveler, for so he was. Some historians, however, think that by using this name he was saying to those who disdained him: "You called me a blockhead; look what a blockhead can do! Can you smart men do as much?" For Leonardo of Pisa, self-educated son of a merchant, was not popular with the professors of his day. But time has shown that he was far and away the greatest mathematician of the thirteenth century.

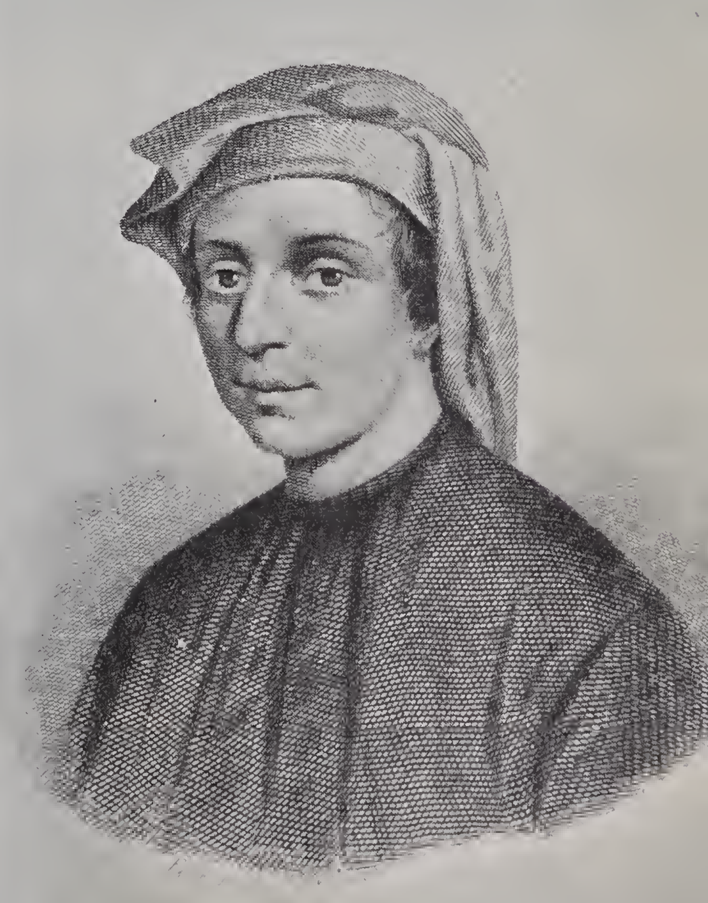
One of the discoveries for which he is remembered is called *Fibonacci's series.* Fibonacci is another name Leonardo used; it is short for *filius Bonacci*, "son of Bonaccio." His series is a sequence of numbers, obtained like this:

$1 + 1 = 2$
$1 + 2 = 3$
$2 + 3 = 5$
$3 + 5 = 8$
$5 + 8 = 13$
$8 + 13 = 21$
$13 + 21 = 34$
$21 + 34 = 55$

Of course, there is no end to the Fibonacci series.

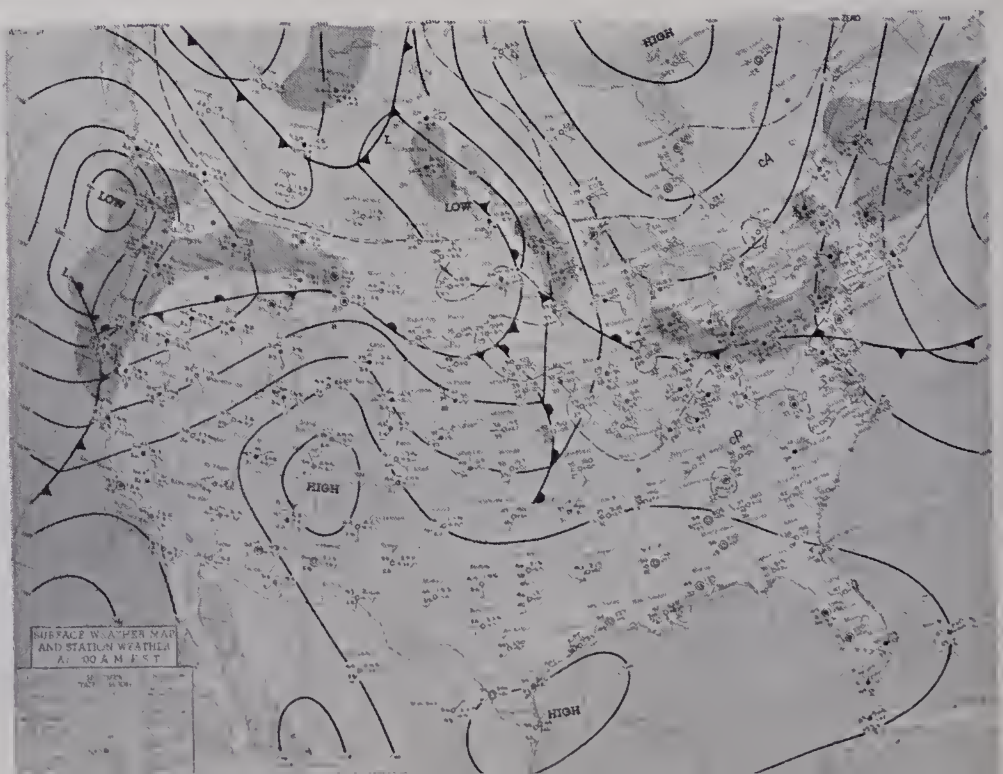
There seems to be no end to the places you find it. Examine some green plant — a weed will do. Start at the bottom of the stalk, and count the leaves on it. When you come to a leaf that is directly over the one you started with, you will find that you have reached some number in the Fibonacci series. Make a similar count of the leaves of head lettuce, or of an artichoke, or of an onion! You will find numbers in Fibonacci's series every time.

Leonardo of Pisa probably did not know most of the implications of his series. Mathematicians today probably do not know all of them. They are still studying the series of numbers discovered by Leonardo the traveler — or blockhead!



Leonardo of Pisa who discovered a number series in the thirteenth century which is not fully explained even today!

CHAPTER 12


HIGH
LOW
LOW
cA
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L
HIGH
cP
HIGH
SURFACE WEATHER MAP AND STATION WEATHER AT 00 A.M. E.S.T.

Functions and Variation

Have you considered the importance of knowing that two things have been paired? The weather bureau (upper and lower photos) can tell you the amount of rainfall, barometric pressure, temperature, and the direction and velocity of the wind during any period for which it has records. A less complex example can be found in a prom in which dancers are grouped into pairs of a boy and a girl, each. The replacement set for one member of each pair is the set of boys; the replacement set for the other member of each pair is the set of girls. By selecting a specified member from one set, you can identify the corresponding partner from the other set, just as the weather bureau can tell you the rainfall for a specified day.

Mathematicians form similar pairs of numbers; when one is designated, the other can be identified. Although this may seem like a simple process, it is one of the unifying concepts of mathematics.

SELECTING PAIRS OF NUMBERS

Alternative definition: A relation is any set of ordered pairs of numbers. See T.M. pg. 38 (1).

12–1 Relations

In graphing equations you formed ordered pairs of numbers. Any pairing of the members of two sets of numbers is a **relation**. Figure 12–1 shows the *graph* of the relation specified by the *roster* $\{(0, 0), (1, 1), (-1, 1), (2, 2), (-2, 2), (3, 3), (-3, 3), \ldots\}$, or by the *rule:* With each integer associate its absolute value. Thus, a relation may be described by a graph, a roster, or a rule.

The two sets of numbers which are paired in a relation are called the *domain of definition* (*domain*) and the *range of values* (*range*) of the relation. In the graph of a relation, points of the horizontal axis represent members of the domain, and points of the vertical axis represent members of the range. In the roster of a relation, the first coordinate of each ordered pair is a member of the domain, and

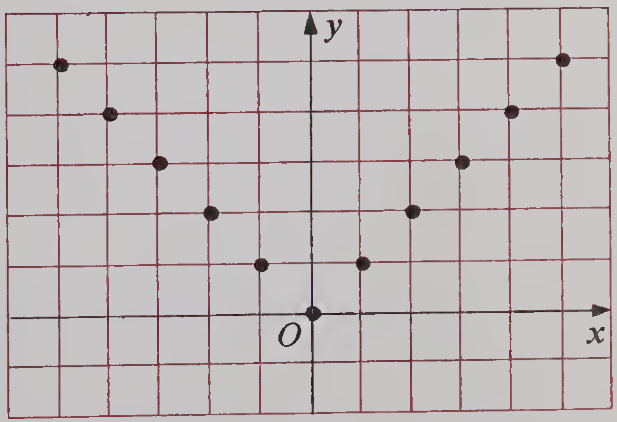


Figure 12–1

Emphasize that the first coordinate belongs to the domain and the second coordinate belongs to the range.

Avoid referring to the open sentence as the relation. See T.M. pg. 38 (2).

the second is a member of the range. In stating the rule for a relation, you must specify the range and domain. In Figure 12–1, the domain is the set of integers, and the range is the set of nonnegative integers.

Frequently you can state a rule as an open sentence: $y = |x|$, where $x \in$ {integers}. Such an open sentence is called a **formula**, and the relation it defines consists of the ordered pairs (x, y) in its solution set. In a formula the domain of the relation is the replacement set of one variable, in this case x, and the range is the replacement set of the other, y.

When a formula is used for a relation and the domain and range are not specified, you agree to include in the domain and range those real numbers for which the formula is meaningful. As division by 0 is meaningless, the domain and range of the relation defined by $y = \frac{1}{x}$ are understood to be {real numbers except 0}. See T.M. pg. 38 (3).

ORAL EXERCISES

Give each relation and its domain and range, and a formula, if possible.

Answers will be given in the order: domain, range, formula

SAMPLE. The price p in cents for n five-cent pencils is five times n.

What you say: The relation is the set of ordered pairs (n, p) for which $p = 5n$.

The domain is {nonnegative integers}.

The range is {nonnegative integral multiples of 5}.

{pos. real nos.}; {pos. real nos.}; $d = 2r$

1. The diameter d of a circle is twice the radius r.

{pos. real nos.}; {pos. real nos.}; $P = 4s$

2. The perimeter p of a square is four times the length s of one side.

For h = {nonneg. int}, for b = {pos. int.}; {nonneg. real nos. ≤ 1}; Av. $= h/b$

3. A batting average is the number of hits h divided by times at bat b.

{nonneg. real nos.}; {nonneg. real nos.}; $\frac{c}{2.54} = i$

4. Divide a length in centimeters c by 2.54 for the length in inches i.

{nonneg. real nos.}; {nonneg. real nos.}; $2.54i = c$

5. Multiply a length in inches i by 2.54 for the length in centimeters c.

{pos. int.}; {even int. > 10}; $c = 10 + 2d$

6. The charge c for a power mower for d days is \$10, plus \$2 a day.

{pos. real nos.}; {pos. real nos.}; $A = s^2$

7. The area A of a square is the square of one side s.

{real nos.}; {nonneg. real nos.}; $y = |n|$

8. Associate with each number n its absolute value $|n|$.

{nonneg. int. ≤ 500}; {nonneg. int. ≤ 500}; $R = n$

9. A stadium has 500 \$1 seats. The receipts r are determined by the number n of seats sold for any game.

{nonneg. int. ≤ 75}; {nonneg. integral multiples of $1000 \leq 75{,}000$}; $p = 1000h$

10. A builder makes \$1000 profit on each of 75 houses. His daily profit p is determined by the number of houses h sold each day.

{ratn. nos. ≥ 13 and < 20}, R = {50}, $c = 50a$, or {ratn. nos. > 0 and < 13}, R = {30}, $c = 30a$

11. A theater charges teen-agers 50¢ and preteens 30¢. The cost of admission c is determined by the patron's age a.

{int. from 100 to 200}; {integral multiples of .15 from 15 to 30}; d($) = .15o

12. A man receives 15¢ for each object he makes. His daily wage d is determined by the number of objects o he makes, which varies from 100 to 200.

In each case, the first row contains the first coordinates of the ordered pair.

Give a formula for each relation, and state the elements of its domain.

13.

t	1	2	3	4	5
d	3	6	9	12	15

d = 3t; {1,2,3,4,5}

14.

h	1	2	3	4
s	35	70	105	140

s = 35h; {1,2,3,4}

15.

x	2	3	4	5
y	6	7	8	9

y = x + 4; {2,3,4,5}

16.

v	4	8	12	16
u	2	4	6	8

$u = \frac{1}{2}v$; {4, 8, 12, 16}

17.

s	4	6	8
r	3	5	7

r = s − 1; {4, 6, 8}

18.

x	3	4	5
y	8	10	12

y = 2x + 2; {3, 4, 5}

WRITTEN EXERCISES

Graph the relation defined by each formula for the domain $\{-2, -1, 0, 1, 2\}$.

A

1. $y = x$

2. $y = x^2 - x$

3. $y = |x|$

4. $y = x^2 - |x|$

State each domain and range as a finite set.

5. $\{(0, 2), (1, 3), (2, 4)\}$

6. $\{(1, -1), (2, 0), (3, 1)\}$

7. $\{(-1, 4), (3, -2), (0, 0)\}$

8. $\{(5, 0), (-2, 3), (5, 1), (3, 6)\}$

9. $\{(-5, 4), (4, -4), (6, 4), (-7, -4)\}$

10. $\{(-6, -1), (-8, -2), (-9, -1), (-10, -2)\}$

For each roster write a formula; then copy and complete the chart.

11. Cost of gasoline:

Gallons	2	?	4	5	10	?	13
Cents	64	96	?	160	320	384	?

12. Parallelogram:

Height	4	5	?	9	10	?	20
Area	28	35	49	?	70	105	?

13.

Meters	1	2	3	4	?
Feet	?	6.56	9.84	?	19.68

14.

Kilograms	1	2	3	7	?
Pounds	?	?	6.6	15.4	33.0

B **15.**

x	1	?	3	4	?
y	0	0	?	0	0

16.

x	−3	−2	−1	4	5
y	3	2	?	4	5

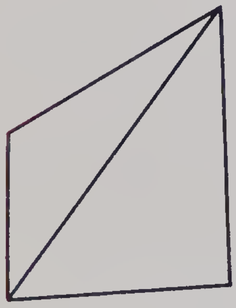
17.

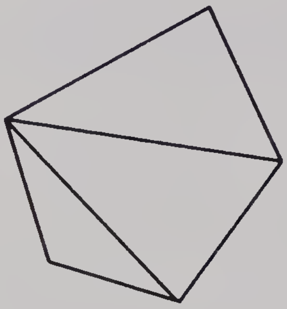
p	−2	−1	1	3	7
q	2	1	1	3	?

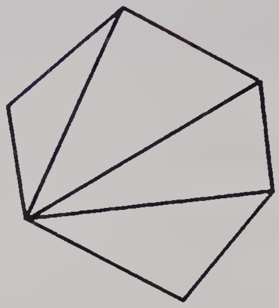
18.

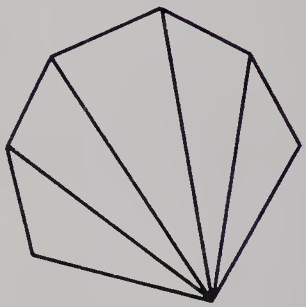
x	1	3	5	7
y	5	9	?	17

In Exercises 19–22 diagonals join one vertex to the other vertices of each polygon.









19. Chart the relation between the number of sides n and the number of triangles t in each figure. Write a formula for this relation.

20. Chart the relation between the number of sides n and the number of diagonals d. Write a formula for this relation.

21. Write a formula for t in terms of d.

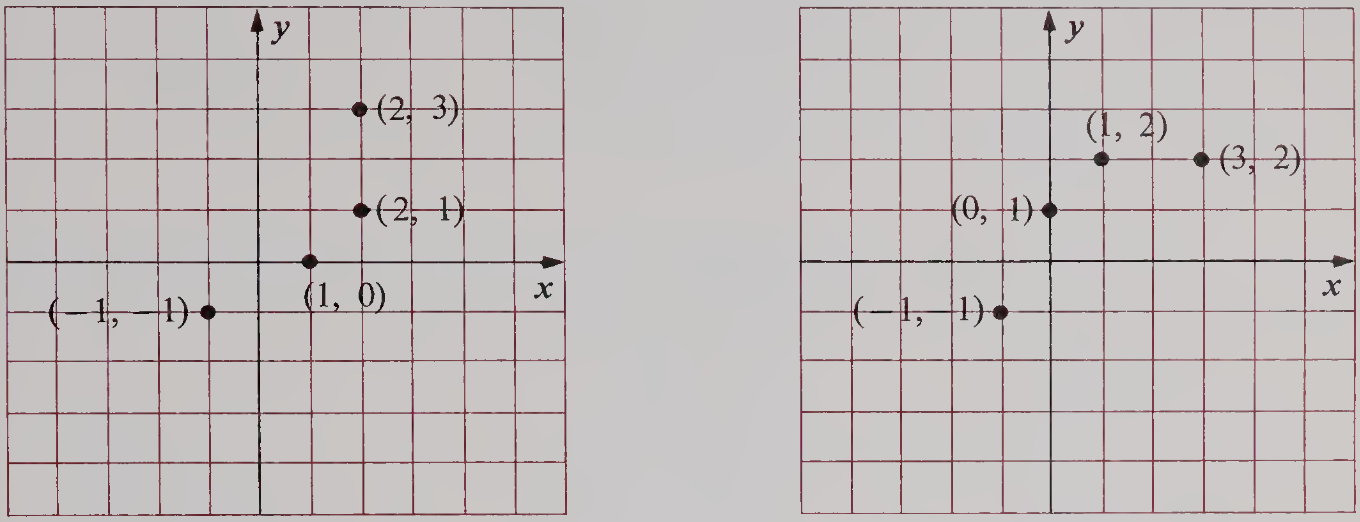
22. Write a formula which pairs the sum s of the angles (in degrees) of each figure with the number of sides n.

12–2 Functions

If you graph the two relations $\{(2, 1), (1, 0), (2, 3), (-1, -1)\}$, and $\{(1, 2), (0, 1), (3, 2), (-1, -1)\}$, (Figure 12–2), you find that two points of the first relation lie on the same vertical line. Its roster has one element of the domain appearing in two ordered pairs. Such is not true of the second relation. Relations of the second kind are called *functions*. Thus, a **function** is a relation which assigns to each element

See 12-2 T.M. pg. 38.

of the domain one and only one element of the range. The latter is called *the value of the function* for the given element of the domain. Relations which are functions are quite common. If a person regularly records his height at different ages, he will never repeat the same age, although eventually he will repeat the same height.

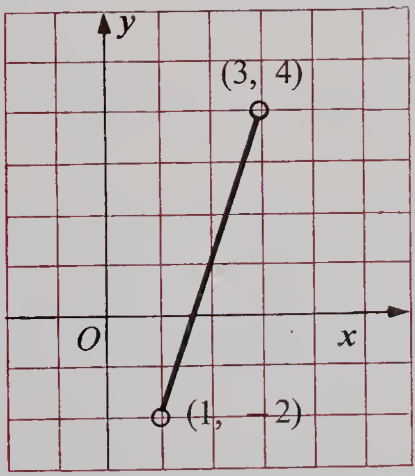


· *Figure 12–2* ·

EXAMPLE 1. **Graph the relation defined by this formula, give its domain and range, and tell whether it is a function:**

$$y = 3x - 5,$$
$$\text{if } 1 < x < 3.$$

Solution:



Domain: $x \in$ {real numbers between 1 and 3}

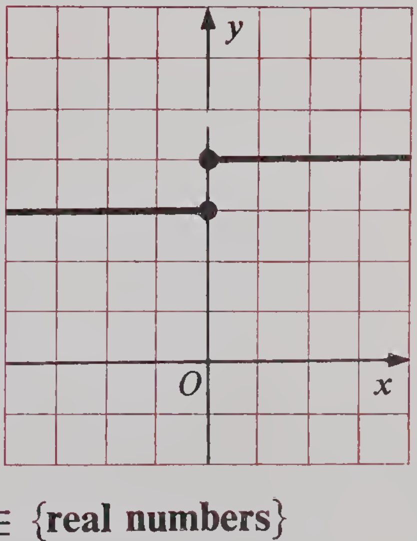
Range: $y \in$ {real numbers between −2 and 4}

This relation is a function.

EXAMPLE 2. **Graph the relation defined by this formula, give its domain and range, and tell whether it is a function:**

$$y = 4 \text{ if } x \geq 0 \text{ and}$$
$$y = 3 \text{ if } x \leq 0.$$

Solution:



Domain: $x \in$ {real numbers}

Range: $y \in \{3, 4\}$

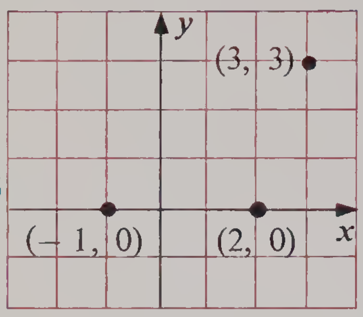
This relation is not a function. Two points lie on the vertical line $x = 0$.

ORAL EXERCISES

Answers will be given in the order: domain, range, function.

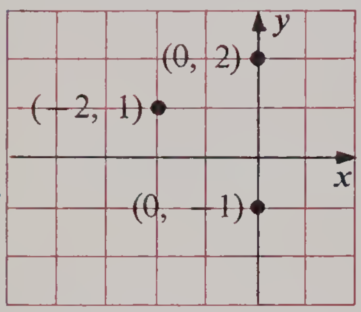
Give the domain and range of each relation, and tell if it is a function.

1.



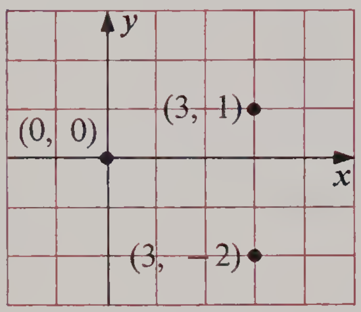
{-1, 2, 3}; {0, 3}; yes

2.



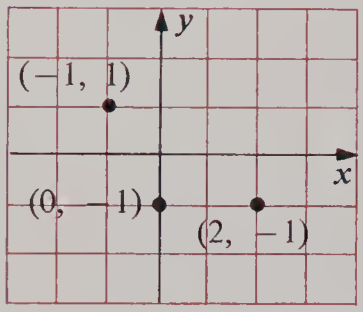
{-2, 0}; {-1, 1, 2}; no

3.



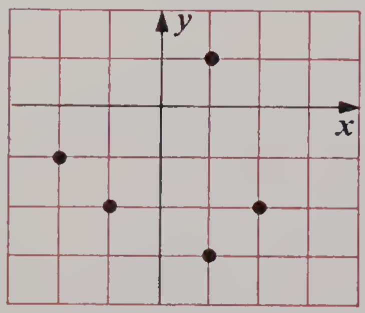
{0, 3}; {-2, 0, 1}; no

4.



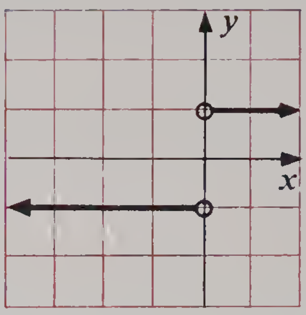
{-1, 0, 2}; {1, -1}; yes

5.



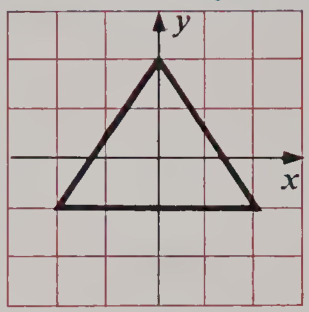
{-2, -1, 1, 2}; {-3, -2, -1, 1}; no

6.



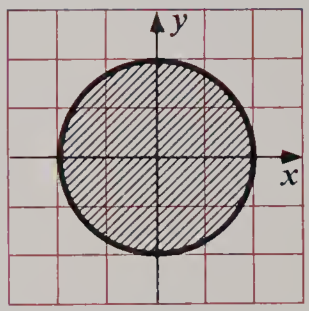
{real nos. excluding 0}; {-1, 1}; yes

7.



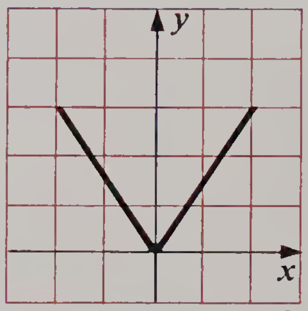
{-2 ≤ real nos. ≤ 2}; {-1 ≤ real nos. ≤ 2}; no

8.



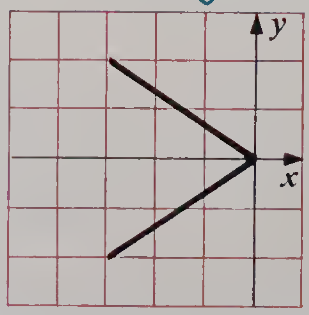
{-2 ≤ real nos. ≤ 2}; {-2 ≤ real nos. ≤ 2}; no

9.



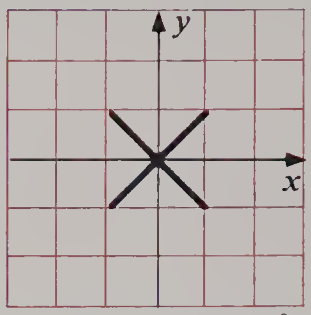
{-2 ≤ real nos. ≤ 2}; {0 ≤ real nos. ≤ 3}; yes

10.



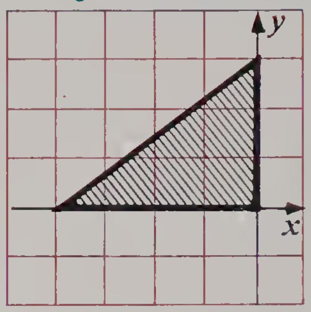
{-3 ≤ real nos. ≤ 0}; {-1 ≤ real nos. ≤ 1}; no

11.



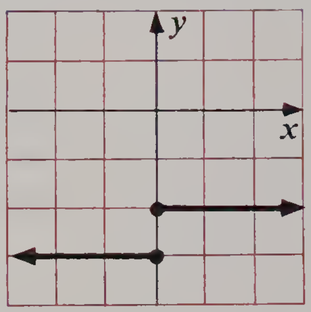
{-1 ≤ real nos. ≤ 1}; {-1 ≤ real nos. ≤ 1}; no

12.



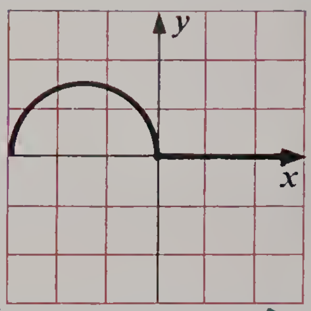
{-4 ≤ real nos. ≤ 0}; {0 ≤ real nos. ≤ 3}; no

13.



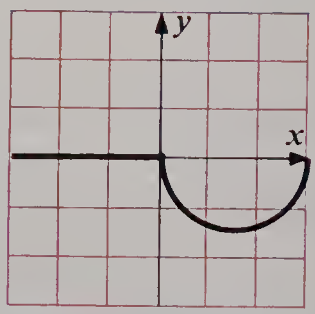
{real nos.}; {-3, -2}; no

14.



{real nos. ≥ -3}; {0 ≤ real nos. ≤ $\frac{3}{2}$}; y

15.



{-3 ≤ real nos. ≤ 3}; {-$\frac{3}{2}$ ≤ real nos. ≤ 0};

WRITTEN EXERCISES

A **1.** The perimeter of a rectangle: $p = 2(l + w)$.

- **a.** Write a formula for the perimeter when $w = 3$, and graph the function defined by the formula.
- **b.** Draw an arrow showing the point on your graph which indicates the perimeter of a rectangle $4\frac{1}{2}$ feet long.
- **c.** Indicate with an arrow the point showing the length of a rectangle whose perimeter is 20.

2. Distance traveled: $d = rt$.

- **a.** Write a formula for finding the distance when $r = 70$, and graph the function defined by your formula.
- **b.** Indicate with an arrow the point on your graph which shows the distance traveled in 5 hours and 20 minutes.
- **c.** Draw an arrow showing the point on the graph which indicates the number of hours required to travel 245 miles.

Write a formula and then graph the function defined by each rule.

3. The number of centimeters is 100 times the number of meters.

4. The number of feet is 3.3 times the number of meters.

5. The number of kilograms is .45 times the number of pounds.

6. The number of U.S. dollars is 2.8 times the number of British pounds.

7. The number of Italian lire is 625 times the number of U.S. dollars.

8. The average density of 100 cubic centimeters of matter is its weight in grams divided by 100.

9. The pressure of water on the bottom of a tank is 62.4 times the depth of water, in feet.

10. The work of pulling a load $3\frac{1}{2}$ feet is $3\frac{1}{2}$ times the force exerted, in pounds.

11. The total number of dollars to be repaid on \$100 borrowed at 5% per year is 100 more than 5 times the number of years of the loan.

12. The total number of dollars to be repaid on \$200 borrowed at $3\frac{1}{2}$% per year exceeds, by 200, 7 times the number of years of the loan.

13. A day's hire of a car, in dollars, is 10, plus .11 times the miles driven.

14. The total cost, in dollars, of printing a magazine is 25, plus .12 times the number of magazines printed.

Graph the function defined by each indicated domain and formula.

B **15.** $y = \frac{1}{2}x - 5;\ -3 \le x < 8$ **16.** $y = 3 - \frac{x}{2};\ -2 < x \le 6$

17. $y = x^3 - 1; x \in \{-2, -1, 0, 1, 2\}$

18. $y = -x^3; x \in \{-2, -1, 0, 1, 2\}$

19. $C = 2\pi r; 0 \leq r \leq 14$ (Use $\frac{22}{7}$ for π.)

20. $C = \pi d; 0 \leq d \leq 7$ (Use $\frac{22}{7}$ for π.)

21. $F = \frac{9}{5}C + 32; -100 \leq C \leq 100$

22. $P = 2(l + 3); l \geq 0$

23. $r = \dfrac{100}{t}; t > 0$

24. $S = 16t^2; t > 0$

C

25. Graph the function which assigns the number 1 to each nonnegative number, and the number 2 to each negative number.

26. Graph the function which assigns the number 3 to each positive number, and the number -2 to each nonpositive number.

27. Graph the function which assigns to each number half its absolute value.

28. Graph the function which assigns to each integer the negative of its absolute value.

VARIATION

12–3 Direct Variation and Proportion

The table shows the velocity v of a stone falling freely for t seconds. The ratio $\frac{v}{t}$ for every pair of numbers is the same, that is, $\frac{v}{t} = 16$, or $v = 16t$. Such a formula describes a *direct variation.*

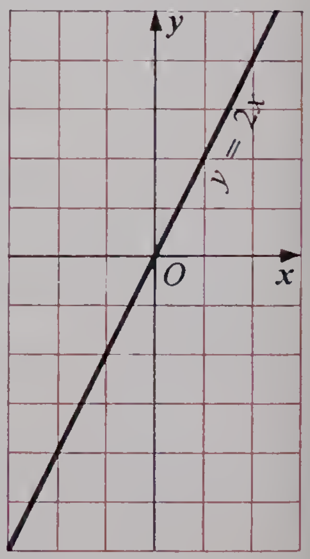
t	v	$\frac{v}{t}$
1	16	$\frac{16}{1} = 16$
2	32	$\frac{32}{2} = 16$
3	48	$\frac{48}{3} = 16$
4	64	$\frac{64}{4} = 16$

A **direct variation** is a function in which the ratio between a number y of the range and the corresponding number x of the domain is the same for all pairs of the function.

$$\frac{y}{x} = k \quad \text{or} \quad y = kx, \text{ where } k \text{ is a constant.}$$

You can say that *y varies directly as x* or *y is directly proportional to x* or *y varies with x.* k is the *constant of proportionality.*

Figure 12–3 is the graph of $y = 2x$. For every value of m the graph of $y = mx$ is a straight line with slope m passing through the origin. Thus, the graph of a direct variation is a straight line passing through the origin and having a slope equal to the constant of proportionality.



· *Figure 12–3* ·

If one ordered pair of a direct variation is (x_1, y_1) (read *x* sub 1, *y* sub 1) and another of the same function is (x_2, y_2), then

$$\frac{y_1}{x_1} = k \quad \text{and} \quad \frac{y_2}{x_2} = k, \quad \text{or} \quad \boxed{\frac{y_1}{x_1} = \frac{y_2}{x_2}}$$

If you wish, mention the alternative form: $y_1 : x_1 = y_2 : x_2$.

Such an equality of ratios is called a *proportion*, and can be read y_1 *is to* x_1 *as* y_2 *is to* x_2. In this proportion, x_1 and y_2 are called the *means* and y_1 and x_2 are called the *extremes*. Because $x_1y_2 = x_2y_1$,

See Sample, page 446.

in any proportion, the product of the means equals the product of the extremes.

EXAMPLE. **A weight of 15 grams stretches a spring 5 centimeters. What weight stretches it 12 centimeters if the elongation is directly proportional to the weight?**

See T.M. pg. 39 (4).

Solution 1: **Let E = elongation in centimeters**

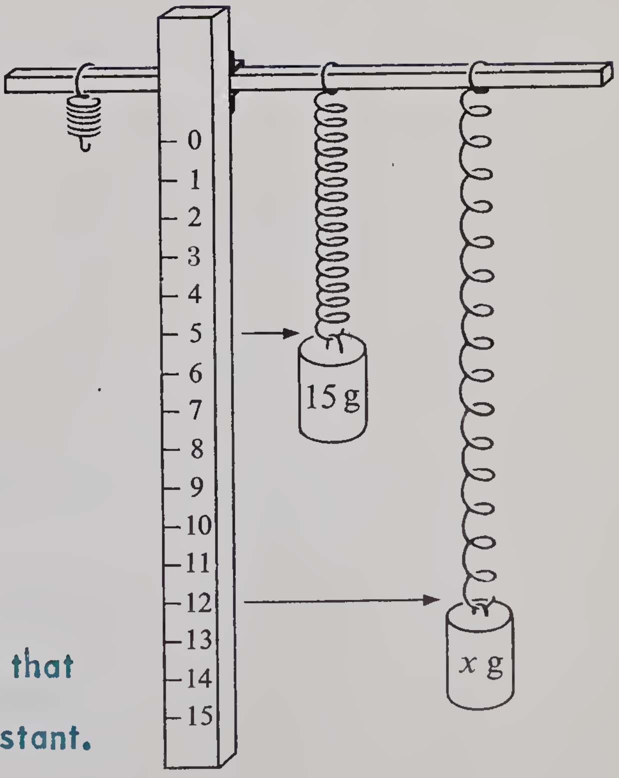
and W = weight in grams.

$$\frac{E_1}{W_1} = \frac{E_2}{W_2}; \ E_1 = 5, W_1 = 15, E_2 = 12$$

$$\frac{5}{15} = \frac{12}{W_2}$$

$$5W_2 = 180$$

$$W_2 = 36$$



Solution 2:

$$E_1 = kW_1 \qquad E_2 = kW_2$$

$$5 = k \cdot 15 \qquad 12 = \tfrac{1}{3}W_2$$

$$\tfrac{1}{3} = k \qquad 36 = W_2$$

Check:

$$\tfrac{5}{15} = \tfrac{1}{3}, \ \tfrac{12}{36} = \tfrac{1}{3}$$

$$\tfrac{1}{3} = \tfrac{1}{3} \ \checkmark$$

Check that $\frac{E}{W}$ is constant.

$\therefore$ **A weight of 36 grams is needed, Answer.**

ORAL EXERCISES

Read each proportion in two ways.

SAMPLE. $\frac{a}{3} = \frac{4}{7}$

What you say: *a* divided by 3 equals four-sevenths, and
a is to 3 as 4 is to 7.

1. $\frac{2}{3} = \frac{x}{6}$ $2:3 = x:6$

2. $\frac{x}{2} = \frac{15}{6}$ $x:2 = 15:6$

3. $\frac{a}{b} = \frac{3a}{3b}$ $a:b = 3a:3b$

4. $\frac{7d}{7b} = \frac{d}{b}$ $7d:7b = d:b$

5. $\frac{ab}{a} = \frac{bc}{c}$ $ab:a = bc:c$

6. $\frac{ac}{bc} = \frac{ad}{bd}$ $ac:bc = ad:bd$

7. $\frac{17}{m} = \frac{34}{5}$ $17:m = 34:5$

8. $\frac{n}{21} = \frac{3}{63}$ $n:21 = 3:63$

9. $\frac{x+3}{4} = \frac{x-3}{3}$ $(x+3)$
$(x-3)$

This is an extension of direct variation.
Translate into formulas expressing direct variation.

SAMPLE. The area of a circle varies directly as the square of its radius. The constant of proportionality is π.

What you say: Let A = area, and let r = radius.

Then $\frac{A}{r^2} = \pi$, or $A = \pi r^2$, or $\frac{A_1}{r_1{}^2} = \frac{A_2}{r_2{}^2}$

k will be used as the constant unless otherwise specified.

Notice that this exercise introduces *direct variation as a power.*

10. The $\overset{h}{\text{heat}}$ required to melt a substance varies with its $\overset{m}{\text{mass}}$. $\frac{h}{m} = k$

11. A shadow's $\overset{L}{\text{length}}$ at any hour varies with the object's $\overset{h}{\text{height}}$. $\frac{L}{h} = k$

12. The $\overset{F}{\text{force}}$ to push an object along a plane varies with the object's $\overset{w}{\text{weight}}$. $\frac{F}{w} =$

13. The annual $\overset{I}{\text{income}}$ from an investment varies with the interest $\overset{r}{\text{rate}}$. $\frac{I}{r} = k$

14. The $\overset{A}{\text{area}}$ of an equilateral triangle varies with the square of its $\overset{s}{\text{side}}$. $\frac{A}{s^2} =$

15. The surface $\overset{A}{\text{area}}$ of a sphere varies with the square of its $\overset{r}{\text{radius}}$. The constant of proportionality is 4π. $\frac{A}{r^2} = 4\pi$

16. The $\overset{P}{\text{pressure}}$ of water varies with its $\overset{d}{\text{depth}}$. $\frac{P}{d} = k$

17. The $\overset{w}{\text{weight}}$ of any liquid varies with its $\overset{V}{\text{volume}}$. $\frac{w}{V} = k$

18. The $\overset{P}{\text{power}}$ to move a ship varies with the cube of its $\overset{s}{\text{speed}}$. $\frac{P}{s^3} = k$

19. The $\overset{d}{\text{distance}}$ needed to stop a car varies with the square of its $\overset{s}{\text{speed}}$. $\frac{d}{s^2}$

20. The wind's $\overset{P}{\text{pressure}}$ on a flat surface varies with the square of its $\overset{v}{\text{velocity}}$. $\frac{P}{v^2} = k$

21. The $\overset{P}{\text{power}}$ an airplane needs varies as the fourth power of its $\overset{v}{\text{velocity}}$. $\frac{P}{v^4} = K$

State whether or not each formula or roster expresses direct variation.

22. $p = 4s$ Yes

23. $\frac{T_1}{T_2} = \frac{V_1}{V_2}$ Yes

24. $\frac{y}{x} = 1$ Yes

25. $xy = k$ No

26. $t = n - 2$ No

27. $\frac{x_1}{x_2} = \frac{y_2}{y_1}$ No

28.

x	1	2	4	8
y	$\frac{1}{2}$	1	2	4

Yes

29.

x	1	2	4	8
y	1	3	7	15

No

WRITTEN EXERCISES

In these direct variations, find the value of the indicated variable.

Ⓐ **1.** $x_1 = 12; x_2 = 8; y_2 = 14; y_1 = ?$

2. $x_1 = 18; y_1 = 8; y_2 = 6; x_2 = ?$

3. $u_1 = 15; z_1 = 35; u_2 = 8; z_2 = ?$

4. $m_1 = 13; m_2 = 9; n_2 = 6; n_1 = ?$

5. $r_1 = 2.6; s_1 = 6.5; s_2 = 4.5; r_2 = ?$

6. $t_1 = 2\frac{1}{2}; t_2 = 6\frac{2}{3}; v_1 = 4\frac{1}{2}; v_2 = ?$

Find all values of the variable for which each proportion is true.

7. $\frac{x-3}{x} = \frac{4}{7}$

8. $\frac{w-4}{w} = \frac{5}{9}$

9. $\frac{y}{y-16} = \frac{5}{3}$

10. $\frac{z}{z-3} = \frac{7}{4}$

11. $\frac{5}{9} = \frac{3x+2}{12}$

12. $\frac{11}{6} = \frac{3x-2}{27}$

Ⓑ **13.** $\frac{12}{x} = \frac{x}{3}$

14. $\frac{x+1}{x} = \frac{x+2}{x-1}$

15. $\frac{x-1}{x+3} = \frac{x+3}{x}$

16. If x varies directly as $y - 2$, and $x = 6$ when $y = 11$, find y when $x = 4$.

17. If y is proportional to $2x + 3$, and $y = 12$ when $x = 6$, find x when $y = 36$.

18. If a, b, and c vary directly as 4, 5, and 10 in that order, and $a + c = 42$, find b.

19. If V varies directly as R^3, and $V = \frac{9\pi}{2}$ when $R = 1\frac{1}{2}$, write the formula for V in terms of R.

Prove each of these properties of proportions if neither x_1 nor x_2 is zero.

SAMPLE. If $\frac{y_1}{x_1} = \frac{y_2}{x_2}$, then $y_1x_2 = x_1y_2$.

Solution: $\frac{y_1}{x_1} = \frac{y_2}{x_2}$	Given
$(x_1x_2)\left(\frac{y_1}{x_1}\right) = (x_1x_2)\frac{y_2}{x_2}$	Multiplication property of equality
$y_1x_2\left(\frac{x_1}{x_1}\right) = x_1y_2\left(\frac{x_2}{x_2}\right)$	Commutative and associative properties
$\therefore\ y_1x_2 = x_1y_2$	Multiplicative property of 1

Also use $\frac{x_1}{x_1} = 1$ and $\frac{x_2}{x_2} = 1$.

20. If $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ and if $y_2 \neq 0$, then $\frac{y_1}{y_2} = \frac{x_1}{x_2}$.

21. If $\frac{y_1}{x_1} = \frac{y_1}{x_2}$ and if $y_1 \neq 0$, then $x_1 = x_2$.

← Assign these exercises. See T.M. pg. 39 (5) for similar exercises.

Ⓒ **22.** If $\frac{x}{y} = \frac{3}{4}$ and $\frac{m}{n} = \frac{1}{3}$, find the values of $\frac{4mx - ny}{2nx - my}$.

23. If $\frac{r}{s} = \frac{3}{5}$ and $\frac{r}{t} = \frac{9}{10}$, find the value of $\frac{(s + t)}{s}$.

24. If $\frac{x + 1}{y - 1} = \frac{5}{1}$ and $\frac{x - 1}{y + 1} = \frac{1}{1}$, find the value of $\frac{x}{y}$.

PROBLEMS

Ⓐ **1.** The ratio of an object's weight on Earth to its weight on Neptune is 5:7. How much would a man who weighs 150 pounds here weigh on Neptune?

2. The ratio of slaked lime to sand in mortar is 2 to 5. How much sand is mixed with 14 bags of the lime in a supply of this mortar?

3. A charge of 3 tons of iron ore yields 2 tons of pig iron. What must the charge of ore be to obtain 78 tons of pig iron?

4. Sixteen tons of sulfur are needed to make 49 tons of sulfuric acid. How many tons of sulfur are needed to make 35 tons of acid?

5. Eighteen grams of hydrochloric acid neutralize 20 grams of lye. How many grams of lye are neutralized by 1080 grams of hydrochloric acid?

6. When an electric current is 20 amperes, the electromotive force is 100 volts. Find the force when the current is 50 amperes.

7. How long is a fifty-pound roll of wire which weighs .3 pounds per foot?

8. Find the resistance of 300 feet of a wire having .00027 ohms of resistance per foot, if the resistance varies directly as the length.

9. What length represents $5\frac{1}{3}$ feet on a map scaled at $\frac{3}{16}$ inch $=$ 1 foot?

10. What is the scale of a drawing in which a 28-inch statue is drawn $1\frac{3}{4}$ inches high? See T.M. pg. 39 (6).

B **11.** Pounds vary directly with cubic feet of a given substance, and k is 170 for limestone and 140 for sandstone. Which is heavier and by how much — a slab of limestone 5 feet by 3 feet by 8 inches, or a slab of sandstone $4\frac{1}{2}$ feet by 4 feet by 6 inches?

12. In grams per cubic centimeter, k for Celluloid is 1.4, and for ivory is 1.9. Find the difference in weight between Celluloid and ivory spheres 42 millimeters in diameter. Use $\pi = \frac{22}{7}$.

13. A diamond's price varies as the square of its weight. If one weighing $\frac{7}{8}$ carat is worth $625, find the cost of a similar diamond of $1\frac{2}{5}$ carats.

14. Rod A has 80 equal divisions, and rod B has 100, although they are the same length. If lengths which equal 60 divisions of rod B are cut from each, how many divisions are cut from rod A?

15. In a circuit of 48 volts a voltmeter registers 60 on its scale of 0–100. What is the maximum number of volts the meter can measure?

C **16.** If 1 inch equals 50 miles on a map, and Wyoming is a rectangle $5\frac{1}{2}$ inches by $7\frac{1}{8}$ inches, calculate its area to the nearest square mile.

17. A commander assigns 175 men per square mile to a rectangular area that is $\frac{7}{8}$ inch by $\frac{1}{2}$ inch on a map where the scale is 1 inch $=$ 4 miles. How many men will be in the area?

18. The liner *United States* displaces about 50,000 tons and is almost 1000 feet long. For a hundred-pound model what should be the scale and the length if the weight varies as the cube of the length?

12–4 Inverse Variation See 12-4 T.M. pg. 40.

Three rectangles whose lengths and widths are (12, 2), (8, 3), and (6, 4) have the same area: $12 \cdot 2 = 24$, $8 \cdot 3 = 24$, $6 \cdot 4 = 24$, or $lw = 24$. Such an open sentence describes *inverse variation.*

An **inverse variation** is a function in which the product of the coordinates of its ordered pairs is a constant. For any ordered pair (x, y) of the function, $xy = k$ or $y = \frac{k}{x}$. You say that ***y varies inversely as x*** or ***y is inversely proportional to x***, because $y = k\left(\frac{1}{x}\right)$, or y varies directly as the inverse of x.

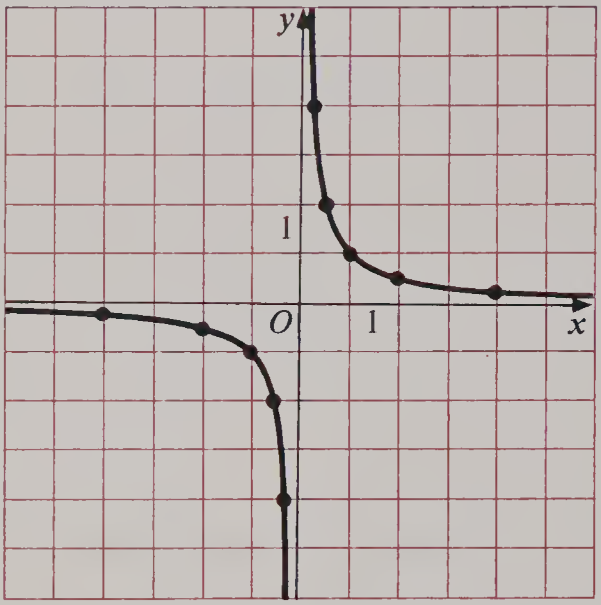
If (x_1, y_1) and (x_2, y_2) are ordered pairs of an inverse variation,

$$x_1y_1 = k \quad \text{and} \quad x_2y_2 = k, \quad \text{or} \quad x_1y_1 = x_2y_2.$$

If neither y_1 nor x_2 is 0,

$$\frac{x_1y_1}{x_2y_1} = \frac{x_2y_2}{x_2y_1} \quad \text{or} \quad \frac{x_1}{x_2} = \frac{y_2}{y_1}.$$

You would not expect the graph of an inverse variation to be a straight line, because its equation, $xy = k$, is not linear; one term, xy, is of the second degree.



x	y
$\frac{1}{4}$	4
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
4	$\frac{1}{4}$

x	y
-4	$-\frac{1}{4}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$-\frac{1}{4}$	-4

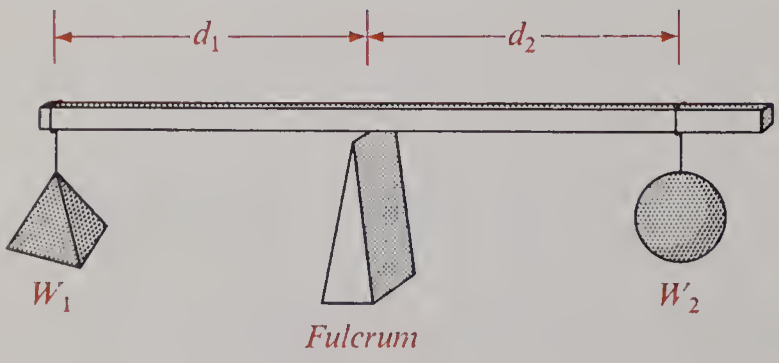
· *Figure 12–4* ·

As x increases, y decreases so that the product is always 1. Because of this, neither x nor y can have the value 0. Therefore, the graph consists of two separate branches which cross neither axis. For every nonzero value of k, the graph of $xy = k$ has this shape and is called a *hyperbola* (hy-**pur**-bo-la). The curve is in the first and third quadrants if k is positive and in the second and fourth quadrants if k is negative. If k were 0, what would be the limitation on the range and domain?

When negative answers are meaningless in practical problems, the range and domain are limited to positive numbers. The graph of such an inverse variation has only one branch.

One instance of inverse variation is the lever (**lee**-ver), a bar pivoted at a point called the fulcrum (**ful**-krum) (Figure 12–5). If weights w_1 and w_2 are placed at distances d_1 and d_2 from the fulcrum, and the lever is in balance, then

$$d_1w_1 = d_2w_2 \text{ or } \frac{d_1}{d_2} = \frac{w_2}{w_1}.$$



· *Figure 12–5* ·

EXAMPLE 1. **If an eighty-kilogram weight is 150 centimeters from the fulcrum of a lever, how far from the fulcrum is a ninety-kilogram weight which balances it?**

Solution: Let $w_1 = 80$, $d_1 = 150$, $w_2 = 90$.

$$d_1w_1 = d_2w_2$$

$$150(80) = d_2 90$$

$$133\tfrac{1}{3} = d_2$$

Check:

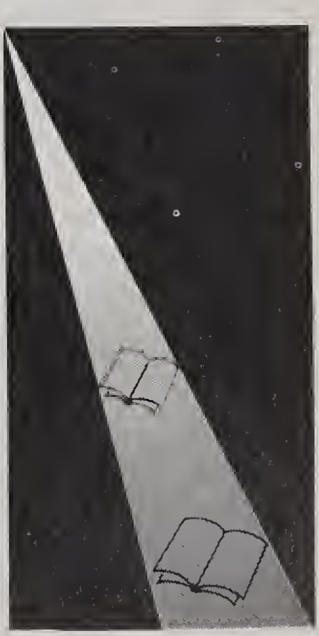
$$\frac{80}{90} \overset{?}{=} \frac{133\frac{1}{3}}{150}$$

$$\tfrac{8}{9} = \tfrac{8}{9} \checkmark$$

$\therefore$ **Distance of ninety kg. weight from fulcrum** $= 133\frac{1}{3}$ **centimeters, Answer.**

The brightness of the illumination of an object varies inversely as the square of the distance from the source of light to the object. If I is the amount of illumination and d is the distance from the light source to the object, $Id^2 = k$. If two objects are illuminated from the same source, $I_1d_1^2 = I_2d_2^2$ or $\dfrac{I_1}{I_2} = \dfrac{d_2^2}{d_1^2}$.

EXAMPLE 2. **The illumination of a book 15 feet from a lamp is 4 foot-candles. Find the illumination of the book 5 feet closer to the lamp.**



Solution: Let $I_1 = 4$, $d_1 = 15$, $d_2 = 10$

$$I_1d_1^2 = I_2d_2^2$$

$$4(15)^2 = I_2 10^2$$

$$9 = I_2$$

Check:

$$\frac{9}{4} \overset{?}{=} \frac{15^2}{10^2}$$

$$\tfrac{9}{4} = \tfrac{225}{100} \checkmark$$

$\therefore$ **In the second position the illumination is 9 foot-candles, Answer.**

ORAL EXERCISES

Translate each statement into a formula expressing inverse variation.

SAMPLE. The number of days required to complete a job varies inversely as the number of men working on it if they work at the same rate.

What you say: Let d = number of days; m = number of men.

$$\therefore dm = k, \text{ or } d = \frac{k}{m}, \text{ or } \frac{d_1}{d_2} = \frac{m_2}{m_1}, \text{ or } d_1m_1 = d_2m_2.$$

1. The base b of a triangle of constant area varies inversely as its altitude a. $ba = k$
2. The altitude a of a parallelogram of constant area varies inversely as its base b. $ab = k$
3. The volume V of a gas at constant temperature varies inversely as its pressure P. $VP = k$
4. Air pressure P is inversely proportional to altitude A. $PA = k$
5. The time t required to move from one point to another is inversely proportional to the rate of motion r. $tr = k$
6. The share s of one person in group expenses e varies inversely as the number of persons n in the group. $sn = e$
7. The temperature t at which water boils varies inversely as the number of feet h above sea level. $th = k$
8. The amount of capital P needed to yield a given income varies inversely as the rate of interest r. $Pr = k$
9. When two gears mesh, their speeds in revolutions per minute, R_1 and R_2, vary inversely as the numbers of teeth on the gears, T_1 and T_2. $\frac{R_1}{R_2} = \frac{T_2}{T_1}$
10. The force F necessary to pry up a rock varies inversely as the length L of the crowbar used. $FL = k$
11. When a generator supplies a fixed voltage to a circuit, the current in amperes I is inversely proportional to the resistance in ohms R. $IR = k$

Which exercises indicate direct variation? Which show inverse variation?
D = direct ; I = inverse

12. $uv = k$ I **13.** $u = \frac{k}{v}$ I **14.** $\frac{u}{v} = k$ D **15.** $kuv = 1$ I **16.** $\frac{v}{u} = \frac{1}{k}$ D

17.

m	3	2	.5
n	10	15	60

I

18. (1.5, .3), (2, .4), (3, .6) D

19. (4, 1.5), (6, 1), (12, .5) I

WRITTEN EXERCISES

In these inverse variations, find the value of the indicated variable.

A

1. $x_1 = 12, x_2 = 15, y_1 = 20, y_2 = ?$
2. $x_1 = \frac{2}{3}, y_1 = \frac{3}{4}, y_2 = 6, x_2 = ?$
3. $p_1 = 1.6, v_1 = 36, p_2 = 1.8, v_2 = ?$
4. $p_1 = .75, v_1 = .4, p_2 = .5, v_2 = ?$
5. If $d = rt$ and r is doubled while d stays constant, how does t change?
6. If $PV = k$ and P is trebled while k stays constant, how does V change?

7. If x varies inversely as $t + 3$, and $x = 6$ when $t = 7$, find t when $x = 15$.

8. If y varies inversely as $2t - 1$, and $y = 8$ when $t = 7.5$, find y when $t = 8.5$.

Ⓑ **9.** How far from a lamp does a book receive 4 times as much illumination as a book 3 feet from the lamp?

10. If F varies inversely as s^2, and $F = 80$ when $s = 100$, find F when $s = 40$.

11. If T is inversely proportional to d^2, what change in T doubles d?

12. If H varies inversely as R^2, what value of R causes H to become one-ninth as much as it is when $R = 8$?

13. If S varies inversely as the cube of L, and $S = 20$ when $L = 3$, write a formula for S in terms of L.

14. If R is inversely proportional to T^2, and $R = 5.5$ when $T = 2$, write a formula for R in terms of T.

PROBLEMS

Use an inverse variation to solve each problem.

Ⓐ **1.** At 40 m.p.h. how fast is a journey which takes 6 hours at 30 m.p.h.?

2. If 8 men do a job in 9 days, how long do 24 men take?

3. How far from the seesaw support must Mary sit to balance John, who sits 4 feet from it, if she weighs 80 and he weighs 100 pounds?

4. Jim, sitting 5 feet from the seesaw support, balances a friend who weighs 110 pounds and sits 6 feet from the support. How heavy is Jim?

5. A weight of 200 grams is 40 centimeters from the center support of a meter stick. Where would a weight of 400 grams balance the stick?

6. How much will each of 6 girls pay to rent a cottage which costs each of 5 girls $36?

7. At what rate does $1400 yield the same annual income as $2100 at 2%?

8. What sum at 3% yields the same yearly income as $1500 at 4%?

9. The altitude of a triangle is 15 inches, and the base, 6 inches. Find the altitude of a triangle of equal area whose base is 2 inches.

10. How many revolutions will a five-inch wheel make in going a distance which takes 42 revolutions of a thirty-inch wheel?

11. The volume of a gas is 40 cubic feet under 3 pounds pressure. What is its volume at the same temperature when the pressure is 8 pounds?

12. If the current is 18 amperes when the resistance is 5 ohms, what is the current when the resistance is 3 ohms?

13. A ten-inch pulley runs at 150 revolutions per minute (r.p.m.). How fast does the five-inch pulley it drives revolve, if the number of r.p.m. varies inversely as the diameter?

14. A gear with 18 teeth makes 100 r.p.m. and meshes with a gear having 12 teeth. What is the speed of the second gear if the number of r.p.m. varies inversely with the number of teeth?

B **15.** Jack weighs 148 pounds and Jill weighs 116. How far from Jack, on a seesaw 13.2 feet long, is the support balancing them?

16. If the gas pressure in a cylinder varies inversely as the distance of the piston from one end, how far was the piston moved to double the pressure?

17. The weight of a body at or above the earth's surface varies inversely as the square of the body's distance from the earth's center. What does a 540-pound projectile weigh 800 miles out from the earth's surface? (Use 4000 miles as the earth's radius.)

18. A three-quarter-inch wire has 12 ohms resistance; how much has the same length of half-inch wire, if resistance varies inversely as the square of the diameter?

See 12-5 T.M. pg. 40.

12–5 Joint Variation and Combined Variation (Optional)

The area A of a triangle depends upon its altitude a and its base b. If a and b are measured in the same units,

$$A = \frac{1}{2}ab \quad \text{or} \quad \frac{A}{ab} = \frac{1}{2}.$$

The area is directly proportional to the altitude and to the base, and you say that the area of a triangle *varies jointly* as its base and altitude.

Joint variation occurs when a variable z varies directly as the product of variables x and y. We say *z varies jointly as x and y*, and for a constant k, write: (Joint variation is always direct variation.)

$$\frac{z}{xy} = k \quad \text{or} \quad z = kxy \quad \text{and} \quad \frac{z_1}{x_1y_1} = \frac{z_2}{x_2y_2} \quad \text{or} \quad \frac{z_1}{z_2} = \frac{x_1y_1}{x_2y_2}.$$

EXAMPLE 1. **The volume of a circular cylinder varies jointly as its height and the square of its radius. If a right circular cylinder of height 10 inches and radius 3 inches has a volume of 90π cubic inches, find the volume of one with a height of 9 inches and a radius of 5 inches.**

Solution: Let $h_1 = 10$, $r_1 = 3$, $h_2 = 9$, $r_2 = 5$, $V_1 = 90\pi$.

$$\frac{V_1}{h_1 r_1^2} = \frac{V_2}{h_2 r_2^2}$$

$$\frac{90\pi}{10(3)^2} = \frac{V_2}{9(5)^2}$$

$$225\pi = V_2$$

Check:

$$\frac{90\pi}{225\pi} \overset{?}{=} \frac{10(3)^2}{9(5)^2}$$

$$\frac{90}{225} = \frac{90}{225} \quad \checkmark$$

The volume of the second cylinder is 225π cubic inches, Answer.

A second variation involving three variables is *combined variation.* **Combined variation** is indicated when a variable z varies directly as one variable x and inversely as another y. For a constant k,

$$z = \frac{kx}{y} \quad \text{or} \quad \frac{zy}{x} = k \quad \text{or} \quad zy = kx \quad \text{and} \quad \frac{z_1 y_1}{x_1} = \frac{z_2 y_2}{x_2} \quad \text{or} \quad \frac{z_1}{z_2} = \frac{x_1 y_2}{x_2 y_1}.$$

(Combined variation is a combination of direct and inverse variation.)

EXAMPLE 2. **The force between two small electrical charges varies jointly as the charges on the bodies and inversely as the square of the distance between them. When the charge on one body is 9 units and on the other 8 units, and they are 6 centimeters apart, the force between them is 2 dynes. Determine the force on the bodies when they are 4 centimeters apart.**

Solution: Let q_1 the charge on the first body $= 9$,

q_2 the charge on the second body $= 8$,

d distance between the two bodies $= 6$,

D distance between the two bodies $= 4$,

f force between the two bodies at 6 cm. $= 2$, and

F force between the two bodies at 4 cm. $= x$.

$$f = \frac{kq_1q_2}{d^2}, \; F = \frac{kq_1q_2}{D^2},$$

$$\frac{F}{f} = \frac{kq_1q_2}{D^2} \cdot \frac{d^2}{kq_1q_2} \quad \text{or} \quad \frac{F}{f} = \frac{d^2}{D^2}$$

$$\frac{x}{2} = \frac{6^2}{4^2}$$

$$x = 2\left(\tfrac{36}{16}\right)$$

$$x = \tfrac{9}{2}$$

Check:

$$2 = \frac{k(9)(8)}{36}, \; \frac{9}{2} = \frac{k(9)(8)}{16}$$

$$\tfrac{72}{72} = k, \; \tfrac{144}{144} = k$$

$$1 = 1 \quad \checkmark$$

$\therefore$ The force when the bodies are 4 cm. apart is $4\frac{1}{2}$ dynes, Answer.

ORAL EXERCISES

j. = jointly, d. = directly, inv. = inversely

Express in words. Assume that k is a constant.

j. as b and h

SAMPLE 1. $\frac{A}{bh} = k$ *Solution:* A varies jointly as b and h.

d. as t, inv. as V

SAMPLE 2. $PV = kt$ *Solution:* P varies directly as t and inversely as V.

j. as I and R

1. $E = kIR$

2. $A = kbh$

d. as T, inv. as V

3. $\frac{V}{T}P = k$

4. $\frac{t}{n}v = k$

j. as b and h

5. $\frac{A_1}{A_2} = \frac{b_1h_1}{b_2h_2}$

6. $\frac{E_1}{E_2} = \frac{I_1R_1}{I_2R_2}$

d. as T, inv. as V

7. $P = k\frac{T}{V}$

8. $V = k\frac{T}{P}$

9. $I = kPrt$

10. $V = klwh$

j. as B and h

11. $V_1:V_2 = B_1h_1:B_2h_2$

12. $A_1:A_2 = l_1w_1:l_2w_2$

d. as n, inv. as v

13. $\frac{t_1}{t_2} = \frac{n_1v_2}{n_2v_1}$

14. $\frac{P_1}{P_2} = \frac{T_1V_2}{T_2V_1}$

15. $R = \frac{12S}{\pi D}$

16. $S = \frac{\pi RD}{12}$

WRITTEN EXERCISES

Translate into formulas.

A

1. The area of a trapezoid varies jointly as its altitude and the sum of its bases.
2. The area of a rectangle varies jointly as the length and the width.
3. The volume of a pyramid varies jointly as the altitude and the area of the base.
4. The volume of a circular cone varies jointly as the altitude and the square of the radius.
5. The number of persons needed to do a job varies directly as the amount of work to be done and inversely as the time in which it must be done.
6. The time required for a journey varies directly as the distance traveled and inversely as the speed.
7. The number of gallons stored in a cylindrical tank varies jointly as the tank's height and the square of the radius of its circular base.
8. The cost of a job varies jointly as the number of men working and the number of days they work.

9. Centrifugal force varies directly as the square of the speed of a moving object and inversely as the radius of its circular path.

10. The pressure necessary to force water through a pipe varies directly as the square of the water's speed and inversely as the pipe's diameter.

PROBLEMS

1. In the formula $H = \dfrac{I^2Rt}{4}$, R remains constant. If I is doubled, and t is made 3 times as large, how is H changed?

2. In the formula $F = \dfrac{mv^2}{r}$, m remains constant, v is halved, and r is doubled. How does F change?

3. W varies jointly as x and y and inversely as the square of z. If $w = 150$, $x = 15$, $y = 18$, and $z = 9$, find (a) the constant k of variation, (b) the equation of relation, and (c) w, when $x = 21$, $y = 12$, and $z = 6$.

4. r varies directly as the cube of s and inversely as t and the square of u. If $r = 4$, $s = 6$, $t = 2$, and $u = 3$, find (a) the constant k of variation, (b) the equation of relation, and (c) t when $r = 9$, $s = 15$, and $u = 5$.

5. If 15 boys pick 360 boxes of berries in 4 hours, how many boys pick 540 boxes in 3 hours?

6. A rod's weight varies jointly as its length and the area of its cross section. If a rod $4\frac{1}{2}$ feet long with a half-inch square cross section is 4.8 pounds, what weight has a similar rod $7\frac{1}{2}$ feet long whose cross section is $\frac{1}{4}$ inch square?

7. When a mass moves at 30 feet per second in a circle whose radius is 6 feet, the centrifugal force is 1260 pounds. Find the force when that mass moves at 40 feet per second in a circle whose radius is 8 feet.

8. The load on a horizontal beam supported at its ends varies directly as the square of the beam's depth and inversely as its length between supports. A beam 10 feet long and 6 inches deep bears 1350 pounds. What load can one 16 feet long and 8 inches deep bear?

9. The cost of operating an appliance varies jointly as the number of watts drawn, the hours of operation, and the cost per kilowatt-hour. A thousand-watt iron operates for 30 minutes for 2¢ at 4¢ per kilowatt-hour. What is the cost of cooking 20 waffles 3 minutes each, if the waffle iron uses 720 watts?

10. The heat developed in an electric wire varies jointly as the wire's resistance, the time the current flows, and the square of the current. In 2 minutes a current of 5 amperes develops 1200 heat units in a wire of 8 ohms resistance. What resistance has a similar wire which develops 6000 heat units with a current of 10 amperes in 5 minutes?

B **11.** The wind pressure on a plane varies jointly as the surface area and the square of the wind's velocity. Under a velocity of 8 miles per hour, the pressure on 1 square foot is $\frac{1}{2}$ pound. What is the velocity when the pressure on a rectangle 2 feet by 3 feet is 27 pounds?

12. The heat lost through a windowpane varies jointly as the difference of the inside and outside temperatures and the window area, and inversely as the thickness of the pane. If 180 heat calories are lost through a pane 50 by 24 centimeters, $\frac{1}{2}$ centimeter thick, in one hour when the temperature difference is 30°C, how many are lost in one hour through a pane .4 centimeter thick having half the area, when the temperature difference is 32°C?

Chapter Summary

Inventory of Structure and Method

1. The ordered pairing of the members of two sets is a **relation** that can be shown by graph, roster, or rule. A **formula** is a rule stated as an open sentence. The domain of definition (domain) and the range of values (range) of the relation must be specified in each case. A **function** is a relation which assigns only one element of the range to an element of the domain.
2. **Direct variation** and **inverse variation** are special types of functions. If k is a constant, equations like $y = kx$ and $y = \frac{k}{x}$ are associated with, respectively, a direct and an inverse variation. In either case, you find the constant of proportionality, k, by substituting in the equation a pair of values for the variable.

Vocabulary and Spelling

relation (*p. 435*)
domain of definition (domain) (*p. 435*)
range of values (range) (*p. 435*)
formula (*p. 436*)
function (*p. 438*)
value of a function (*p. 439*)
direct variation (*p. 442*)
constant (*p. 442*)
constant of proportionality (*p. 442*)
proportion (*p. 443*)

means (*p. 443*)
extremes (*p. 443*)
inverse variation (*p. 447*)
hyperbola (*p. 448*)
joint variation (*p. 452*)
combined variation (*p. 453*)

Chapter Test

12–1 **Give the domain and range, with a formula, if possible.**

1. The weight p, in pounds, of n cubic feet of water is 62.4 times n.

2.

x	0	1	2	3	4	5
y	−1	1	3	5	7	9

3. Give the roster of $s = 2x - x^2$, when $-4 < x \leq 2$.

12–2 **4.** Is the following relation a function? Justify your answer.
$\{(3, -1), (1, -1), (0, 0), (1, 1), (3, 1)\}$

5. Graph $y = 3 - 2x$, if $|x| \leq 3$. Give its domain and range, and tell whether it is a function.

12–3 **6.** If $t = 5.6$ when $r = 3.2$, and t varies directly as r, find r when $t = 4.2$.

7. A freely falling body falls a distance directly proportional to the time it is falling. If a body falls 576 feet in 6 seconds, find the constant of proportionality and how far it falls in 10 seconds.

12–4 **8.** If P is inversely proportional to V, and $P = 12$ when $V = 8$, find V when $P = 16$.

9. The intensity H of heat radiation on a surface varies inversely as the square of the distance d from the heat source to the surface. If $H = 2.7$ when $d = 4$, find H when $d = 3$.

10. If m men do a job in h hours, how many can do it in 2 hours less? ($h > 2$).

12–5 (Optional)

11. The volume of a pyramid varies jointly as its altitude and the area of its base. A pyramid whose base is a four-inch square and whose altitude is 6 inches has a volume equal to 32 cubic inches. Find the volume of one 8 inches high with a three-inch square base.

12. If y varies directly as n and inversely as t, and $y = 42$ when $n = 14$ and $t = 18$, find n when $y = 36$ and $t = 21$.

Chapter Review

12–1 Relations *Pages 435–438*

1. Any pairing of the elements of two sets of numbers is a _?_.

2. A relation may be shown by _?_, _?_, or _?_.

3. The first coordinate of the roster of a relation is a member of the _?_; the second is a member of the _?_.

4. A formula is a rule stated as an _?_ sentence.

5. In stating the rule for a relation, always specify the _?_.

6. Using $\{3, 1, 0, -1, -3\}$ as the domain, give the roster of the relation $m = t^2 - 3t$.

7. Give the domain and range of relation shown by the roster below. Find a formula for the relation.

x	−2	−1	0	1	2	3
y	2	1	0	1	2	3

12–2 Functions *Pages 438–442*

8. A function is a kind of _?_.

9. In a function, an element of the domain may appear _?_ in the roster; an element of the range may appear _?_ times.

10. Two or more points in the graph of a _?_ may lie on the same vertical line.

11. A vertical line may not cut the graph of a _?_ more than once.

Is the given relation a function? Justify your answer.

12. $\{(-1, -2), (0, -3), (1, -2), (2, 1)\}$

13.

x	−2	2	2	7	7
y	0	2	−2	3	−3

12–3 Direct Variation and Proportion *Pages 442–447*

14. A direct variation is a kind of _?_.

15. When the _?_ of each pair of numbers of a function is the same throughout, the function is a _?_ _?_.

16. A direct variation may be shown by $\frac{y}{x} = k$ or $y =$ _?_.

17. The constant k is called the _?_ of _?_.

18. If r varies directly as s, then $r_1 : r_2 =$ _?_ : _?_.

19. An equality of two ratios is called a _?_.

20. In $a:b = c:d$, the means are __?__ and __?__; the extremes are __?__ and __?__.

21. In a proportion, the __?__ of the means equals the product of the __?__.

22. The graph of a direct variation is a __?__ __?__, whose slope is the __?__ of __?__, and whose y-intercept is __?__.

23. If c is directly proportional to n, and $c = 30$ when $n = 12$, find n when $c = 40$.

24. The lift on an airfoil L is directly proportional to the square of the air speed s. If $L = 756$ when $s = 450$, find L when $s = 300$.

25. On a map, $1\frac{7}{8}$ inches represents 30 miles. Find the actual distance between two points $6\frac{1}{4}$ inches apart on the map.

12–4 Inverse Variation *Pages 447–452*

26. An inverse variation is a kind of __?__.

27. An inverse variation is shown by $xy = k$ or $y =$ __?__, $x \neq$ __?__.

28. If r varies inversely as s, then $r_1:r_2 =$ __?__ : __?__.

29. The graph of an inverse variation is called a __?__.

30. If n varies inversely as z, then n varies directly as the __?__ of z.

31. Joan, weighing 108 pounds, sits 12 feet from the pivot of a seesaw. How many feet from the pivot should Bob sit in order to balance the seesaw if he weighs 144 pounds?

32. The force F of gravity on an earth satellite varies inversely as the square of its distance s from the center of the earth. If $F = 120$ when $s = 60{,}000$, find F when $s = 40{,}000$.

33. If x is inversely proportional to y and $x = 1.5$ when $y = 6$, find the rule specifying the function.

12–5 Joint Variation and Combined Variation (Optional)

Pages 452–456

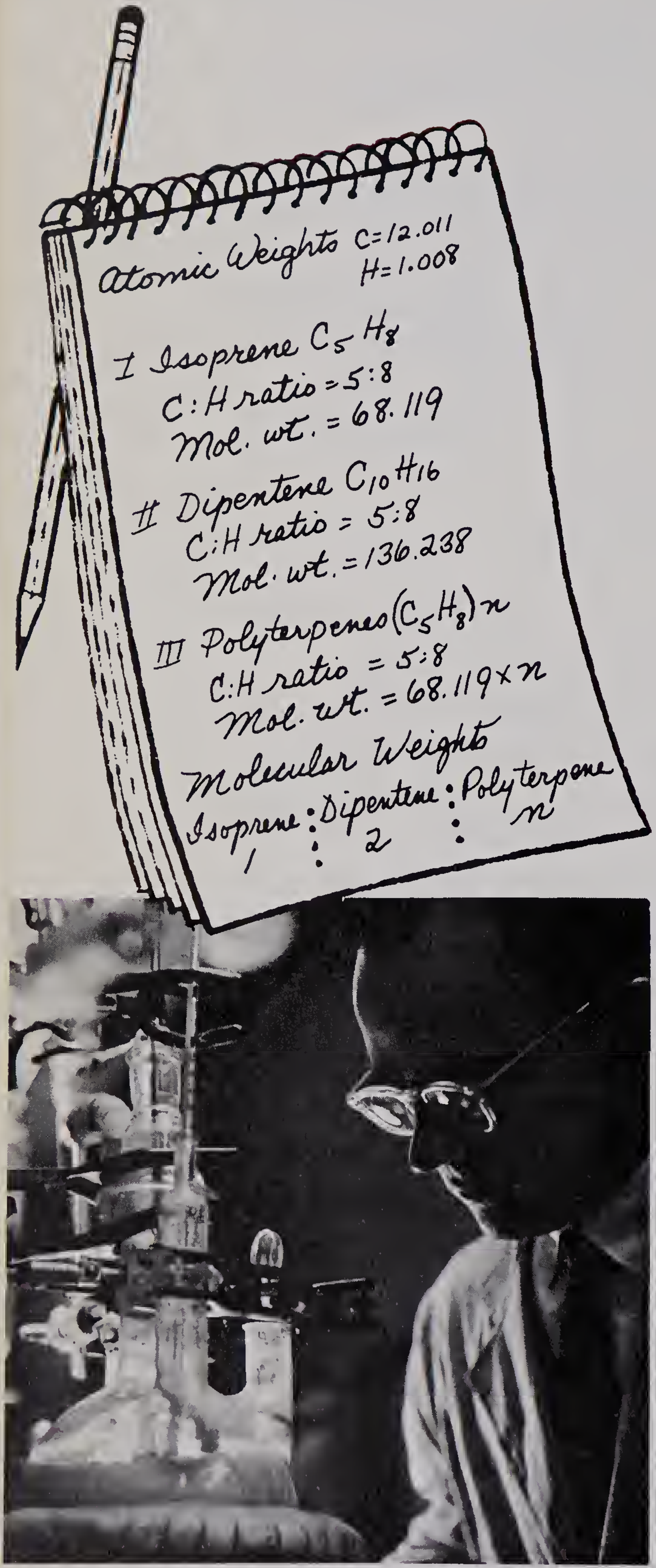
34. When a variable varies jointly as other variables, it varies __?__ as the __?__ of the other variables.

35. The centripetal force f on a body varies jointly as its mass m and the square of its speed v. If $f = 1{,}350{,}000$ when $m = 60$ and $v = 300$, find m when $f = 1{,}920{,}000$ and $v = 400$.

36. In a combined variation, a variable varies __?__ as one variable and __?__ as another.

37. If 12 men can dig a trench 150 feet long in 3 hours, how fast can 8 men dig a similar trench 200 feet long?

Petroleum Chemists and Mathematics


Atomic Weights C=12.011
H=1.008
I Isoprene C_5H_8
C:H ratio = 5:8
Mol. wt. = 68.119
II Dipentene $C_{10}H_{16}$
C:H ratio = 5:8
Mol. wt. = 136.238
III Polyterpenes $(C_5H_8)n$
C:H ratio = 5:8
Mol. wt. = 68.119×n
Molecular Weights
Isoprene : Dipentene : Polyterpene
1 : 2 : n

Nylon, paraffin, acrylic fibers, plastics, and high-power fuel oils are so common a part of modern life that we rarely stop to consider their origin. Yet, if we do reflect, we realize that each of these invaluable products was the result of research in petroleum chemistry.

Petroleum in its natural state cannot be used as a fuel. Called *crude oil*, it is a complex mixture of gases and liquids, known as hydrocarbons. The common fuel oils are most easily obtained by distillation from raw petroleum. Chemists like the one in the photograph work continually to improve the quality and increase the yield of fuel from petroleum.

Polymerization, the process which gives us many plastics and synthetic fibers, is probably the major development in petroleum research to date. It is a process whereby two or more molecules of the same substance are combined to form one large molecule. Although the new substance retains the same proportion of elements as the original substance, it has a higher molecular weight and a different set of physical properties.

The work pad shows the mathematical relationships between two polymer groups and the molecules from which they were formed. The original substance, isoprene, has five carbon atoms to eight hydrogen atoms and a molecular weight of $5 \times 12.001 + 8 \times 1.008$. The first polymer, dipentene, also has a ratio of five carbon : eight hydrogen, but since each dipentene molecule is formed from two isoprene molecules, the molecular weight of dipentene is twice that of isoprene. The polyterpenes, the third of these polymers, are important because of their rubber-like characteristics. They are composed of *n* number of isoprene units (*n* is an element of a set of whole numbers), and their molecular weight is *n* times that of isoprene.

Extra for Experts

Linear Programing

Graphing sets of linear inequalities can help you to make decisions such as this:

> Each day Mr. Kay needs at least 40 milligrams of niacin, 12 milligrams of riboflavin, and 24 milligrams of thiamine. He can buy *A* pills for 2¢ each, containing 6 milligrams of niacin, 1 milligram of riboflavin, and 8 milligrams of thiamine, or *B* pills at 3¢ each, containing 8 milligrams of niacin, 4 milligrams of riboflavin, and 3 milligrams of thiamine. What combination of pills will satisfy his minimum needs at the smallest cost daily?

Let x = number of *A* pills used daily, and
y = number of *B* pills used daily.
If C = total daily cost, in cents,

$$C = 2x + 3y, \text{ or } y = -\frac{2}{3}x + \frac{C}{3}$$

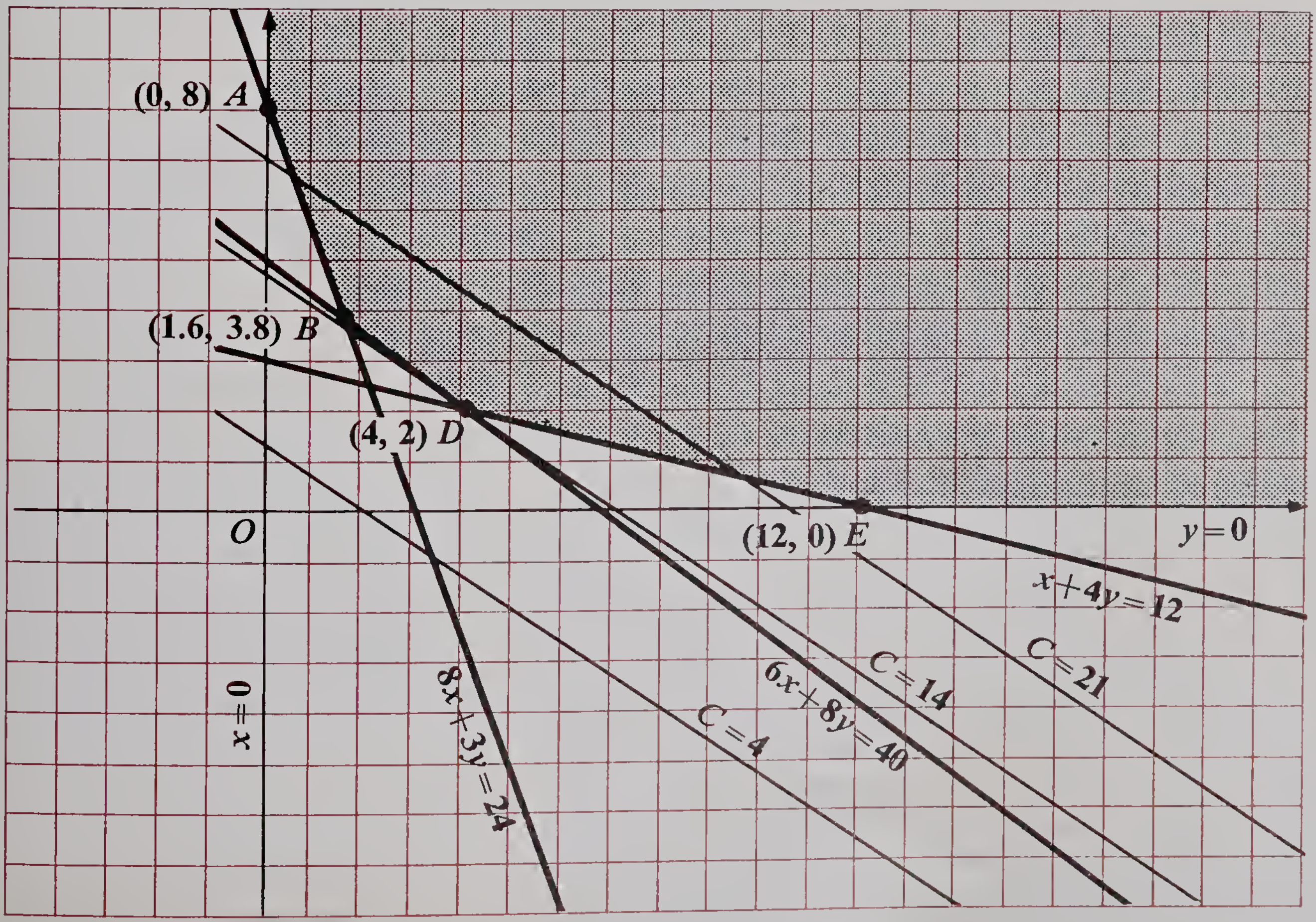
You are trying to minimize C (find its smallest value) within the limits of these inequalities (*constraints*):

$$\left.\begin{array}{ll}\textbf{1.} & 6x + 8y \geq 40\\ \textbf{2.} & x + 4y \geq 12\\ \textbf{3.} & 8x + 3y \geq 24\end{array}\right\}$$

The total amount of each vitamin must equal at least the daily need.

$$\left.\begin{array}{ll}\textbf{4.} & x \geq 0\\ \textbf{5.} & y \geq 0\end{array}\right\}$$

Mr. Kay cannot use a negative amount of either pill.



The graphs of these inequalities indicate the points in their common solution set (see figure, shaded area). Because the points satisfying all the constraints lie within or on its boundary line, this region is called the *feasible region.*

For any value of C, such as $C = 21$, the graph of the solution set of $C = 2x + 3y = 21$ is a straight line. For a smaller value of C, like $C = 4$, the graph is a line parallel to the graph of $C = 21$, with a smaller y-intercept. Different values of C give a family of parallel lines, each having y-intercept, $\frac{C}{3}$. As the figure suggests, the line of this family, having the smallest y-intercept and, therefore, smallest C, and containing a point of the feasible region, must intersect this region at a corner point. Substituting the coordinates of the corner points in $C = 2x + 3y$ shows that point $D(4, 2)$ gives the least value, $C = 14$, since at the other corners C has the values 24, 14.6, 24. Thus, the cheapest combination is four A pills and two B pills, with a total daily cost of 14¢.

Because the constraints, as well as the variable C, are linear in x and y, this is called a *linear programing* problem. Under such conditions a linear expression takes on its maximum and minimum values at corner points.

Questions

1. Graph the set of points defined by $y \geq \frac{x}{2} + 1, y \leq 4$, and $y \geq -3x + 8$.
2. Over the polygon obtained in Question 1, determine the points where $2.5x + .8y$ has its maximum and minimum values, which are __?__ and __?__.
3. Find x and y, maximizing $R = 4x + y$ subject to the constraints, $x \geq 0, y \geq 0, x + 2y \leq 6$, and $x + y \leq 3$.
4. A druggist wishes to display Toanup and Freshall bath salts on 18 inches of a shelf 10 inches deep with 10 inches of space above it. Toanup is at least four times as popular as Freshall, and a box of Toanup takes 35 cubic inches of space, while one of Freshall takes 40 cubic inches. If his profit on a box of Toanup is 28¢ and on a box of Freshall, 35¢, how many of each should he display to realize the maximum profit when all are sold? Find the maximum profit.
5. Machine A runs for an hour for \$1, producing 120 bolts and 50 screws in that time. Machine B runs for an hour at \$1.20 and produces 80 bolts and 80 screws. With a combined running time of no more than 15 hours, how long should each of the machines run to produce an order of 1000 bolts and 750 screws at the minimum operating cost?

THE HUMAN EQUATION

A Friendship Pact and What Came of It

It was a great honor to study with the Imam Mowaffek. Did not all Persia know that everyone who did so attained fame and fortune, honor and happiness? Any boy would be proud to be accepted as his pupil. Surely Nizam and Hasan and Omar were proud.

But the three boys couldn't quite believe that they were all destined for success. So one day they made a friendship pact, pledging each other, as Nizam later reported in his autobiography, "that to whomsoever this fortune falls, he shall share it equally with the rest." Fortune favored Nizam. About the year 1050, he became Vizier to the Sultan; that is, the chief administrative officer in all Persia. True to his boyhood pact, he shared his good fortune with Hasan and Omar.

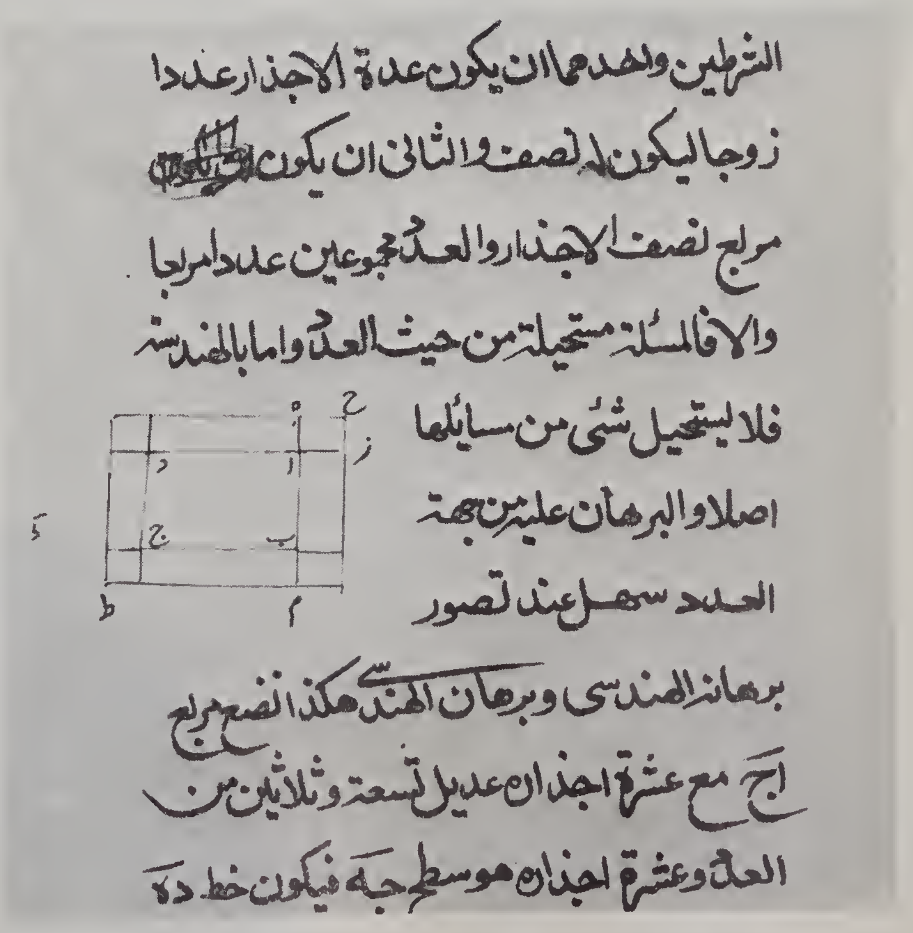
Hasan asked for and was given a position in the government. But he was jealous of Nizam, and tried to get his job for himself. As a result, the Sultan banished Hasan from his court. Hasan became the leader of a gang of outlaws. When he died, he was a fugitive from justice.

Omar's request was also granted — a place to live and freedom to study. He used his freedom well. He made thousands of observations of the heavens, recorded the results in tables that other astronomers could use, and reformed the calendar. And he wrote a really excellent book on algebra. When Omar died, in 1123, he was famous as a mathematician and a scientist.

Omar is still famous today — primarily as a poet. For he was the Omar Khayyam who wrote the well-known *Rubaiyat*.

Do you recognize the problem in the multiplication of polynomials on this page of Omar Khayyam's algebra? Here, also, is some of his poetry:

Ah, but my computations, people say,
Reduced the year to better reckoning? Nay
'Twas only striking from the calendar
Unborn tomorrow and dead yesterday.



الشرطين واحدهما ان يكون عدة الاجذار عددا
زوجا ليكون له نصف والثاني ان يكون [illegible]
مربع نصف الاجذار والعدد مجموعين عددا مربعا
والا فالمسئلة مستحيلة من حيث العدد واما بالهندسة
فلا يستحيل شي من مسائلها
اصلا والبرهان عليه من جهة
العدد سهل عند تصور
برهانه الهندسي وبرهان الهندسي هكذا نضع مربع
اج مع عشرة اجذار عديل تسعة وثلاثين من
العدد وعشرة اجذار هو سطح جه فيكون خط ده

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CHAPTER 13

Quadratic Equations and Inequalities

How long a cable should an engineer plan for a suspension bridge? What are the orbits of planets, asteroids, and satellites as they flash through the sky? What is the trajectory a missile follows to its destination (radar screen, bottom)? Mathematically, all of these problems have one common characteristic — their solutions involve quadratic equations.

Just as the natural world is not made up entirely of figures with straight lines, the world of algebra is not confined to linear equations. The study of quadratic equations will extend your insight into the properties and relationships of numbers; to the types of problems you can solve will be added those involving relationships of the second degree.

GENERAL METHODS OF SOLVING QUADRATIC EQUATIONS

13–1 The Square-Root Property

A quadratic equation can be put into the standard form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. At present your chief tool in solving such equations is factoring. If the equation is of the form $ax^2 + c = 0$, it is a pure quadratic and may be solved by using the ***property of square roots of equal numbers:***

Ask the class for proof. See 13-1 T.M. pg. 40.

If r and s are real numbers $r^2 = s^2$ if, and only if, $r = s$ or $r = -s$.

EXAMPLE 1. Solve $y^2 - 12 = 0$.

Solution:

$$y^2 - 12 = 0$$
$$y^2 = 12$$

$$\begin{array}{r|l} y = \sqrt{12} \text{ or } & y = -\sqrt{12} \\ y = 2\sqrt{3} & y = -2\sqrt{3} \end{array}$$

Check:

$$\begin{array}{r|l} (2\sqrt{3})^2 - 12 \stackrel{?}{=} 0 & (-2\sqrt{3})^2 - 12 \stackrel{?}{=} 0 \\ 12 - 12 = 0 \checkmark & 12 - 12 = 0 \checkmark \end{array}$$

$\therefore$ The solution set is $\{2\sqrt{3}, -2\sqrt{3}\}$, Answer.

EXAMPLE 2. Solve $64x^2 + 1 = 0$.

Emphasize again that equations are solved in terms of a specific replacement set for the variable.

Solution:

$$\begin{aligned} 64x^2 + 1 &= 0 \\ 64x^2 &= -1 \\ x^2 &= -\tfrac{1}{64} \end{aligned}$$

Since negative numbers have no square roots in the set of real numbers, $64x^2 + 1 = 0$ is not solvable in the real number system.

This method also solves quadratic equations having a trinomial square as one member and a nonnegative constant as the other.

EXAMPLE 3. Solve $t^2 - 6t + 9 = 25$.

Solution:

$$\begin{aligned} t^2 - 6t + 9 &= 25 \\ (t - 3)^2 &= 25 \\ t - 3 &= \pm\sqrt{25} \end{aligned}$$

$$\begin{array}{r|r} t - 3 = 5 & t - 3 = -5 \\ t = 8 & t = -2 \end{array}$$

Do the elements of $\{-2, 8\}$ check as the roots of the given equation?

ORAL EXERCISES

Give the roots of these equations.

1. $r^2 = 4$ ± 2
2. $s^2 = 9$ ± 3
3. $z^2 = \frac{1}{9}$ $\pm \frac{1}{3}$
4. $k^2 = \frac{1}{16}$ $\pm \frac{1}{4}$
5. $x^2 - 49 = 0$ ± 7
6. $y^2 - 64 = 0$ ± 8
7. $g^2 - 5 = 0$ $\pm\sqrt{5}$
8. $p^2 - 8 = 0$ $\pm 2\sqrt{2}$
9. $4y^2 = 1$ $\pm \frac{1}{2}$
10. $25x^2 = 1$ $\pm \frac{1}{5}$
11. $81s^2 = 16$ $\pm \frac{4}{9}$
12. $49t^2 = 4$ $\pm \frac{2}{7}$
13. $r^2 - 2r + 1 = 0$ $1, 1$
14. $v^2 + 2v + 1 = 0$ $-1, -1$
15. $8u^2 - \frac{1}{2} = 0$ $\pm \frac{1}{4}$
16. $9t^2 - \frac{1}{4} = 0$ $\pm \frac{1}{6}$

WRITTEN EXERCISES

Solve each equation.

A

1. $9x^2 - 25 = 0$
2. $25x^2 - 9 = 0$
3. $27 - y^2 = 0$
4. $125 - y^2 = 0$
5. $27m^2 - 3 = 0$
6. $8m^2 - 2 = 0$
7. $4v^2 - \frac{1}{25} = 0$
8. $\frac{1}{9} - 4v^2 = 0$
9. $3w^2 - 16 = 0$

10. $2w^2 - 121 = 0$
11. $(x - 1)^2 = 4$
12. $(x + 1)^2 = 9$
13. $(y + 2)^2 = 1$
14. $(y - 2)^2 = 1$
15. $4(z + 3)^2 = 25$
16. $9(z - 5)^2 = 49$
17. $(r - \frac{1}{3})^2 = 36$
18. $(r + \frac{1}{7})^2 = 25$
19. $(s + \frac{3}{4})^2 = 4$
20. $(s - \frac{3}{5})^2 = 25$
21. $y^2 + 10y + 25 = 9$
22. $y^2 - 12y + 36 = 4$
23. $t^2 - \frac{2}{11}t + \frac{1}{121} = 1$
24. $t^2 + \frac{2}{9}t + \frac{1}{81} = 1$

B

25. $(t - 6)^2 = 5$
26. $(t + 6)^2 = 7$
27. $(k + 1)^2 = 3$
28. $(k - 1)^2 = 2$
29. $r^3 - 4r = 0$
30. $9s - s^3 = 0$
31. $\frac{1}{5}z^3 - 5z = 0$
32. $4z - \frac{1}{4}z^3 = 0$
33. $k^3 - 5k = 0$
34. $k^3 - 7k = 0$
35. $m^4 + 4m^3 + 4m^2 = 0$
36. $m^4 - 6m^3 + 9m^2 = 0$

C

37. $x^2 - 2\sqrt{2}x + 2 = 0$
38. $x^2 + 2\sqrt{3}x + 3 = 0$
39. $x^2 - 2\sqrt{2}x + 2 = 9$
40. $x^2 - 2\sqrt{3}x + 3 = 4$
41. Is $x^2 + 7 = 0$ solvable over the set of real numbers? Why?
42. Is $x^2 + \frac{9}{25} = 0$ solvable in the real number system? Why?

13–2 Checking Solution Sets

For a teaching suggestion, see T.M. pg. 41 (1).

A method of checking roots of quadratic equations involves the property of factors whose product is zero. If r and s are the roots of a quadratic equation, you know that $x - r = 0$ and $x - s = 0$. Therefore, you can write

$$(x - r)(x - s) = 0 \quad \text{or} \quad x^2 - (r + s)x + rs = 0.$$

This latter equation is identical to one in the form $x^2 + bx + c = 0$, if $-(r + s) = b$ or $r + s = -b$, and $rs = c$. This is the ***property of the sum and product of the roots of a quadratic equation.***

If the roots of a quadratic equation of the form $x^2 + bx + c = 0$ are r and s, then $r + s = -b$ and $rs = c$.

EXAMPLE 1. Is $\{2, 3\}$ the solution set of $x^2 - 5x + 6 = 0$?

Solution:

$$x^2 - 5x + 6 = 0;\ b = -5,\ c = 6.$$
$$r + s = 2 + 3 = 5 = -(-5) = -b\ \checkmark$$
$$rs = 2 \cdot 3 = 6 = c\ \checkmark$$

$\therefore$ $\{2, 3\}$ is the solution set of $x^2 - 5x + 6 = 0$, **Answer.**

Emphasize that the coefficient of x^2 is 1. Ask students to state the property for $ax^2 + bx + c = 0$. See T.M. pg. 41 (2).

EXAMPLE 2. Is $\{-3 + \sqrt{10}, -3 - \sqrt{10}\}$ the solution set of $x^2 + 6x - 1 = 0$?

Solution:

Is the sum of the roots -6?

$$-3 + \sqrt{10} - 3 - \sqrt{10} \overset{?}{=} -6$$
$$-6 = -6 \checkmark$$

Is the product of the roots -1?

$$(-3 + \sqrt{10})(-3 - \sqrt{10}) \overset{?}{=} -1$$
$$9 - 10 = -1 \checkmark$$

$\therefore \{-3 + \sqrt{10}, -3 - \sqrt{10}\}$ is a solution set of $x^2 + 6x - 1 = 0$, **Answer.**

The numerals $-3 + \sqrt{10}$ and $-3 - \sqrt{10}$ may be written jointly as $-3 \pm \sqrt{10}$ (read -3 plus or minus $\sqrt{10}$).

Pupils should also check by substitution to learn the virtue of using the relations between roots and coefficients in checking.

ORAL EXERCISES

Give the sum and the product of the roots of each equation.

1. $x^2 - 20x + 9 = 0$ 20; 9
2. $x^2 + 17x + 10 = 0$ −17; 10
3. $x^2 + x + 3 = 0$ −1; 3
4. $x^2 - x + 2 = 0$ 1; 2
5. $y^2 - 3y - 1 = 0$ 3; −1
6. $y^2 - 19y - 4 = 0$ 19; −4
7. $2n^2 + 8n - 22 = 0$ −4; −11
8. $3m^2 + 3m - 15 = 0$ −1; −5
9. $z^2 + 4z = 2$ −4; −2
10. $z^2 + 10z = 10$ −10; −10

Is the set given for each equation its solution set?

11. $x^2 + 2x - 8 = 0$; $\{-4, 2\}$ Yes
12. $A^2 - 5A = 0$; $\{5, 0\}$ Yes
13. $2A^2 + 6A = 0$; $\{3, 0\}$ No
14. $x^2 - 8x - 20 = 0$; $\{10, -2\}$ Yes
15. $x^2 - 3x = 4$; $\{4, -1\}$ Yes
16. $r^2 + 5r = 14$; $\{-7, 2\}$ Yes
17. $9 - n^2 = 0$; $\{\pm 3\}$ Yes
18. $2n^2 = 32$; $\{\pm 4\}$ Yes

WRITTEN EXERCISES

Determine whether or not the set given for each equation is its solution set.

A

1. $2t^2 - 7t - 4 = 0$; $\{4, -\frac{1}{2}\}$
2. $2x^2 - x - 6 = 0$; $\{2, -1\frac{1}{2}\}$
3. $4x^2 - 8x - 21 = 0$; $\{\frac{7}{2}, -\frac{3}{2}\}$
4. $4x^2 - 4x - 15 = 0$; $\{\frac{3}{2}, -\frac{5}{2}\}$
5. $u^2 - 18u = 7$; $\{9 \pm 2\sqrt{11}\}$
6. $v^2 - 20v = 25$; $\{10 \pm 5\sqrt{5}\}$
7. $w^2 + 14w = 49$; $\{\pm 7\}$
8. $x^2 = 2 - 10x$; $\{5 \pm 3\sqrt{3}\}$
9. $3x^2 + 18x = 33$; $\{-3 \pm 2\sqrt{5}\}$

10. $x^2 = 9 - 12x$; $\{6 \pm 9\sqrt{15}\}$ **11.** $n^2 - n - 3 = 0$; $\left\{\frac{1 \pm \sqrt{13}}{2}\right\}$

12. $n^2 + n - 9 = 0$; $\left\{\frac{-1 \pm \sqrt{35}}{2}\right\}$

Find a quadratic equation having the given solution set.

B **13.** $\{5, 2\}$ **15.** $\{-4\}$ **17.** $\{\sqrt{7}, -\sqrt{7}\}$

14. $\{-2, 3\}$ **16.** $\{0, -3\}$ **18.** $\{-\sqrt{3}, \sqrt{3}\}$

19. $\{1 + \sqrt{2}, 1 - \sqrt{2}\}$ **21.** $\{3\sqrt{5}\}$

20. $\{3 - \sqrt{5}, 3 + \sqrt{5}\}$ **22.** $\{-2, -\frac{3}{2}\}$

13–3 Completing a Trinomial Square

If you can transform a quadratic equation into one having a trinomial square as a member, you can find its solution set.

This is an important technique to learn. See 13-3 T.M. pg. 41.

EXAMPLE 1. Solve $x^2 + 2x - 8 = 0$.

Solution:

1. Write an equivalent equation with the constant term as right member.
$$x^2 + 2x - 8 = 0$$
$$x^2 + 2x = 8$$

2. Add to both members the number making the left member a trinomial square.
$$x^2 + 2x + 1 = 8 + 1$$
$$(x + 1)^2 = 9$$

3. Use the property of square roots.
$$x + 1 = \pm 3$$

4. Form two linear equations.
$$x + 1 = 3 \quad\Big|\quad x + 1 = -3$$

5. Solve.
$$x = 2 \quad\Big|\quad x = -4$$

6. Check.
$$2^2 + 2(2) - 8 \overset{?}{=} 0 \quad\Big|\quad (-4)^2 + 2(-4) - 8 \overset{?}{=} 0$$
$$0 = 0 \checkmark \quad\Big|\quad 0 = 0 \checkmark$$

$\therefore$ **The solution set is $\{-4, 2\}$, Answer.**

The only unfamiliar step in this solution is the second, called **completing a trinomial square.** To apply the method of this example to any quadratic equation in the form $x^2 + bx = k$, you must be able to determine what to add to $x^2 + bx$ to produce a trinomial square.

Analyze the following trinomials which are squares of binomials.

$$(x + 5)^2 = x^2 + 2\cdot(5)x + (5)^2 \qquad (x - n)^2 = x^2 + 2(-n)x + (-n)^2$$
$$(x - 3)^2 = x^2 + 2(-3)x + (-3)^2 \qquad \left(x + \frac{b}{2}\right)^2 = x^2 + 2\left(\frac{b}{2}\right)x + \left(\frac{b}{2}\right)^2$$

Do you see that in each case the constant term is the square of half the coefficient of the linear term?

EXAMPLE 2. **What value of c makes $m^2 - \frac{2m}{5} + c$ a trinomial square?**

Solution: **Half the coefficient of the linear term is $\frac{1}{2}(-\frac{2}{5}) = -\frac{1}{5}$**
$c = (-\frac{1}{5})^2 = \frac{1}{25}$

Check: Is $m^2 - \frac{2m}{5} + \frac{1}{25}$ a trinomial square?

$$\begin{aligned} m^2 - \frac{2m}{5} + \frac{1}{25} &= \left(m - \frac{1}{5}\right)\left(m - \frac{1}{5}\right) \\ &= \left(m - \frac{1}{5}\right)^2 \checkmark \end{aligned}$$

$\therefore c = \frac{1}{25}$, **Answer.**

Stress this.

To solve an equation whose quadratic term has a coefficient other than 1, you first may use the division property of equality and divide each term by the coefficient of the quadratic term.

EXAMPLE 3. **Solve $3x^2 - 2x - 9 = 0$.**

Solution:

$$\begin{aligned} 3x^2 - 2x - 9 &= 0 \\ x^2 - \tfrac{2}{3}x - 3 &= 0 \\ x^2 - \tfrac{2}{3}x &= 3 \\ x^2 - \tfrac{2}{3}x + \tfrac{1}{9} &= 3 + \tfrac{1}{9} \\ (x - \tfrac{1}{3})^2 &= \tfrac{28}{9} \\ x - \tfrac{1}{3} &= \pm\sqrt{\tfrac{28}{9}} \end{aligned}$$

$$\begin{array}{l|l} x - \frac{1}{3} = \sqrt{\frac{28}{9}} & x - \frac{1}{3} = -\sqrt{\frac{28}{9}} \\ x = \dfrac{1}{3} + \dfrac{2\sqrt{7}}{3} & x = \dfrac{1}{3} - \dfrac{2\sqrt{7}}{3} \\ x = \dfrac{1 + 2\sqrt{7}}{3} & x = \dfrac{1 - 2\sqrt{7}}{3} \end{array}$$

Check: **Is the sum of the roots $-(-\frac{2}{3})$?**

$$\frac{1+2\sqrt{7}}{3} + \frac{1-2\sqrt{7}}{3} \overset{?}{=} -\left(\frac{-2}{3}\right)$$

$$\frac{2}{3} = \frac{2}{3} \checkmark$$

Is the product of the roots -3?

$$\left(\frac{1+2\sqrt{7}}{3}\right)\left(\frac{1-2\sqrt{7}}{3}\right) \overset{?}{=} -3$$

$$-3 = -3 \checkmark$$

$\therefore$ The solution set is $\left\{\frac{1+2\sqrt{7}}{3}, \frac{1-2\sqrt{7}}{3}\right\}$, Answer.

For computational purposes, you sometimes need rational approximations of such roots. To approximate them to the nearest tenth, use a two-decimal-place approximation of $\sqrt{7}$, and perform the indicated operations.

$$\frac{1+2\sqrt{7}}{3} \doteq \frac{1+2\ (2.64)}{3} = \frac{1+5.28}{3} = \frac{6.28}{3} = 2.09 \doteq 2.1$$

$$\frac{1-2\sqrt{7}}{3} \doteq \frac{1-2\ (2.64)}{3} = \frac{1-5.28}{3} = \frac{-4.28}{3} = -1.42 \doteq -1.4$$

$\therefore$ To the nearest tenth, the roots are 2.1 and -1.4, Answer.

ORAL EXERCISES

What value of c will make each trinomial a square?

1. $x^2 + 4x + c$ 4
2. $x^2 + 8x + c$ 16
3. $n^2 - 14n + c$ 49
4. $n^2 - 12n + c$ 36
5. $y^2 + y + c$ $\frac{1}{4}$
6. $y^2 - y + c$ $\frac{1}{4}$
7. $z^2 - 3z + c$ $\frac{9}{4}$
8. $u^2 + 5u + c$ $\frac{25}{4}$
9. $r^2 - .2r + c$ .01
10. $s^2 - .6s + c$ .09
11. $z^2 + 1.2z + c$ .36
12. $b^2 + 1.8b + c$ .81
13. $t^2 + \frac{1}{6}t + c$ $\frac{1}{144}$
14. $v^2 - \frac{v}{5} + c$ $\frac{1}{100}$
15. $h^2 - \frac{5}{4}h + c$ $\frac{25}{64}$
16. $w^2 + \frac{4w}{3} + c$ $\frac{4}{9}$

WRITTEN EXERCISES

Solve by completing the square. Give irrational roots in radical form, and to the nearest tenth. Students may use Table 4, here, and wherever approximations are sought.

A

1. $x^2 + 2x = 7$
2. $x^2 + 4x = 14$
3. $y^2 + y - 6 = 0$
4. $y^2 - 4y + 3 = 0$

5. $m^2 - 8m + 2 = 0$
6. $n^2 + 6n + 4 = 0$
7. $a^2 + 7a + 5 = 0$
8. $b^2 - 5b + 3 = 0$
9. $m^2 - 3m = 0$
10. $m^2 + 2m = 0$
11. $v^2 = 20v - 19$
12. $x^2 = 24x - 23$
13. $2r^2 - 10r - 10 = 0$
14. $3p^2 + 9p - 81 = 0$
15. $4y^2 + 12y + 9 = 0$
16. $9y^2 - 6y + 1 = 0$
17. $3n^2 + 7n = 1$
18. $5n^2 - 8n = 1$

B

19. $s^2 + \frac{2}{3}s = 0$
20. $t^2 - \frac{3}{2}t = 0$
21. $z^2 + \frac{5z}{2} = 25$
22. $w^2 + \frac{2w}{5} = 3$
23. $\frac{1}{y} + \frac{1}{y - 2} = 2$
24. $\frac{1}{x + 1} + \frac{1}{x} = 1$
25. $\frac{3}{s + 1} + \frac{1}{s - 1} = 2$
26. $\frac{2}{t + 1} - \frac{1}{t - 1} = 3$
27. $\frac{2m}{m + 3} - \frac{m}{m - 3} + \frac{9}{m^2 - 9} = 0$
28. $\frac{n}{n + 2} - \frac{2n}{2 - n} = \frac{3}{n^2 - 4}$

PROBLEMS

Recall earlier discussion (page 269) of sensible answers to problems involving quadratic equations.

Give irrational answers to the nearest tenth. Reject inappropriate roots.

A

1. The dimensions of a rectangle can be represented by consecutive even integers. Its area is 224 square inches. Find its dimensions.
2. A rectangular foyer is 10 feet longer than it is wide. Its floor area is 119 square feet. Find its length and width.
3. To cover two floors completely takes 690 square feet of carpet. One floor is 3 feet longer than it is wide. The other floor is 2 feet wider than the first is long, and its length is twice the length of the first. Find the dimensions of the floors.
4. Two tin squares together have an area of 325 square inches. One square is 5 inches longer than the other. Find the side of each.
5. Two buses travel at right angles from one spot. In one hour they are 50 miles apart. If one goes 10 m.p.h. faster than the other, what is the rate of each?
6. Some pupils buy a \$3 gift and share the cost equally. With 5 more pupils, each would give 2¢ less. How large is each share?
7. Two boys join a team, thus lowering by \$2 each member's share of \$120 spent on equipment. What is each share now?

8. A plane goes 600 miles with a wind of 30 m.p.h. in $\frac{1}{2}$ hour less than it returns against the wind. What is its rate in still air?

9. A dealer bought a number of old trucks for \$4000. Although one truck was useless, the dealer made a \$200 profit on each of the rest and so broke even on the transaction. How many trucks did he buy?

10. A dealer bought some items for \$10.40. He kept 4 and sold the rest at a gain of 60¢ each, with a total profit of \$2.20. How many did he sell?

13–4 The Quadratic Formula

See T.M. pg. 41 (3).

For a given set of coefficients a, b, and c, the quadratic equation $ax^2 + bx + c = 0$ can be transformed to express the variable x directly in terms of a, b, and c by completing the square. Study carefully the following parallel treatment of the *standard quadratic equation* and of a special quadratic equation.

$ax^2 + bx + c = 0$	$5x^2 + 8x + 1 = 0$
$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	$x^2 + \frac{8}{5}x + \frac{1}{5} = 0$
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	$x^2 + \frac{8}{5}x = -\frac{1}{5}$
$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$	$x^2 + \frac{8}{5}x + (\frac{4}{5})^2 = -\frac{1}{5} + (\frac{4}{5})^2$
$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$	$(x + \frac{4}{5})^2 = -\frac{1}{5} + \frac{16}{25}$
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	$(x + \frac{4}{5})^2 = \frac{11}{25}$
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	$x + \frac{4}{5} = \pm\sqrt{\frac{11}{25}}$
$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$	$x = -\frac{4}{5} \pm \sqrt{\frac{11}{25}}$
$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$	$x = -\frac{4}{5} \pm \frac{\sqrt{11}}{5}$
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$x = \frac{-4 \pm \sqrt{11}}{5}$

Students should write in the fraction line of the formula, i.e. $x = \frac{\qquad}{\qquad}$ and then fill in. This prevents errors like $x = -b \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$.

Class should explain why requiring $\sqrt{b^2 - 4ac}$ to be a real number requires tha

$b^2 - 4ac \geq 0.$

The last step is actually two sentences written as one and is called the *quadratic formula.* If either expression is taken as the value of x and substituted in the general quadratic equation, the resulting sentence is $0 = 0$. In developing the quadratic formula, notice the assumptions that $a \neq 0$, and that $\sqrt{b^2 - 4ac}$ is a real number ($b^2 - 4ac \geq 0$).

To solve any quadratic equation of the form $ax^2 + bx + c = 0$, substitute the coefficients in the quadratic formula, and evaluate.

EXAMPLE. **Solve $5x^2 - 8x + 1 = 0$ by using the quadratic formula.**

Solution:

$$5x^2 - 8x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}; \ a = 5, b = -8, c = 1$$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(5)(1)}}{2(5)}$$

$$x = \frac{8 \pm \sqrt{64 - 20}}{10} = \frac{8 \pm \sqrt{44}}{10}$$

$$x = \frac{8 \pm 2\sqrt{11}}{10} = \frac{4 \pm \sqrt{11}}{5}$$

Check: **Is the sum of the roots $-\left(\frac{-8}{5}\right)$? Is the product of the roots $\frac{1}{5}$?**

$\therefore$ **The solution set is $\left\{\frac{4 + \sqrt{11}}{5}, \frac{4 - \sqrt{11}}{5}\right\}$, Answer.**

Recall the standard form:
$ax^2 + bx + c = 0.$

ORAL EXERCISES

State the values of a, b, and c for these quadratic equations.

1. $2x^2 + 4x + 1 = 0$ 2; 4; 1
2. $3y^2 - 7y - 3 = 0$ 3; -7; -3
3. $z^2 + 12z - 9 = 0$ 1; 12; -9
4. $u^2 - 14u + 1 = 0$ 1; -14; 1
5. $17w - 8w^2 = -1$ 8; -17; -1
6. $19x - 4x^2 = 3$ 4; -19; 3
7. $7y^2 = 12y - 3$ 7; -12; 3
8. $5z^2 = 8z + 2$ 5; -8; -2
9. $u^2 - 14 = 0$ 1; 0; -14
10. $3v^2 - 17 = 0$ 3; 0; -17
11. $2w^2 - 5 = 0$ 2; 0; -5
12. $x^2 + 9x = 0$ 1; 9; 0
13. $y^2 - 7y = 0$ 1; -7; 0
14. $5z^2 = 0$ 5; 0; 0
15. $12v^2 = 0$ 12; 0; 0
16. $0 = 4t^2 - t$ 4; -1; 0

WRITTEN EXERCISES

Pupils may use Table 4.

Use the quadratic formula to solve each equation. Give irrational roots in simplest radical form, and correct to tenths.

A

1. $3x^2 + 5x + 1 = 0$
2. $4x^2 + 7x + 2 = 0$
3. $2x^2 - 8x + 3 = 0$
4. $4x^2 - 6x + 1 = 0$
5. $2x^2 - 5x - 12 = 0$
6. $6x^2 + x - 35 = 0$
7. $x^2 + 4x = 3$
8. $x^2 + 6x = 4$
9. $x^2 = 2x + 1$
10. $x^2 = 11 - x$
11. $20x^2 - 17x = -3$
12. $10x^2 - 17x = -3$
13. $3x^2 - x = 0$
14. $5x^2 - 17 = 0$

Factor these polynomials over the set of real numbers. See T.M. pg. 41 (4).

B

15. $x^2 - 2x - 2$
16. $x^2 - 2x - 4$
17. $y^2 + 6y + 3$
18. $y^2 + 8y + 13$

Show that these equations have no real roots.

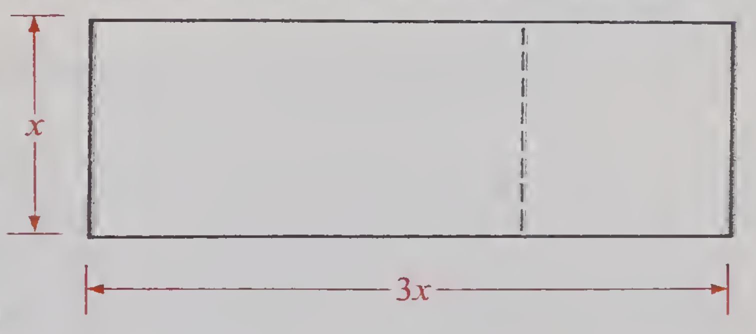
19. $x^2 - 2x + 2 = 0$
20. $x^2 - 2x + 4 = 0$
21. $\sqrt{y} = y + 1$
22. $2\sqrt{y} = y + 3$

PROBLEMS

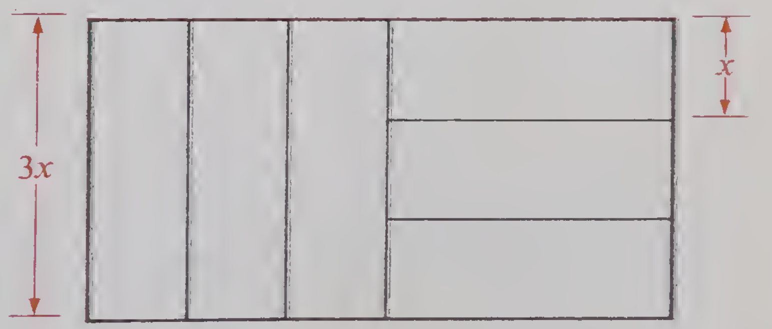
Pupils may use Table 4.

Give irrational answers to the nearest tenth. Reject inappropriate roots.

1. A rectangular floor of 147 square feet is three times as long as it is wide. It is divided into a rectangle twice as long as it is wide and a square (see diagram). Find the dimensions of each division.



2. A section of a wood floor is 72 square inches in area. This section has six rectangular pieces of equal area, three laid vertically and three, horizontally (see diagram). The length of each piece is three times its width; find these dimensions.



3. A square table top has a two-inch border. If two-thirds of its area is within the border, what are the dimensions of the table top?
4. One hundred square tiles would cover a floor now covered by 150 square tiles, 1 inch shorter on a side. How long is each small tile?
5. The perimeter of a triangle is 2 feet. Two sides form a right angle and are in the ratio of 3 to 4. Find the lengths of all three sides.
6. A group hikes east along a road. Another group starts cycling north at the same time from that point. The cyclers travel 7 miles an hour faster than the hikers. The groups are 13 miles apart at the end of 1 hour. Find the rate of each group.

B

7. Motor trouble reduced a bus's usual speed by 10 miles per hour, lengthening the time of its journey of 400 miles by 2 hours. What is the bus's usual speed?
8. A rectangular paper is 11 inches longer than it is wide. Eight-inch squares are cut from each corner, and the ends folded to form an open box whose volume is 2800 cubic inches. Find the paper's dimensions.
9. In a town with 1000 daily bus riders, the fare is 25¢. If the company raises the fare, for each increase of 1¢, it loses 4 riders.
   - **a.** What fare increase would yield the company $44 more daily?
   - **b.** How many answers are possible and how many are practical?
10. A telephone company has a net profit of $16 a year for each of its 1000 subscribers. This profit is decreased 1¢ for each additional subscriber.
    - **a.** How many new subscribers would increase the annual profit by $900?
    - **b.** How many answers are possible?

Distinguish between quadratic functions and quadratic equations. See 13-5 T.M. pg. 42.

13–5 The Nature of the Roots of a Quadratic Equation (Optional)

If to each real number x you assign the number y given by $y = x^2 + 24x + 140$, the resulting quadratic function has as its graph the parabola farthest left in Figure 13–1 (page 477). In that figure also appear the graphs of other quadratic functions:

$$y = -x^2 + 26x - 169 \quad \text{and} \quad y = \tfrac{1}{2}x^2 - 16x + 134$$
$$y = x^2 - x - 6 \qquad y = -x^2 - 7 \qquad y = x^2$$

Since the x-axis is the line $y = 0$, by replacing y with 0 in the formula for the function and solving the resulting quadratic equation, you can determine the x-intercepts of the graph of the function. Emphasize that the abscissas of the points where the graph of the function intersects the x-axis are th roots of the corresponding quadratic equation in x.

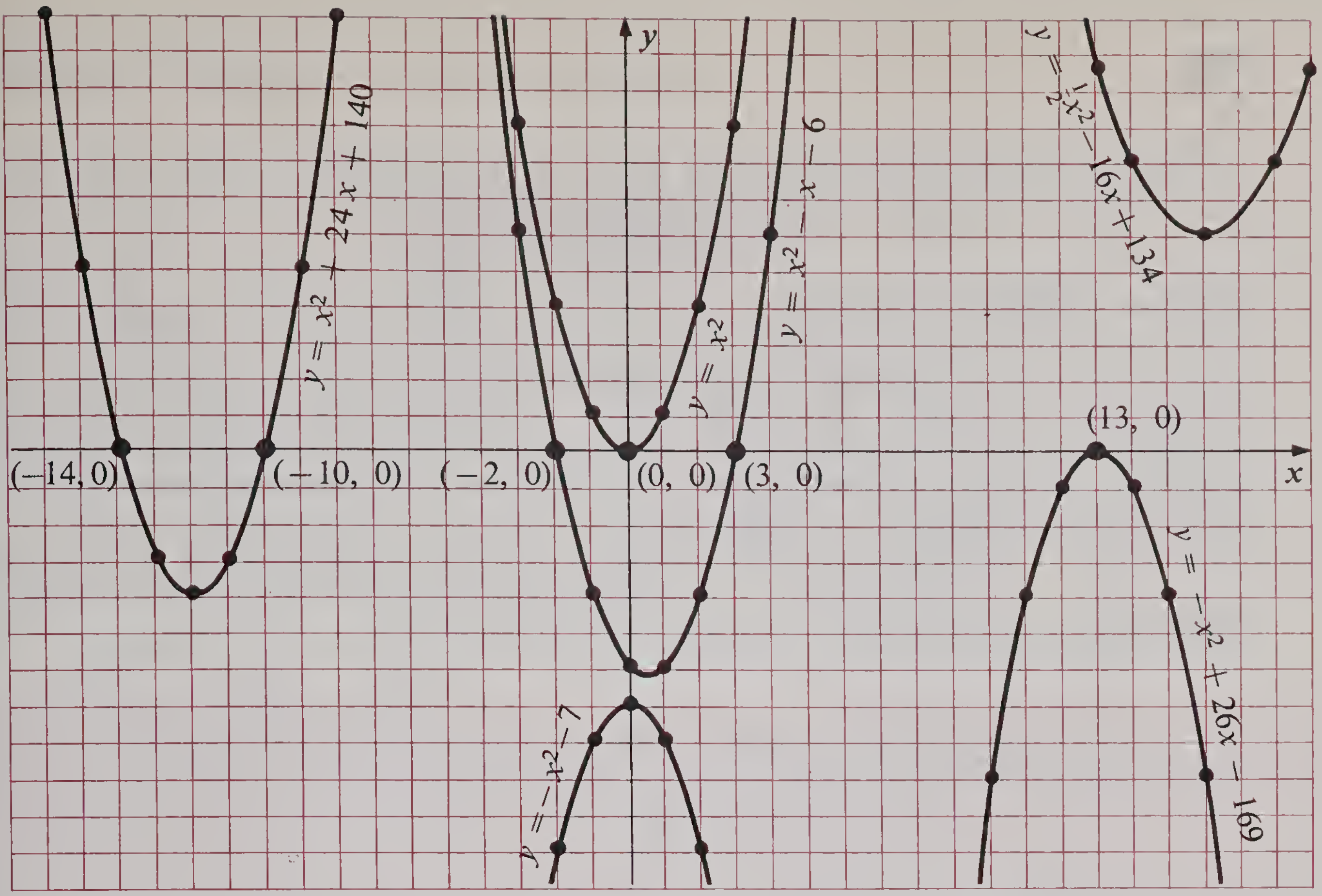


Figure 13–1

Case 1	*Case* 2	*Case* 3
$y = x^2 + 24x + 140$	$y = -x^2 + 26x - 169$	$y = \frac{1}{2}x^2 - 16x + 134$
$0 = x^2 + 24x + 140$	$0 = -x^2 + 26x - 169$	$0 = \frac{1}{2}x^2 - 16x + 134$
$x = \dfrac{-24 \pm \sqrt{576 - 560}}{2}$	$x = \dfrac{-26 \pm \sqrt{676 - 676}}{-2}$	$x = 16 \pm \sqrt{256 - 268}$
$x = -12 \pm 2$	$x = 13 \pm 0$	$x = 16 \pm \sqrt{-12}$
$x = -10$ or $x = -14$	$x = 13$ or $x = 13$	But $\sqrt{-12}$ does not exist in the set of real numbers.

The three cases are analyzed in the following chart:

	Number of points in common with the x-axis	Number of different real roots of the equation	Value of $b^2 - 4ac$
Case 1	2	2	a positive number
Case 2	1	1, a double root	zero
Case 3	0	0	a negative number

Do you see that the value of $b^2 - 4ac$ is the key to these cases? If $b^2 - 4ac > 0$, then $\sqrt{b^2 - 4ac}$ is positive. Thus, $ax^2 + bx + c = 0$ has two different roots for $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a} \neq \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

But if $b^2 - 4ac = 0$, then $\sqrt{b^2 - 4ac} = 0$ and you find that $\dfrac{-b + 0}{2a} = \dfrac{-b - 0}{2a} = \dfrac{-b}{2a}$, so that the roots are equal. But, for $b^2 - 4ac < 0$, no real root exists, because square roots of negative numbers do not exist in the real number system.

A quadratic equation with real coefficients can have

1. **two different real roots,**
2. **a double real root, or**
3. **no real roots.**

Because the value of $b^2 - 4ac$ distinguishes the three cases it is called the **discriminant** of the quadratic equation.

WRITTEN EXERCISES

Determine the nature of the roots of these equations graphically and by use of the discriminant.

A

1. $x^2 + 2x - 3 = 0$
2. $x^2 + 4x - 5 = 0$
3. $2x^2 + 8x + 15 = 0$
4. $-2x^2 + 6x + 8 = 0$
5. $-\frac{1}{2}x^2 - 5x + 6 = 0$
6. $\frac{1}{2}x^2 - 7x + 6 = 0$
7. $-x^2 - x - 1 = 0$
8. $-x^2 + x - 1 = 0$
9. $3x^2 - 4x + 2 = 0$
10. $4x^2 + 6x + 1 = 0$

Determine whether these polynomials can be factored over the set of real numbers. If they can, find the factors.

B

11. $x^2 + 1 - 2x$
12. $x^2 + 4x - 4$
13. $-2x^2 + 4x - 1$
14. $-3x^2 - 5 - 3x$
15. $u^2 + \frac{1}{4}u - \frac{1}{8}$
16. $u^2 - \frac{1}{6}u + \frac{1}{6}$
17. $\frac{1}{8}y - \frac{1}{8}y^2 - \frac{3}{2}y^3$
18. $\frac{1}{15}z + \frac{1}{5}z^2 - \frac{2}{3}z^3$

PROBLEMS

Find the roots in the most efficient way; reject inappropriate roots. Find irrational roots to the nearest tenth. Use the indicated formulas.

A

1. A ball is thrown down from the top of the Statue of Liberty, h (288) feet above the ground. In how many seconds t does it strike the ground if it starts falling with a velocity v of (a) 48 feet per second? (b) 112 feet per second? (c) 30 feet per second? $h = vt + 16t^2$

2. How high h is a circular arch with a radius r of 30 feet and a span s of 36 feet? $s^2 = 8rh - 4h^2$

3. The greatest number of lines l that can join a number of points p is 10. Find the number of points, if $l = \dfrac{p(p-1)}{2}$.

4. How many vertices are in a figure, if only 6 lines connect them?

5. What is the escape velocity v of a rocket fired from the earth's surface, if the radius r of the earth is 4000 miles? $v^2 = r\left(\dfrac{64.4}{5280}\right)$

6. Find the circular orbital speed v of (a) *Sputnik* I, if $h = 500$ miles; (b) the moon, if $h = 236{,}000$ miles; and (c) Discoverer XIII, if $h = 425$ miles. $v^2 = \dfrac{gR^2}{R + h}$; $g = 32.2$; $R = 4000$

7. At what speed v should you be driving 50 feet from an intersection in order to stop at the intersection? $d = .045v^2 + 1.1v$

The distance d between two points (x_1, y_1) and (x_2, y_2) is given by $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

8. Find the ordinates of the two points on the line $x = 7$ which are 13 units from the point (2, 1). $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

9. Find the distance between the points (4, −1) and (−2, 3).

10. Show that points (−2, −2), (4, 0), and (7, 1) are on one straight line.

11. How far d can a man in an airplane at an altitude of 4 miles see to the horizon? $d^2 = h(h + 8000)$

12. At what altitude can a pilot see 446 miles to the horizon?

THE SOLUTION OF QUADRATIC INEQUALITIES

13–6 Solving Quadratic Inequalities (Optional)

To solve quadratic equations by factoring, you use the property of factors whose product is zero. To solve a *quadratic inequality*, you use the ***property of the nonzero product of two real numbers:***

> A product is greater than zero if, and only if, both factors are greater than zero or both are less than zero; and a product is less than zero if, and only if, one factor is greater than zero and the other, less than zero.

EXAMPLE 1. **Graph the solution set of $x^2 - 2x > 3$.**

This is the familiar "law of signs" (page 135) stated in terms of the order relation. See 13-6 T.M. pg. 42.

Solution:

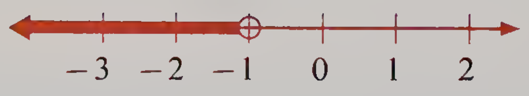
$$x^2 - 2x > 3$$
$$x^2 - 2x - 3 > 0$$
$$(x + 1)(x - 3) > 0$$

Both factors are less than zero.

$x + 1 < 0$ and $x - 3 < 0$

$x < -1$ and $x < 3$

The only numbers which satisfy both conditions satisfy $x < -1$.



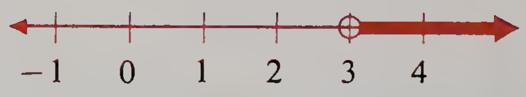
or

Both factors are greater than zero.

$x + 1 > 0$ and $x - 3 > 0$

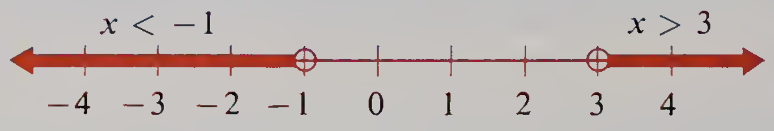
$x > -1$ and $x > 3$

The only numbers which satisfy both conditions satisfy $x > 3$.



Stress that two conditions are to be satisfied in each case.

∴ The solution set is



, Answer.

EXAMPLE 2. **Graph the solution set of $x^2 - 2x \leq 3$.**

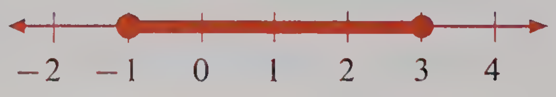
Solution:

$$x^2 - 2x - 3 \leq 0$$
$$(x + 1)(x - 3) \leq 0$$

$x + 1 \geq 0$ and $x - 3 \leq 0$

$x \geq -1$ and $x \leq 3$

The numbers satisfying both conditions satisfy $-1 \leq x \leq 3$.



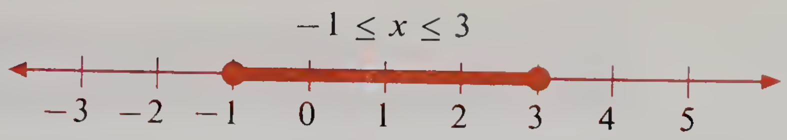
or

$x + 1 \leq 0$ and $x - 3 \geq 0$

$x \leq -1$ and $x \geq 3$

$x \in \emptyset$

No number satisfies both conditions.

∴ The solution set is  , Answer.

ORAL EXERCISES

1. $(x + 2)(x - 1) > 0$

a. If $x + 2 > 0$, then $x - 1 \overset{>}{?} 0$.

b. If $x + 2 < 0$, then $x - 1 \overset{<}{?} 0$.

c. If $x + 2 > 0$ and $x - 1 > 0$, $x > \overset{1}{?}$.

d. If $x + 2 < 0$ and $x - 1 < 0$, $x < \overset{-2}{?}$.

2. $(x - 1)(x - 2) > 0$

a. If $x - 1 > 0$, then $x - 2 \overset{>}{?} 0$.

b. If $x - 1 < 0$, then $x - 2 \overset{<}{?} 0$.

c. If $x - 1 > 0$ and $x - 2 > 0$, $x > \overset{2}{?}$.

d. If $x - 1 < 0$ and $x - 2 < 0$, $x < \overset{1}{?}$.

3. Consider $x^2 - x - 2 < 0$ or $(x + 1)(x - 2) < 0$.

a. If $x + 1 > 0$, then $x - 2 \overset{<}{?} 0$.

b. If $x + 1 > 0$ and $x - 2 < 0$, then $\overset{-1}{?} < x < \overset{2}{?}$.

c. If $x + 1 < 0$, then $x - 2 \overset{>}{?} 0$.

d. If $x + 1 < 0$ and $x - 2 > 0$, then $x < \overset{-1}{?}$ and $x > \overset{2}{?}$.

e. Since x cannot satisfy the two conditions in (d) simultaneously, the solution set is ?. $\emptyset$

4. Consider $x^2 - 3x \ge 0$ or $x(x - 3) \ge 0$.

a. If $x \ge 0$, then $x - 3 \overset{\ge}{?} 0$.

b. If $x \ge 0$ and $x \ge 3$, then $x \ge \overset{3}{?}$.

c. If $x \le 0$, then $x - 3 \overset{\le}{?} 0$.

d. If $x \le 0$ and $x \le 3$, then $x \le \overset{0}{?}$.

5. Consider $x^2 - 9 \le 0$ or $(x + 3)(x - 3) \le 0$.

a. If $x + 3 \ge 0$, then $x - 3 \overset{\le}{?} 0$.

b. If $x + 3 \ge 0$ and $x - 3 \le 0$, then $\overset{-3}{?} \le x \le \overset{3}{?}$.

c. If $x + 3 \le 0$, then $x - 3 \overset{\ge}{?} 0$.

d. If $x + 3 \le 0$ and $x - 3 \ge 0$, then $\overset{3}{?} \le x \le \overset{-3}{?}$. $\therefore x \in \emptyset$

WRITTEN EXERCISES

Graph the solution set of each inequality.

A

1. $x^2 - 5x + 4 > 0$
2. $x^2 - 5x + 4 < 0$
3. $x^2 + 4x + 3 < 0$
4. $x^2 + 4x + 3 > 0$
5. $y^2 + y > 2$
6. $y^2 + y < 2$
7. $2n^2 < n + 15$
8. $5m^2 > 12 - 4m$
9. $r^2 + 6 \geq 7r$
10. $s^2 - 5s \leq 6$
11. $x^2 \geq 9$
12. $x^2 \leq 25$

B

13. $x^2 + 2x + 1 > 0$
14. $x^2 - 6x + 9 \geq 0$
15. $x^2 - 4x + 4 < 0$
16. $x^2 + 10x + 25 \leq 0$
17. $x^2 > 0$
18. $x^2 < 0$

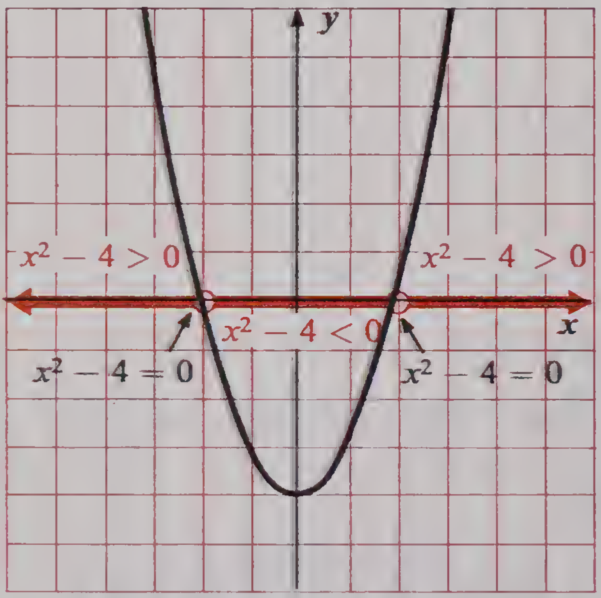
Find the values of x for which each expression is a real number.

C

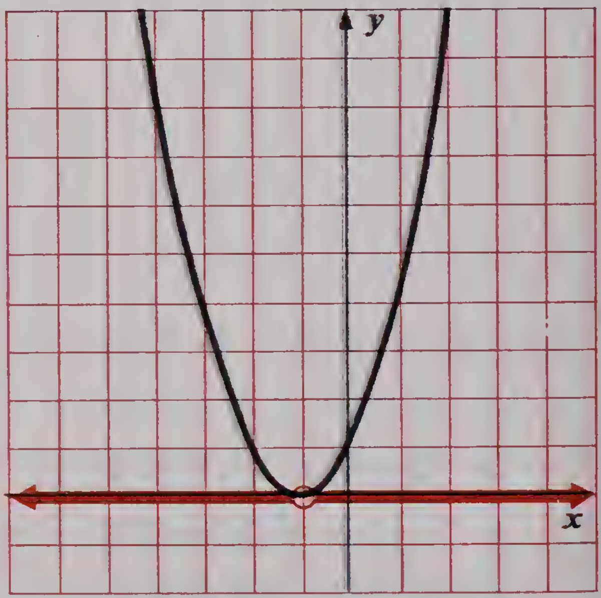
19. $\sqrt{x^2 - 2x - 35}$
20. $\sqrt{x^2 - 12x + 35}$
21. $\sqrt{x^2 + 5x}$
22. $\sqrt{x^2 - x}$

13–7 Using Graphs of Equations to Solve Inequalities (Optional)

Figure 13–2 shows the graph of $y = x^2 - 4$. The abscissas of points at which $y = 0$ form the solution set of $x^2 - 4 = 0$. The values of x for which $y > 0$ give the solution set of $x^2 - 4 > 0$. And the values of x for which $y < 0$ give the solution set of $x^2 - 4 < 0$. Do you see that you can find the solution sets of these open sentences by determining the values of x for which the graph of the equation is on, above, or below the x-axis?



· *Figure 13–2* ·



· *Figure 13–3* ·

Pupils sometimes find the graphic solution simpler than the method of 13-6. Actually drawing the graph of $y = ax^2 + bx + c$ is unnecessary. See 13-7 T.M. pg. 42.

Figure 13–3 shows the graph of $y = x^2 + 2x + 1$. How many x-intercepts does it have? Does the point $(-1, 0)$ satisfy $y = 0$, $y > 0$, or $y < 0$? Which sentence is satisfied by values of x to the left of $(-1, 0)$? to the right of $(-1, 0)$? Do you see that no value of x will make $y < 0$, since the graph does not go below the x-axis?

Notice that the values of x for which the graph of $y = ax^2 + bx + c$ lies above the x-axis form the solution set of $ax^2 + bx + c > 0$, while the x-coordinates of the points on the graph below the x-axis comprise the solution set of $ax^2 + bx + c < 0$.

ORAL EXERCISES

1. If $y = x + 1$,

a. What is the value of $x + 1$ when $x = -1$? 0

b. is $x + 1 > 0$ or is it < 0 when $x > -1$? >0

2. If $y = x - 2$,

a. What is the value of $x - 2$ when $x = 2$? 0

b. is $x - 2 > 0$ or is it < 0 when $x < 2$? <0

3. Consider the equation $y = (x - 1)(x - 4)$.

a. What are the x-intercepts of the graph of the equation? $1, 4$

Is $(x - 1)(x - 4) > 0$ or is it < 0 when

b. $x < 1$? >0 **c.** $4 > x > 1$? <0 **d.** $x > 4$? >0

4. If $y = (x - 1)^2$,

a. What are the x-intercepts of the graph of the equation? 1

Is $(x - 1)^2 > 0$ or is it < 0 when

b. $x < 1$? >0 **c.** $x > 1$? >0 **d.** $x \neq 1$? >0

5. Examine $y = x^2 - 1$ in the form $y = (x + 1)(x - 1)$.

a. What are the x-intercepts of the graph of the equation? $-1, 1$

Is $x^2 - 1 > 0$ or is it < 0 when

b. $x < -1$? >0 **c.** $-1 < x < 1$? <0 **d.** $x > 1$? >0

6. Examine $y = x^2 - 4$ in the form $y = (x + 2)(x - 2)$.

a. What are the x-intercepts of the graph of the equation? $-2, 2$

Is $x^2 - 4 > 0$ or is it < 0 when

b. $x < -2$? >0 **c.** $-2 < x < 2$? <0 **d.** $x > 2$? >0

7. If $y = x(1 - x)$,

a. What are the x-intercepts of the graph of the equation? 0, 1

Is $x(1 - x) > 0$ or is it < 0 when

b. $x < 0$? <0 **c.** $0 < x < 1$? >0 **d.** $x > 1$? <0

8. If $y = 2x(2 - x)$,

a. What are the x-intercepts of the graph of the equation? 0,2

Is $2x(2 - x) > 0$ or is it < 0 when

b. $x < 0$? <0 **c.** $0 < x < 2$? >0 **d.** $x > 2$? <0

WRITTEN EXERCISES

Graph these equations, and mark the sections of the x-axis each determines as solution sets for $y = 0$, $y > 0$, $y < 0$.

Ⓐ **1.** $x^2 + 3x + 2 = y$
2. $x^2 - 4x + 3 = y$
3. $x^2 - 8 - 2x = y$
4. $x^2 + 3x + 10 = y$
5. $3x^2 - 5x + 2 = y$
6. $4x^2 + 13x - 3 = y$
7. $x^2 - 1 = y$
8. $x^2 - 4 = y$
9. $x^2 - 2x = y$
10. $x - 2x^2 = y$
11. $x^2 - 3x = y$
12. $x - 3x^2 = y$

Ⓑ **13.** $x^2 - 2x + 1 = y$
14. $x^2 + 8x + 16 = y$
15. $4x^2 + 1 - 4x = y$
16. $9x^2 + 6x + 1 = y$

Find the values of x for which each expression is a real number.

Ⓒ **17.** $\sqrt{x^2 - 3x - 10}$
18. $\sqrt{x^2 - 3x}$

Chapter Summary

Inventory of Structure and Method

1. A quadratic equation containing no linear term can be solved by using the property of square roots of equal numbers: If r and s are real numbers, $r^2 = s^2$ if, and only if, $r = s$ or $r = -s$.

2. In a quadratic equation of the form $x^2 + bx + c = 0$, the **sum of the roots** is equal to $-b$, the opposite of the coefficient of the linear term;

the **product of the roots** equals the constant term c. These facts may be used to **check** the solution set of a quadratic equation.

3. To solve a quadratic equation in one variable by the method of **completing the square:** transform it into an equivalent equation whose quadratic term has coefficient 1; write it in the form $x^2 + bx = -c$; add to each member $\left(\frac{b}{2}\right)^2$, thus making the left member a square; apply the property of square roots of equal numbers; and solve the resulting linear equations. A **rational approximation** of an irrational root can be calculated by substituting a rational approximation for each radical which is accurate to one more place than desired in the root and performing the operations. Expressions such as $\frac{a \pm b}{c}$ represent two expressions $\frac{a + b}{c}$ and $\frac{a - b}{c}$.

4. The roots of the **standard quadratic equation** $ax^2 + bx + c = 0$ are given by the **quadratic formula:**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

5. To solve a quadratic equation in one variable graphically, write it in the standard form $ax^2 + bx + c = 0$, and find ordered pairs (x, y) satisfying $ax^2 + bx + c = y$. Then draw a parabola through the points represented by those ordered pairs. Determine the abscissas of the points of intersection of the parabola $(y = ax^2 + bx + c)$ with the x-axis $(y = 0)$.

6. To solve a quadratic inequality in one variable, first transform it into an inequality whose right member R is zero, and then factor the left member L. (If $L > 0$, the factors must be both positive or both negative; if $L < 0$, the factors must be opposite in sign.) Solve the resulting linear inequalities.

7. The solution set of $ax^2 + bx + c > 0$ is the set of values of x for which the graph of $y = ax^2 + bx + c$ lies above the x-axis. The solution set of $ax^2 + bx + x < 0$ is the set of values of x for which the graph of $y = ax^2 + bx + c$ lies below the x-axis.

Vocabulary and Spelling

completing a trinomial square (*p. 469*)

general quadratic equation (*p. 473*)

quadratic formula (*p. 474*)

nature of roots (*p. 476*)

discriminant (*p. 478*)

quadratic inequality (*p. 479*)

Chapter Test

13–1 Solve, using the property of square roots of equal numbers.

1. $\frac{1}{8} - 2y^2 = 0$ **2.** $4(x - \frac{1}{2})^2 = 9$ **3.** $\frac{1}{4} - n + n^2 = 0$

13–2 Are the roots given for each equation correct?

4. $x^2 - 22x = 11;\ x \in \{11 \pm 2\sqrt{33}\}$

5. $x^2 + 30x = -29;\ x \in \{1.1, -31.1\}$

13–3 **6.** Find the number c which completes the square in $x^2 - 1.4x + c$, and write the trinomial as the square of a binomial.

7. Transform $x^2 - 3x = 9$ into an equation whose left member is shown as the square of a binomial.

8. Solve $32x - 2x^2 - 16 = 0$ by completing the square. Express the roots in simplest radical form and to the nearest tenth.

13–4 **9.** Using the quadratic formula, find the roots of $4x^2 + 12x + 7 = 0$ in simplest radical form and to the nearest tenth.

10. A rectangular picture, 18 inches by 24 inches, has a frame of uniform width. The area of both picture and frame is 720 square inches. How wide is the frame?

13–5 (Optional) How many real roots do these equations have?

11. $x^2 - 3x - 5 = 0$ **13.** $x^2 + x + 3 = 0$

12. $4x^2 + 4x + 1 = 0$ **14.** $5x^2 - 3x + 7 = 0$

13–6 **15.** (Optional) Graph the solution set of $x^2 - 3x > 4$.

13–7 **16.** (Optional) Graph $y = x^2 + 4x + 3$, and from the graph determine the solution set of $x^2 + 4x \leq -3$.

Chapter Review

13–1 The Square-Root Property *Pages 465–467*

1. If $u^2 = v^2$, you know that _?_ = _?_ or _?_ = _?_.

Solve by using the square-root property.

2. $a^2 = 4$ **3.** $b^2 - 64 = 0$ **4.** $(c + 1)^2 = 25$

13-2 Checking Solution Sets *Pages 467–469*

5. When one member of the equation is zero and x^2 has a coefficient of 1, the sum of the roots of a quadratic equation equals the __?__ of the __?__ of the __?__ term, and the product of the roots equals the __?__ term.

Use the sum and product method to check the solution sets.

6. $x^2 - 4x = 4; \{2, -2\}$
7. $x^2 + 26x + 19 = 0; \{-13 \pm 5\sqrt{6}\}$

13-3 Completing a Trinomial Square *Pages 469–473*

State the value of k which makes each trinomial a square, and express the trinomial as the square of a binomial.

8. $s^2 - 20s + k$ 9. $y^2 + y + k$ 10. $z^2 + .5z + k$

Solve by completing the square. Express irrational roots in simplest radical form and to the nearest tenth.

11. $x^2 + 2x = 23$
12. $2x^2 - 12x = 24$
13. $y^2 + 5y - 3 = 0$
14. $y^2 - 6y + 9 = 20$
15. A man rows 18 miles downstream and back in 8 hours, in a current of 3 miles per hour. Find his rate of rowing in still water.
16. A car and a train each travel 400 miles. The train goes 10 m.p.h. faster than the car and arrives 2 hours sooner. What is the rate of each vehicle?

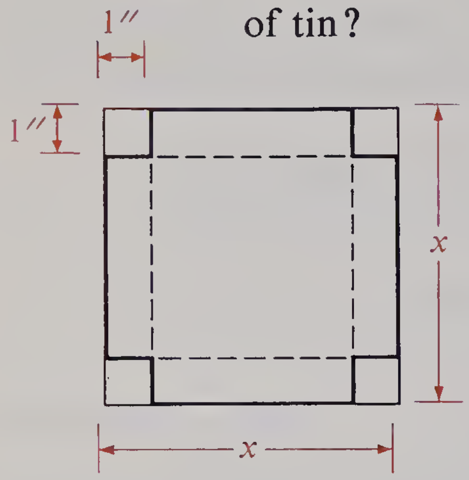
13-4 The Quadratic Formula *Pages 473–476*

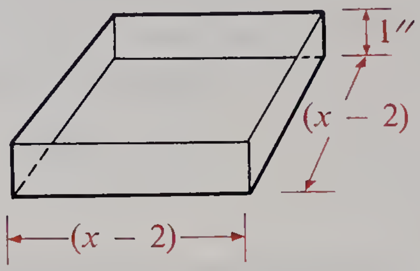
17. In a quadratic equation, the coefficient of the quadratic term may be any real number except __?__.
18. The linear term in a quadratic equation may have any __?__ number as its coefficient.
19. The roots of $ax^2 + bx + c = 0$ are __?__ and __?__.

Solve by the quadratic formula. Express the irrational roots in simplest radical form and correct to tenths.

20. $2x^2 + 11x + 3 = 0$
21. $x^2 - 18x + 4 = 0$
22. $3x^2 - 14x - 5 = 0$
23. $5x^2 + 8x = 3$
24. A dealer sold 200 radios a month at $35 each. For each price increase of $1, he sells two fewer radios each month. What price increase will yield $1100 more monthly?

25. A tray with a volume of 16 cubic inches was made from a square of tin by cutting one-inch squares from the corners and turning up the sides (see sketches). What size was the square of tin?





13-5 The Nature of the Roots of a Quadratic Equation (Optional)

Pages 476–479

26. An equation in the form $ax^2 + bx + c = 0$ can be solved by graphing the equation __?__.

27. The graph of $x^2 + 3x + 1 = y$ crosses the x-axis at __?__ points.

28. How many real roots does each equation have?

a. $2x^2 - 7x + 2 = 0$ **c.** $x^2 + 3x + 2 = 0$

b. $2x^2 = 8x - 8$ **d.** $x^2 + 3x = -3$

29. If $h = vt - 16t^2$, in how many seconds t will a ball thrown with a velocity v of 48 feet per second reach a height h of 32 feet?

30. If $s^2 = 8rh - 4h^2$, how high h to the nearest tenth of a foot, is an arch with a radius r of 20 feet and a span s of 22 feet?

31. If $c = \frac{1}{2}t(t - 1)$, how many telephones t are served by a switchboard with 630 connections c?

13-6 Solving Quadratic Inequalities (Optional)

Pages 479–482

32. If $(x + 1)(x - 2) > 0$, then $x + 1 > 0$ and $x - 2$ __?__ 0, or $x + 1 < 0$ and $x - 2$ __?__ 0.

33. If $(x + 3)(x - 1) \leq 0$, then $x + 3 \geq 0$ and $x - 1$ __?__ 0, or $x + 3 \leq 0$ and $x - 1$ __?__ 0.

34. There are __?__ values of x that satisfy both $x + 3 \leq 0$ and $x - 1 \geq 0$.

Solve each quadratic inequality.

35. $x^2 + 3x + 2 > 0$ **37.** $x^2 - 25 \geq 0$

36. $x^2 \leq x + 12$ **38.** $x^2 - 1 < 0$

13-7 Using Graphs of Equations to Solve Inequalities (Optional)

Pages 482–484

Items 39–40 refer to the solution of $x^2 - x > 2$.

39. The number line segments $x < -1$ and $x > 2$ represent abscissas of points for which $y \underline{\ ?\ } 0$.

40. If $-1 < x < 2$, $(x + 1)(x - 2) \underline{\ ?\ } 0$, and the values of x are members of the solution set of $x^2 - x \underline{\ ?\ } 2$.

Solve each of the following inequalities by graphing a quadratic equation in two variables.

41. $x^2 + 4x + 3 \geq 0$

42. $x^2 < x + 6$

43. $x^2 - 9 > 0$

44. $x^2 - 4 \leq 0$

Extra for Experts

The Factor Theorem

Knowing the factors of the left member of a polynomial equation like $2x^3 - 7x^2 - 7x + 30 = 0$, you can determine its roots. But if you know the roots, can you determine the factors of the left member? The answer is yes, and it usually is explained in terms of the ***factor theorem.***

If a polynomial equation in standard form is satisfied by a specific value a of x, then $x - a$ is a factor of the polynomial.

EXAMPLE 1. **Factor $2x^3 - 7x^2 - 7x + 30$.**

Solution: **Set $2x^3 - 7x^2 - 7x + 30 = 0$, and test values of x in it.**

$$\text{When } x = 1: \quad 2(1)^3 - 7(1)^2 - 7(1) + 30 = 18 \neq 0$$

$$x = 2: \quad 2(2)^3 - 7(2)^2 - 7(2) + 30 = 4 \neq 0$$

$$x = 3: \quad 2(3)^3 - 7(3)^2 - 7(3) + 30 = 0 \checkmark$$

$$\therefore x - 3 \text{ is a factor of } 2x^3 - 7x^2 - 7x + 30.$$

$$\frac{2x^3 - 7x^2 - 7x + 30}{x - 3} = 2x^2 - x - 10 = (x + 2)(2x - 5)$$

$$\therefore 2x^3 - 7x^2 - 7x + 30 = (x - 3)(x + 2)(2x - 5), \text{ Answer.}$$

The quadratic formula can help you factor a quadratic polynomial. In equations with integral coefficients like $x^3 + 2x^2 - 14x + 5 = 0$, where the coefficient of x^3 is 1, all integral roots of the equation are factors of the constant term 5. This is so because if the equation is factorable the left member will be $(x - a)(x - b)(x - c)$, where a, b, and c are real numbers, and the constant term will equal $(-a)(-b)(-c)$. If there is no constant term, 0 is a root of the equation, and $x - 0$ is a factor of the left member.

EXAMPLE 2. **Factor $y^3 + 2y^2 - 14y + 5$ into linear factors.**

Solution: **Check $\{1, -1, 5, -5\}$ in $y^3 + 2y^2 - 14y + 5 = 0$.**

$(1)^3 + 2(1)^2 - 14(1) + 5 \neq 0$; $(-1)^3 + 2(-1)^2 - 14(-1) + 5 \neq 0$

$(5)^3 + 2(5)^2 - 14(5) + 5 \neq 0$; $(-5)^3 + 2(-5)^2 - 14(-5) + 5 = 0$ ✓

$$\frac{y^3 + 2y^2 - 14y + 5}{y - (-5)} = \frac{y^3 + 2y^2 - 14y + 5}{y + 5} = y^2 - 3y + 1$$

Apply the factor theorem to $y^2 - 3y + 1$:

$y^2 - 3y + 1 = 0$; **by the quadratic formula,** $y = \dfrac{3 \pm \sqrt{5}}{2}$

$$\therefore y^3 + 2y^2 - 14y + 5 = (y + 5)\left(y - \frac{3 + \sqrt{5}}{2}\right)\left(y - \frac{3 - \sqrt{5}}{2}\right)$$

Questions

1. Show by substitution that each polynomial has the indicated factor.
   a. $n^{156} - 2n^{41} + 1$; $n - 1$
   b. $x^{71} - 3x^{28} + 4$; $x + 1$
   c. $x^3 - x^2 - x + 10$; $x + 2$
   d. $3t^4 - 16t^3 + 19t^2 + 5t + 3$; $t - 3$

2. Factor these polynomials.
   a. $x^3 - 7x + 6$
   b. $2y^3 + 3y^2 - 2y - 3$
   c. $n^4 + 16n^3 - 14n^2 - 24n - 9$
   d. $2m^4 + 2m^3 - 17m^2 - 23m + 6$

3. Solve these equations. Leave irrational roots in radical form.
   a. $x^3 - 4x^2 - x - 4 = 40$
   b. $3z^3 - 5z^2 = 3z - 2$
   c. $2z^3 - 14z = 3z^2 - 15$
   d. $5y^2 + 40y = 7y^3 + 48 - y^4$

4. Find k so that:
   a. $x + 1$ is a factor of $2x^3 + kx - 4$.
   b. $x - 2$ is a factor of $x^3 + kx^2 - 14x + 2k$.

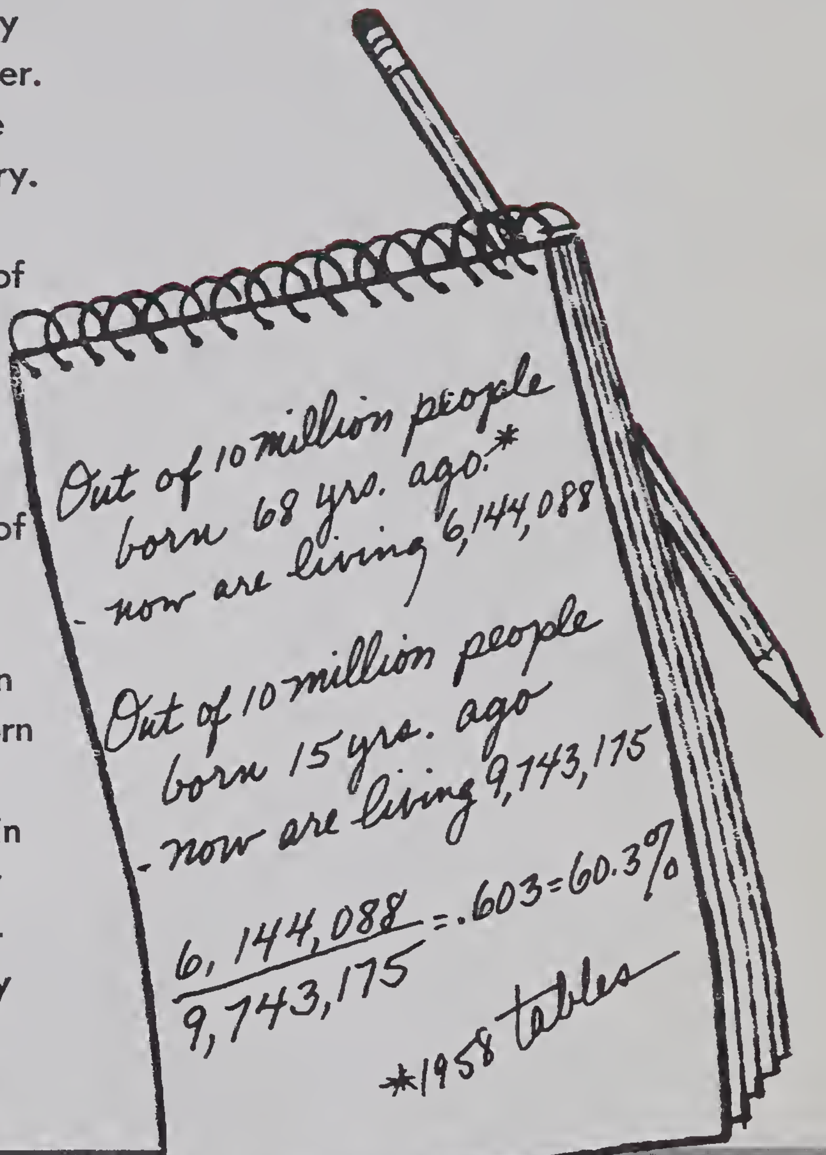
5. Show these statements to be true.
   a. $x - y$ is a factor of $x^n - y^n$.
   b. $x + y$ is a factor of $x^n + y^n$ when n is an odd integer, but not when n is an even integer.

Actuaries and Mathematics

if insurance plans are to be financially sound, premiums must be charged in proportion to the risk, the likelihood that the insurance company will have to pay out money to the policyholder. Determining the degree of risk and fixing the rates accordingly is the main job of an actuary.

Actuarial science relies heavily on the branch of mathematics known as the theory of probability and statistics. Shown in the photograph is a class of actuarial trainees who are learning to compute the value of annuities. In classes such as these, actuaries learn to interpret data gathered on millions of people, covering birth, death, and accident rates, illness, disability, and unemployment.

Analyzing statistics on more than 10 million people, actuaries determined that babies born in 1941 had an average life expectancy of approximately 62 years, while babies born in 1958 had an average expectancy of nearly 70 years. From statistics like these, the actuary can determine the probability that a boy now 15 years old will live to be sixty-eight. This kind of problem, important in adjusting the premium for pension, old-age, or life insurance, is illustrated on the work pad. The actuary uses two facts: (1) 6,144,088 of ten million people born 68 years ago are still living, and (2) 9,743,175 of the ten million people born 15 years ago are still living. He finds that there is a sixty per cent probability that the 15-year-old boy will live to be 68.

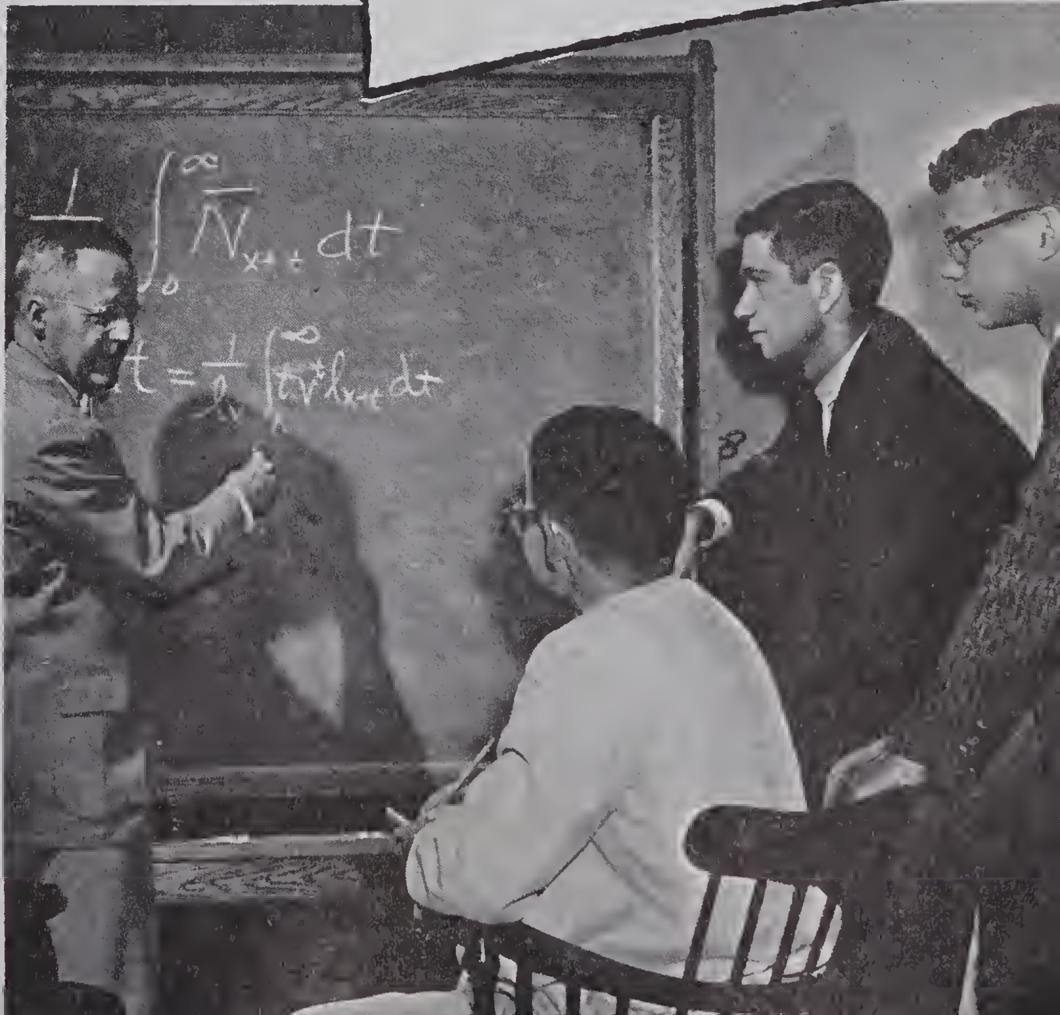


Out of 10 million people
born 68 yrs. ago.*
- now are living 6,144,088

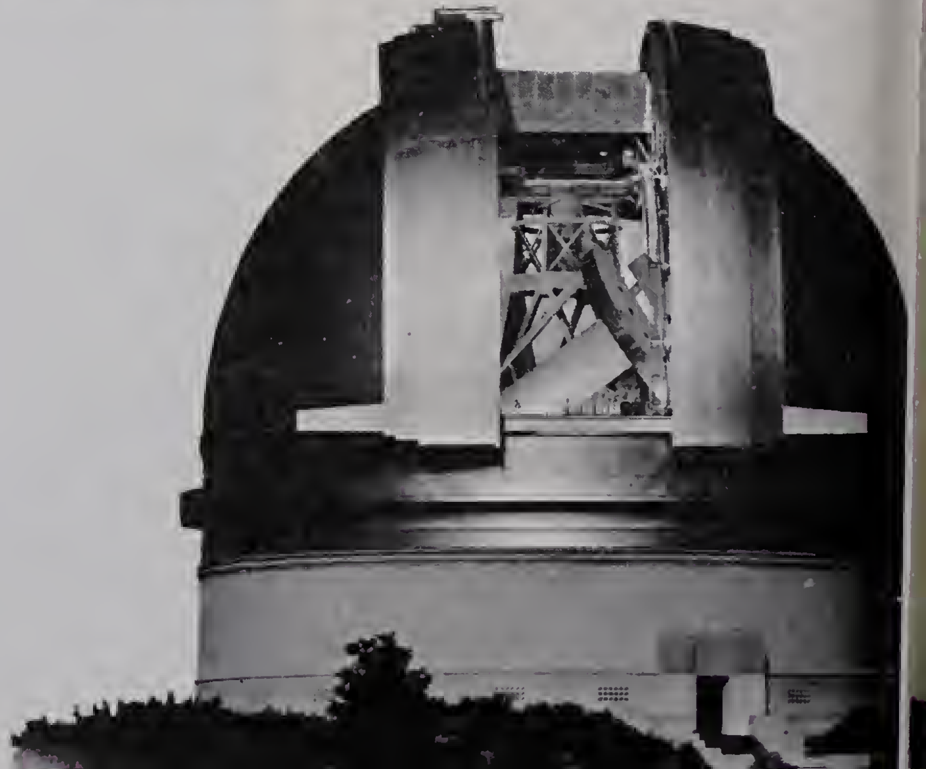
Out of 10 million people
born 15 yrs. ago
- now are living 9,743,175

$\frac{6,144,088}{9,743,175} = .603 = 60.3\%$

*1958 tables


$\int_0^\infty \bar{N}_{x+t}\,dt$
$t = \frac{1}{l}\int_0^\infty t v^t l_{x+t}\,dt$
8

CHAPTER 14



Geometry and Trigonometry

Looking ahead in mathematics may be looking ahead into your future. Through the screen of Cartesian coordinates you see mathematical symbols extending into infinity, representing the boundlessness of mathematics (upper photo). Man's knowledge of this field is limited only by his imagination; its challenges are as broad as the constellations of the heavens.

As you look into your own future, can you see the role mathematics may play in it? The astronomers who work in the observatory (lower left) require a knowledge of mathematics greater than that you now possess. They did not learn their mathematics as part of their jobs. They learned it in order to get their jobs. Since many occupations which are challenging require a knowledge of mathematics, you should plan to include it in your schooling.

GEOMETRY

14–1 The Structure of Geometry

Up to this time, all of your study in this book has been concerned directly or indirectly with the properties of various sets of *numbers*. As you were told on page 1, your first aim was to become acquainted with some of the properties of the numbers of arithmetic. Later you learned about the properties of other sets of numbers: directed numbers, rational numbers, real numbers. To help you understand some of these properties, you sometimes used sets of *points*. When you made graphs, for instance, you associated a pair of numbers (x, y) with a point P.

You used points, lines, and other sets of points (parabolas, for example) to help you understand the properties of sets of numbers, but of course it is perfectly possible to have as your aim the study of the properties of sets of points themselves, without being particularly concerned with sets of numbers of any kind. This is the object of the study of *Geometry:* to learn about the properties of points, lines, triangles, squares, rectangles, circles, and other figures made up of points. Just as your aim in algebra is to learn about the properties of sets of numbers, so your aim in studying geometry would be to learn about the properties of sets of points.

Try to stress this point of view; to understand it is more important than to memorize the axioms.

Fortunately, whether you are learning about sets of numbers or sets of points, you do not have to list and memorize all the properties in which you are interested. The various properties are related to one another, and if you know certain basic properties, others can be worked out. In other words, in both algebra and geometry you deal with a structure, not just a miscellaneous collection of observations.

Like numbers, geometric points and sets of points such as lines, planes, and circles, are abstract concepts, not concrete objects. However, every mathematician helps his thinking by representing points by dots, lines by strokes, and planes by flat surfaces, as shown in Figure 14–1.

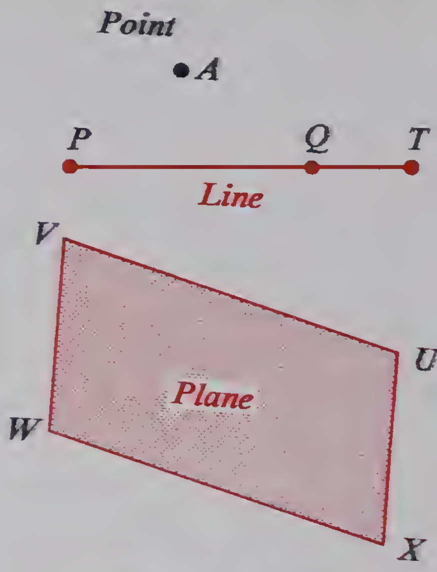


Figure 14–1

14–2 Geometric Assumptions

Your study of geometry begins by stating certain relationships that will be taken for granted. These statements are called *axioms*, and they are assumed to hold for the points and sets of points you will be studying. Other properties and relationships will be proved on the basis of the axioms. Such proved relationships, as you know, are called *theorems*.

Here are six axioms you will need:

I. Every line is a set of points containing at least two different points.

II. Any two different points belong to one and only one line.

From these two axioms you know at once that you can identify any line by naming any two of its points; there is no other line that contains them both. Thus, you can call the line shown in Figure 14–1: *PQ* or *QT* or *PT* or *TP* or *TQ* or *QP*. Points that belong to or lie on the same line are called **collinear** (ko-**lin**-e-ar); thus, *P*, *Q*, and *T* are collinear points.

From these two axioms you know, too, that two different lines cannot have more than one point in common. For if points *H* and *K* lie on line *l* and also lie on line *m*, then *l* and *m* must be two names for the same line, by Axiom II.

III. Every plane is a set of points containing at least three points not all on the same line (or, you might say, containing three noncollinear points).

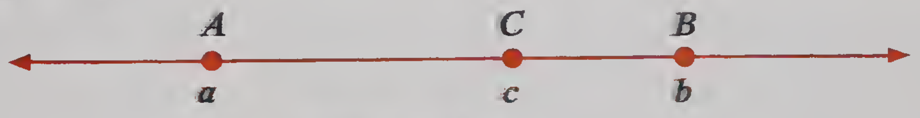
IV. Any three noncollinear points belong to one and only one plane.

From Axioms III and IV you know at once that you can identify any plane by naming any three of its points: there is no other plane that contains all three. Thus, you can call the plane shown in Figure 14–1: *UVW* or *UXW* or *XUV*. Points or lines that belong to or lie on the same plane are called **coplanar** (ko-**plain**-ar) points or lines.

From these two axioms you know, too, that two different planes cannot have more than one line in common. For if lines *l* and *m* lie on plane *M*, and also lie on plane *N*, there must be three distinct points, say *A* and *B* on *l* and *C* on *m*, all three of which belong to *M* and also to *N*. By Axiom IV, therefore, *M* and *N* must be two names for the same plane.

You already know that it is important to be able to put coordinates on a number line. This is the way you used sets of points to help you understand the properties of sets of numbers. You therefore repeatedly assumed the following axiom:

V. There is a one-to-one correspondence between real numbers and points of a line such that if A and B are points on the line with coordinates a and b, the *distance* between A and B (AB) is the absolute value of the difference of their coordinates, that is $AB = |a - b|$.



· *Figure 14–2* ·

From this axiom you know that a line extends indefinitely in both directions without holes or gaps. You know this because of the properties of real numbers you have learned. If you have a point B with coordinate b, you can always find a real number larger than b. The point belonging to this number must be to the right of B. Thus, since the set of real numbers has no largest number, the set of points being paired with the real numbers, that is, the line, must extend indefinitely to the right. Since the set of real numbers has no smallest number, the line must extend indefinitely to the left. Since the real numbers are dense (see page 407), that is, between any two real numbers there is another, there can be no gaps or holes in the line.

You can use this axiom also to define precisely what is meant when you say one point is *between* two others:

Definition. If point C lies on line AB, then C is **between** A and B if $a < c < b$ or $b < c < a$, where the real numbers a, b, and c are coordinates of points A, B, and C, respectively.

Definition. The **line segment** determined by points A and B is the set of points consisting of A and B and the points on the line between them. A and B are the **end points** of the line segment $\overline{AB}$.

The length of a line segment is the distance between its end points. Thus, the length of segment $\overline{AB}$ is AB.

The last axiom we need at this stage is one about lines and planes:

VI. If two points of a line are in a plane, every point of the line is in that plane.

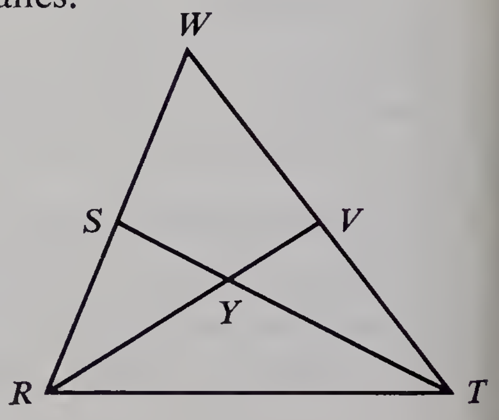
This axiom immediately tells you two things: (1) that a plane does indeed contain lines; (2) that, since the lines it contains extend indefinitely, a plane also extends indefinitely.

WRITTEN EXERCISES

Tell which axiom justifies each statement.

Ⓐ

1. If P is a point on line m, there is another point on m.
2. If P and Q are different points on line m and if line n also contains P and Q, then m and n are the same line.
3. Any three points lie in a plane.
4. Any two points are collinear.
5. If A, B, and C are noncollinear points, then AB, AC, and BC are different lines.
6. If A, B, C, and D are points not all in the same plane, then ABC, BCD, ACD, and ABD designate different planes.
7. If L and T are two different points of line m and if P is a point not on m, there is a plane containing P and m.



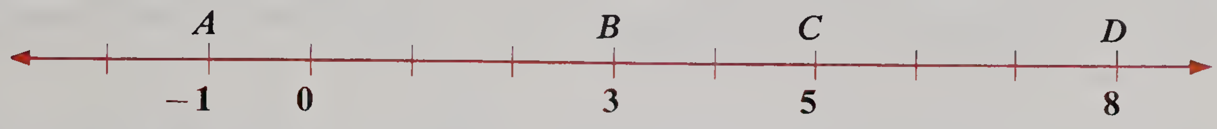
Which point (or points) makes each statement about the adjoining figure true?

8. Points S, W, and ___?___ are collinear.

9. Points Y, T, and $\underline{\ ?\ }$ are collinear.

10. Points R, V, and $\underline{\ ?\ }$ are noncollinear.

Given collinear points with coordinates as indicated in the figure, find each distance.



11. AC **12.** CD **13.** DB **14.** CA **15.** $AB + BC$ **16.** $AC + CD$

Diagram the points K, L, M, and P on a line in the given relationships. In each case, at least two orders are possible.

17. $KM + LM = KL, P \in \overline{KM}$

18. $PL + LP = MP, L \in \overline{KP}$

19. $K \in \overline{LP}, LP < LM$

20. $KL > KP$ and $PM < PL$

14–3 Rays and Angles

A line is a set of points. Clearly there are many subsets that can be selected from this set of points. For instance, the segment $\overline{QR}$ in Figure 14–3 is one such subset; so is any other segment, say $\overline{TS}$.

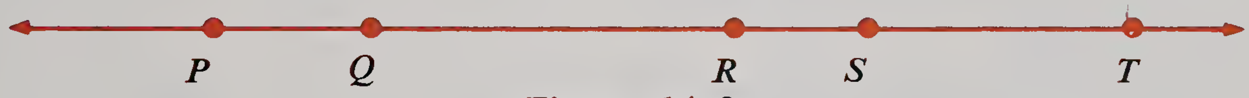


Figure 14–3

But you do not need to limit yourself to segments. You can make up a subset consisting only of the three particular points P, R, and S. Another possibility is the subset consisting of point Q and all points to its right.

This last subset is an example of a particularly important idea: the set consisting of a point on a line and all the points on one side of it. A subset of this sort is called a *ray*. In a sense a ray is intermediate to a segment and the whole line. A segment has two end points; a line has no end point, and extends indefinitely in both directions; a ray has one end point and extends indefinitely in one direction.

Clearly, if you take a particular point on a line, say R, there are two rays having R as end point: the subset including R, S, T and all other points to the right of R is one such ray; the other is the subset including R, Q, P, and all other points to the left of R. To distinguish them, the first is called the ray RS, and the other is the ray RP. Do you see that RS and RT are two names for the same ray?

In the study of geometry, it is considered very important not merely to describe a set of points or a figure that is being named, but to give a formal definition that distinguishes the entity being defined from every-

Since a line is determined by two points, clearly a ray is determined by its end point and any other point on it.

thing else under discussion. Here is a definition of a ray:

> **Definition.** If a point P with coordinate p is taken on a line, then the set of points consisting of P and all points whose coordinates equal or exceed p is called a **ray proceeding from P**. Similarly, the set of points consisting of P and all points whose coordinates equal or are less than p is a **ray proceeding from P**.

The term, *a ray having P as end point*, means exactly the same thing as *a ray proceeding from P*.

Another important set of points in geometry is the set called an angle.

> **Definition.** An **angle** is a set of points consisting of two rays proceeding from the same point. The point is called the **vertex** of the angle and the rays are called the **sides**. Either of the rays may be called the initial ray or initial side, and the other one the terminal ray or terminal side. An angle is merely a set of points. See T.M. p.43(1).

The union of the two rays may or may not be a line. (See Figure 14–4.) If it is, the angle is called a *straight angle*.

To name an angle, use the sign ∠ (read angle) followed by a letter naming its vertex, as $\angle G$ or $\angle K$ in Figure 14–4. When two or more angles have the same vertex, use ∠ followed by three letters, naming first a point on one side, then the vertex and finally, a point on the other side, as $\angle MHP$ or $\angle RHP$. Why would the designation $\angle H$ be ambiguous?

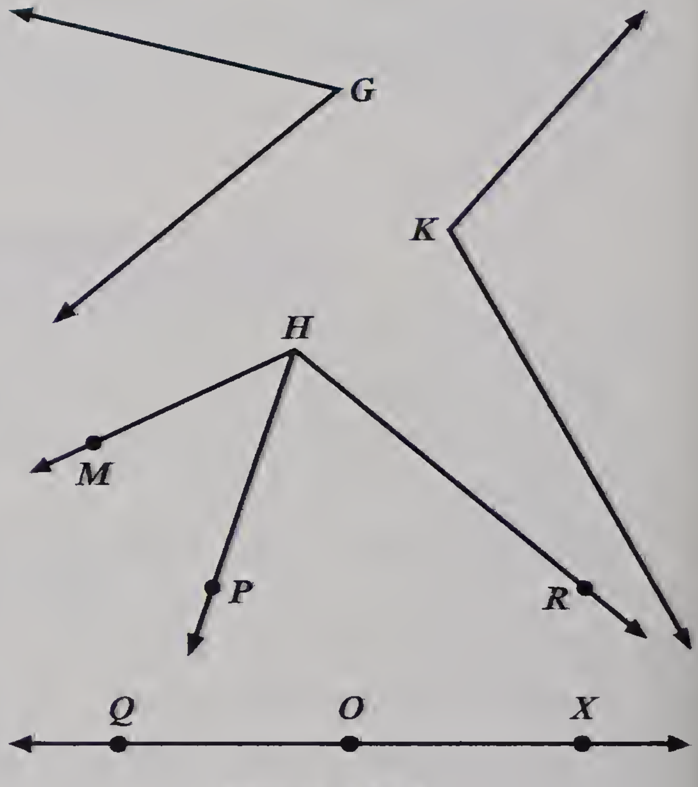


Figure 14–4

WRITTEN EXERCISES

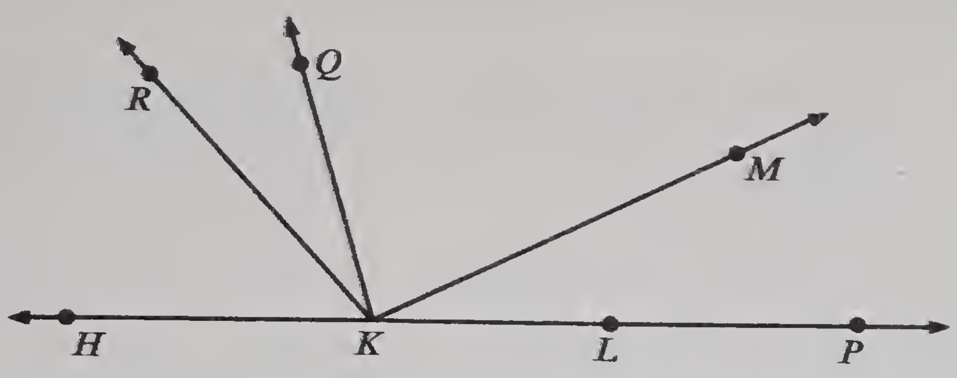
Exercises 1–9 refer to the figure at the top of page 499.

A

1. Which angle has ray KP as its initial ray and ray KM as its terminal ray?
2. Which angle has ray KQ as its initial ray and ray KH as its terminal ray?
3. Name two angles, one of which has KQ as its initial ray and the other KQ as its terminal ray.

A straight angle is a line with a particular point on it distinguished as the vertex of the angle.

Ex. 1–9



4. Is KL the same ray as KP?
5. Is KL the same ray as LK?
6. Is the angle PKR the same as the angle LKR?
7. What kind of angle is angle LKH?
8. What is the initial ray and the terminal ray of angle PLK?
9. Name five rays proceeding from point K.

On a scaled line indicate the graph of each open sentence, and identify the graph as a ray, a point, a line segment, a line, or none of these.

10. $x \geq 2$
11. $x = 1$
12. $x < 5$
13. $|x| \geq 0$
14. $|x| \leq 1$
15. $-2 \leq x < 1$
16. $|x| \leq 2$
17. $|x| \geq 2$

14–4 Triangles

The definition below should be regarded as a sample of a careful definition; it is not necessary for the students to memorize it.

You will have a good deal to do with triangles in the work of this chapter. You know of course what a triangle looks like, but in geometry, figures must be defined, even those you can already recognize. The definition of a triangle is as follows:

Definition. If A, B, and C are three noncollinear points, the set of points comprising the segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is a **triangle**. The segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are called the **sides** of the triangle and the angles having rays AB and AC, BA and BC, CA and CB respectively as sides are called the **angles of the triangle**.

You also know informally from your earlier work the meaning of the terms: "right angle," and "measure of an angle." In the next section these will be carefully defined. For completeness, two properties of triangles you have already used are listed here, ahead of the definitions.

If one of the angles of a triangle is a right angle, the triangle is called a *right triangle*. The side opposite the right angle is called the *hypotenuse*.

Make sure that these two terms are truly understood.

You have already used one of the most important properties of a right triangle, the Pythagorean Theorem (p. 411). This theorem states:

THEOREM: **In any right triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the other two sides.**

You will need this theorem repeatedly.

You have also used this property of a triangle:

THEOREM: **The sum of the measures of the angles of a triangle is 180°.**

The students must know the facts in these two theorems.

WRITTEN EXERCISES

By using the Pythagorean Theorem, determine which of the following triangles are right triangles. Draw rough sketches.

A

1. $\triangle ABC$, where $AB = 5$, $BC = 3$, and $AC = 4$.
2. $\triangle JKL$, where $JK = 13$, $JL = 5$, and $LK = 12$.
3. $\triangle DEF$, where $DE = 1$, $EF = 2$, and $DF = 3$.

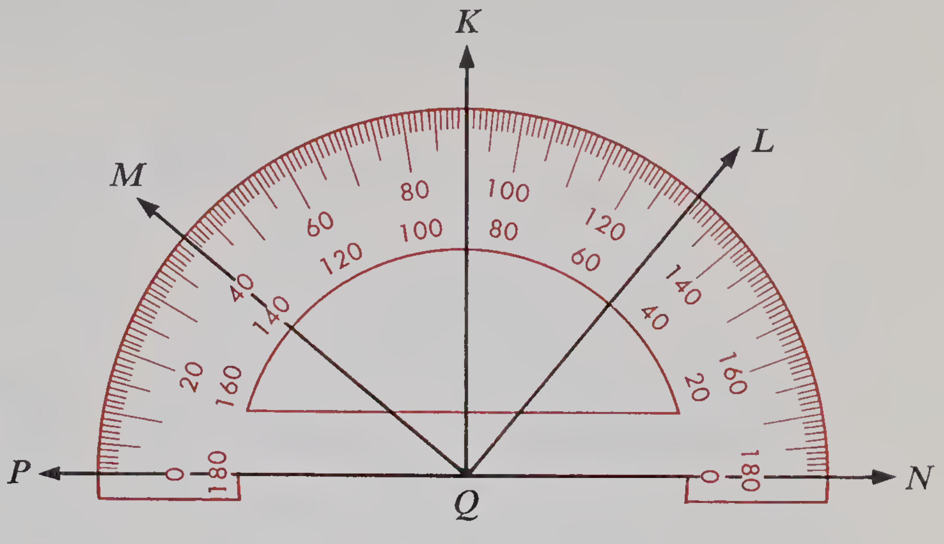
Are the following right triangles?

4. $\triangle GHK$, where $GH = 6$, $HK = 7$, and $GK = 9$?
5. $\triangle PQR$, where $PQ = 17$, $QR = 15$, and $PR = 8$?
6. $\triangle XYZ$, where $XY = 2$, $YZ = 3$, and $XZ = 2$?
7. $\triangle ABC$ is a right triangle with $AC = BC$ and $AB = 2$. Find AC and BC.
8. In $\triangle ABC$, the measures of $\angle A$ and $\angle B$ are each 45°. What is the measure of $\angle C$?
9. In $\triangle ABC$, the measure of $\angle A$ is 40° and of $\angle B$ is 60°. What is the measure of $\angle C$?
10. In $\triangle ABC$, the measure of $\angle A$ is 30° and of $\angle B$ is 60°. What is the measure of $\angle C$?
11. Why can there not be a triangle whose angle measures are 30°, 40°, and 50°?
12. Can there be a triangle each of whose angles measures 60°?

14–5 The Measurement of Angles

If you draw any line in any plane, the set of points comprising the plane is separated into three subsets: the points on one side of the line, the points on the other side of the line, and the line itself. Either of the first two sets is called a half-plane, and the line is called the *edge* of either half-plane. The union (see p. 60) of a half-plane and its edge is called a *closed half-plane.*

In any plane, draw a line *PN* and choose a fixed point *Q* on it. Choose one of the closed half-planes into which *PN* separates the plane. Since it is closed, that is, contains the edge *PN*, the half-plane contains the point *Q*, and a set of rays proceeding from *Q*, such as *QN*, *QL*, *QK*, and many more. By placing a protractor as indicated in Figure 14–5, each ray can be associated with a number.



· *Figure 14–5* ·

The important matter for geometry is not the protractor; it is the one-to-one correspondence between the rays and certain real numbers. This is emphasized in the following axiom:

← Stress this.

VII. In a half-plane the set of rays with a common end point in the edge of the half-plane can be put into one-to-one correspondence with the set of real numbers between 0 and 180, inclusive, in such a way that any angle whose sides are rays of the given set has a measure equal to the absolute value of the difference between the numbers corresponding to its sides.

← Stress this.

Ray	*Number*
QN	0
QL	50
QK	90
QM	140
QP	180

To see more clearly what this means, look closely at Figure 14–5. The table at the right shows the numbers corresponding to certain rays. You will notice that another correspondence could have been assigned in which *QP* corresponded to 0 and *QM* to 40. It is customary, however, for the purposes for which you are going to use this correspondence to use the inner circle of numbers on the protractor, that is, to *assign the number 0 to the ray on the extreme right* when the edge is taken horizontally.

What is the measure of angle *LQK*? You can find it, by using the protractor, to be 40. This may be written $m\angle LQK = 40$. Do you see that you have actually taken the difference of the numbers corresponding to the two rays making up the angle?

QK corresponds to 90 and QL to 50; $m\angle LQK = 90 - 50 = 40$.

Look now at the set of angles having as sides ray QN and any other ray proceeding from Q. Since the number corresponding to ray QN is 0, the measure of any one of these angles will be the number corresponding to the other ray. Thus,

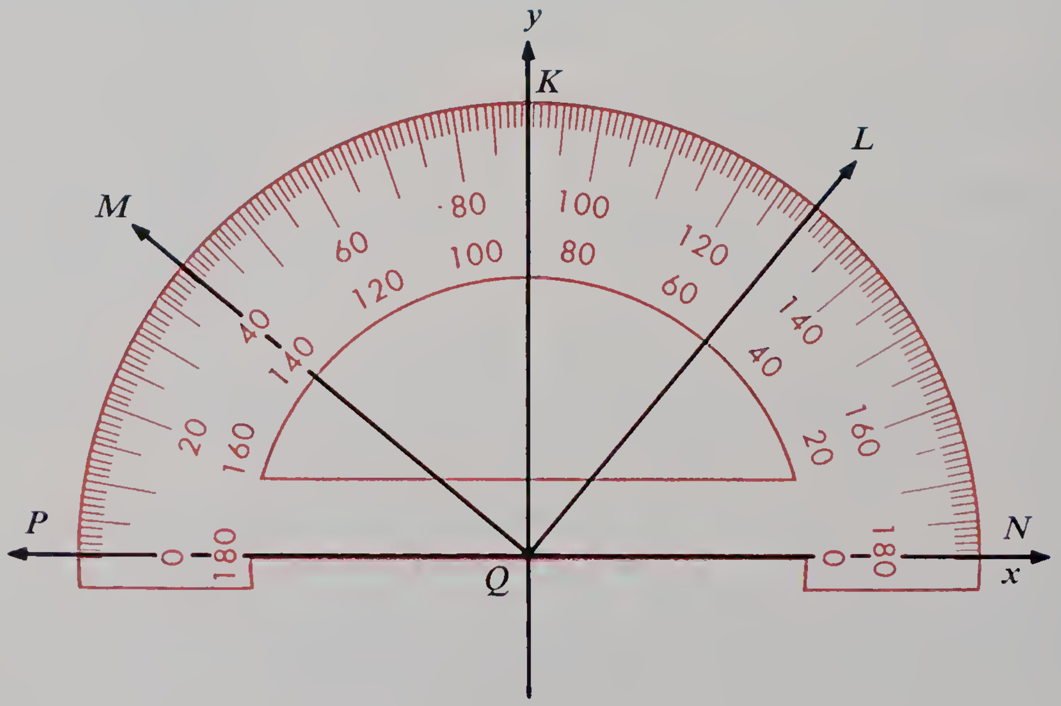
for $\angle NQM$, ray QM corresponds to 140, ray QN corresponds to 0

and $m\angle NQM = 140 - 0 = 140$.

Do you see that you can draw an angle, one side of which is ray QN, having as its measure any specified number between 0 and 180? Moreover, $\angle NQP$, which as you have already learned is a straight angle, since the union of its sides is a line, has measure 180. It is agreed also to call the ray QN an angle with measure zero. By making this agreement, you can say that the set of measures of the angles having ray QN as one side is the set of all real numbers equal to or greater than 0 and equal to or less than 180. An angle of measure 90 is called a **right angle**; an angle whose measure is between 90 and 180 is called an **obtuse angle**; an angle whose measure is greater than 0 or less than 90 is called an **acute angle**

Now superimpose on Figure 14–5 a rectangular coordinate system with Q as origin, QN as the positive x-axis, and QK as the positive y-axis. The subset of angles with ray QN as one side is now the subset of angles that have the positive x-axis as one side. Any angle in this subset is said to be in *standard position*. In other words, an angle is in *standard position* if its vertex is at the origin of a rectangular coordinate

The notion of standard position will be needed in the trigonometry section.



· *Figure 14–6* ·

Note that angles in standard position must be named by giving a point on the

initial side first.

system, and if one side is the ray which is the positive X-axis. The other side falls in Quadrant I or II (or on an axis), since all the angles discussed here have measure 180 or less.

TRIGONOMETRY

See T.M. p.44(2)

14–6 Tangent of an Angle

It is possible to set up another correspondence between real numbers and this particular set of rays through the origin, since each one of the rays through the origin (except OY) can be identified by its slope (page 343). Ray OA, for example, passes through the point (5, 3). Its slope is therefore $\frac{3-0}{5-0} = \frac{3}{5}$. There is no other ray proceeding from O with this particular slope; the ray is therefore completely identified by giving its slope.

Review the meaning of slope. Emphasize that this is another correspondence between angles and numbers but not another way of measuring angles.

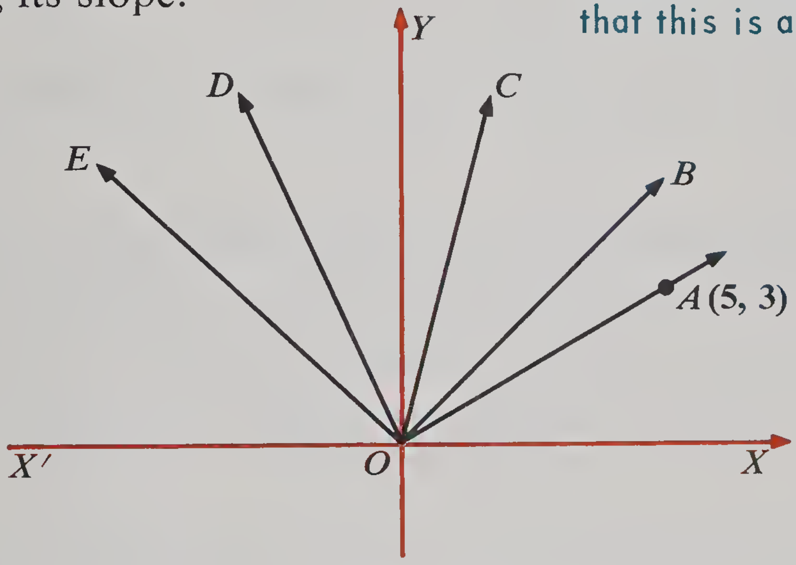


Figure 14–7

Since the slope of a line was defined to be $\frac{\text{difference of ordinates}}{\text{difference of abscissas}}$ for two points on the line, and since all the rays in which you are interested have (0, 0) as one such point, the slope of any of these rays is $\frac{y}{x}$, where (x, y) is any point on the ray.

You will recall that the slope of any ray in Quadrant I is positive; the slope of OX is 0, the slope of OB is 1; the slope of the rays between OX and OB, like OA, is less than 1; the slope of rays between OB and OY is greater than 1; OY itself has no slope.

Similarly, the rays in the second quadrant all have negative slopes; those between OY and OE are less than -1, that is, negative and in absolute value greater than 1, for example, -2. Those between OE and OX' fall between -1 and 0.

In your brief review of slope you will probably wish to include the ideas in these last two paragraphs.

Cols. 1 and 2 show the first correspondence that was set up. Cols. 1 and 3 show the second correspondence. Now we set up Cols. 2 and 3 as a third correspondence, and give it a name.

Thus, for every real number, negative or positive, there is one and only one ray through the origin having that number as its slope. However, there are two rays, OX and OX', having 0 slope, and there is one ray, OY, having no slope.

The two procedures for assigning numbers to rays proceeding from the origin may profitably be compared. A number assigned by the protractor method is called the degree measure of the angle. Hereafter such measures of angles will be labeled as degrees.

Ray	*Angle-measure*	*Slope*	*Ray*	*Angle-measure*	*Slope*
OX	0°	0	OR	105°	−3.73
OM	15°	.27	OS	120°	−1.73
ON	30°	.58	OE	135°	−1
OB	45°	1.00	OT	150°	−.58
OP	60°	1.73	OU	165°	−.27
OQ	75°	3.73	OX'	180°	0
OY	90°	—			

It is clear that the last two columns of this table can be related directly; in other words a correspondence can be set up directly between the measure of an angle in standard position and the slope of its terminal ray. The name given to this correspondence is *tangent*. To indicate that the tangent of 45° is 1.00, we write tan 45° = 1.00.

Definition. The **tangent** of an angle in standard position is the slope of its terminal ray.

Since any angle can be put in standard position, this definition will enable you to compute the tangent of any angle A. Place the angle in standard position, choose any point (x_1, y_1) on the terminal ray, or if more convenient, two points (x_1, y_1) and (x_2, y_2), neither $(0, 0)$, and compute the slope:

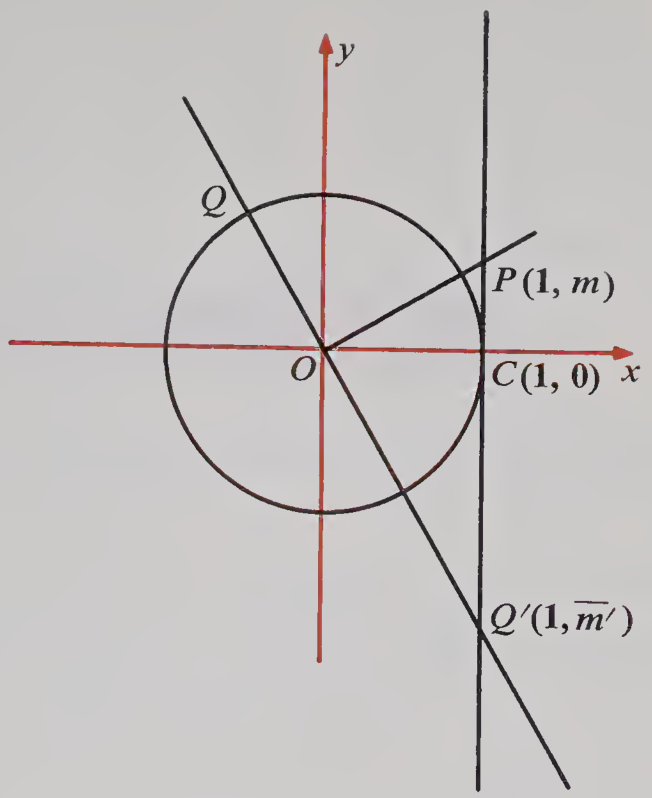
$$\tan A = \frac{y_1}{x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Practice with many simple exercises to make sure this is understood before going on.

Now draw on the coordinate axes a circle with center at the origin and radius 1 (called a unit circle), and the vertical line through the point $C(1, 0)$ as in Figure 14–8.

Since any point on a ray through the origin can be used to find the slope of the ray, you can agree to use the point P where the ray intersects the vertical line. The coordinates of this point are $(1, m)$, where m is the slope of the ray.

This paragraph begins the introduction of the "line value" as a concrete picture of the tangent of an angle.



· *Figure 14–8* ·

You can see this, either by writing the equation of a line through (0, 0) with slope m,

$$y = mx, \text{ and letting } x = 1 \text{ in this equation,}$$

or by noting that, for point $P(1, y)$,

$$m = \frac{y - 0}{1 - 0}, \text{ that is, the ordinate of } P, \text{ which is } y, \text{ equals } m.$$

In this figure, then, there is a line segment $\overline{CP}$ which has as its length the same number as the slope of the ray OP. This number is also the *tangent of angle COP*. You have, so to speak, a picture of the tangent of any angle in standard position.

You can see how the lengths of the line segments above the x-axis cut off on the vertical line $x = 1$ by rays from the origin would agree with the numbers listed in the table on page 504. Angles less than 45° clearly have tangents less than 1, and angles between 45° and 90° have tangents greater than 1; the greater the angle, the greater the tangent.

If the angle is greater than 90°, for example $\angle COQ$, the ray OQ does not cut the line $x = 1$. But the line QQ' of which the ray OQ is a part does cut the line $x = 1$, and of course line OQ' and ray OQ necessarily have the same slope. You can therefore use the point Q' where the *line* OQ cuts the line $x = 1$ to find the slope of *ray* OQ, even though Q' is not a point of the ray. The coordinates of Q' are $(1, m')$, where m' is negative and equal in absolute value to the length of the segment CQ'.

Notice the nice picture this makes of the fact that rays from the origin have positive slopes in Quadrant I and negative slopes in Quadrant II.

You can now see that the ordinate of every point on the vertical line $x = 1$ is the tangent of some angle, namely, the angle formed by the positive x-axis and the ray lying in the upper half-plane along the line joining the point in $x = 1$ to the origin.

WRITTEN EXERCISES

Find the tangent of each of the following angles, using the data shown in the figure.

A

1. $\angle XOA$

2. $\angle XOB$

3. $\angle XOC$

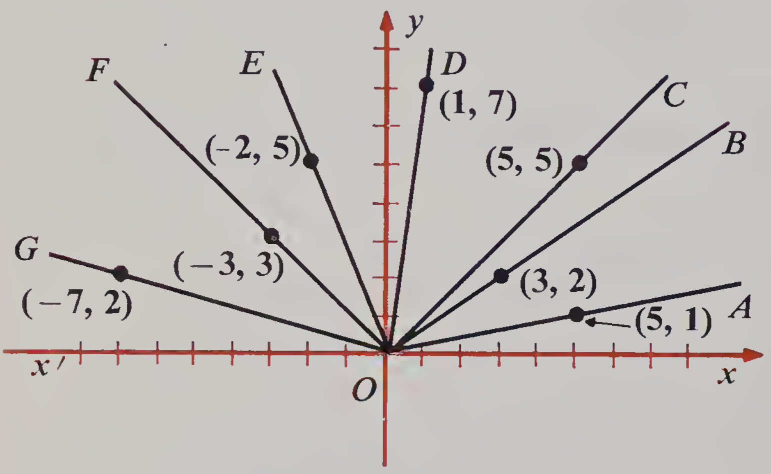
4. $\angle XOD$

5. $\angle XOE$

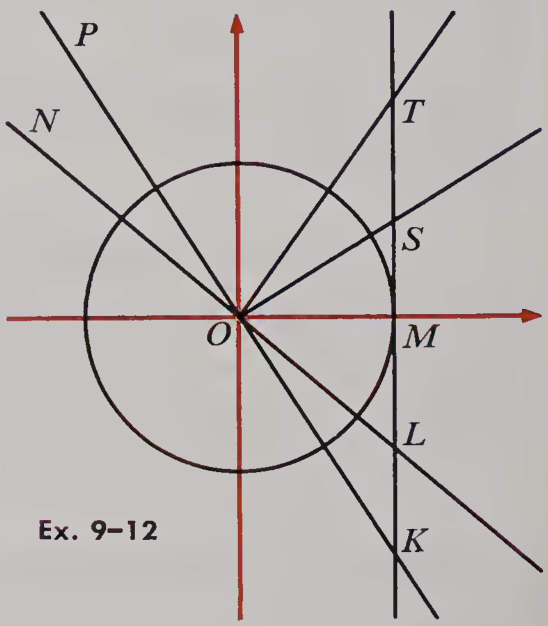
6. $\angle XOF$

7. $\angle XOG$

8. $\angle XOX'$



Ex. 1–8



Ex. 9–12

9. In the accompanying figure $MT = \tan MOT$. How long is segment OM?

10. For what angle does segment MS represent the tangent?

11. What segment represents $\tan MOP$?

12. $m\angle MON = 135°$. What are the coordinates of point L?

First, lines are found whose lengths are sin A and cos A. Then sine and cosine are defined as the lengths of these lines.

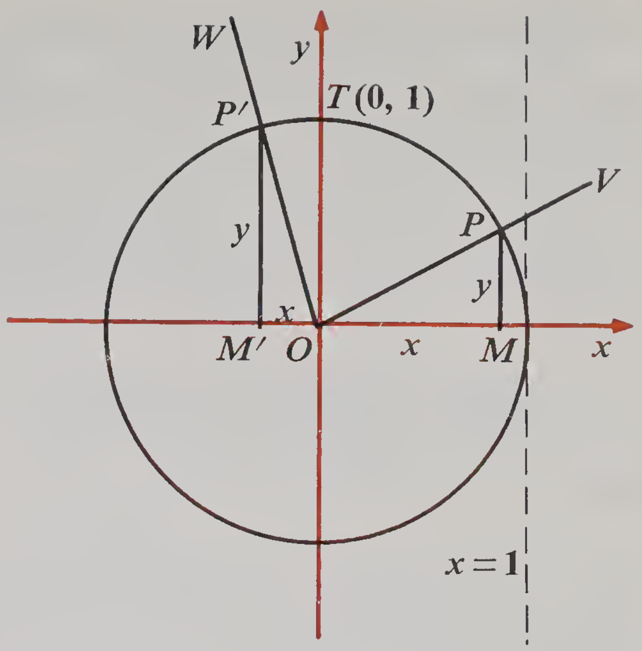
14–7 Sine and Cosine of an Angle

Finally, the conventional definitions are proved as theorems.

When you wanted a line segment equal in length to the tangent of an angle so that you could, so to speak, picture the tangent, you took a particular point on the terminal ray of the angle, namely, the point where the ray (or the line containing it) cut the vertical line $x = 1$.

By taking a different point on the terminal ray, two other very important line segments can be found.

Consider any angle in standard position, for example, $\angle XOV$ or $\angle XOW$. This time consider the point where the ray intersects the unit circle, P or P'. The line segments in which you are interested are OM and MP, or for $\angle XOW$, OM' and $M'P'$. The lengths of these line segments are nothing but the coordinates of P or P'. They are given special names, however, when they are thought of in terms of the angle rather than in terms of the point. y is called the *sine* of the angle and x is called the *cosine* of the angle.



· *Figure 14–9* ·

Definition. The **sine** of an angle in standard position is the ordinate of the point on the terminal ray whose distance from the origin is 1.

Definition. The **cosine** of an angle in standard position is the abscissa of the point on the terminal ray whose distance from the origin is 1.

From these definitions you can immediately note five very important facts.

1. *The sine of an angle is never greater than 1.* You see this because every ordinate except the ordinate of $T(0, 1)$ is less than the radius of the circle.

Make sure the students see these facts; mere memorization is not enough.

Similarly, you can see that

2. *The cosine of an angle is never greater than 1.*

3. *The sine and the cosine of any angle between 0° and 90° are both greater than 0 (positive).* You see this because the coordinates of any point in the first quadrant are positive.

4. $\sin 90° = 1$ and $\cos 90° = 0$. You see this by using the coordinates of point T.

5. The sine of any angle between 90° and 180° is greater than 0 (positive), and the cosine of any angle between 90° and 180° is less than 0 (negative). You see this because in the case of (x, y), the ordinates of any point in the second quadrant, $x < 0$ and $y > 0$.

Since, in Figure 14–10, $OM = \cos A$ and $MP = \sin A$, the coordinates of point P may be written $P(\cos A, \sin A)$.

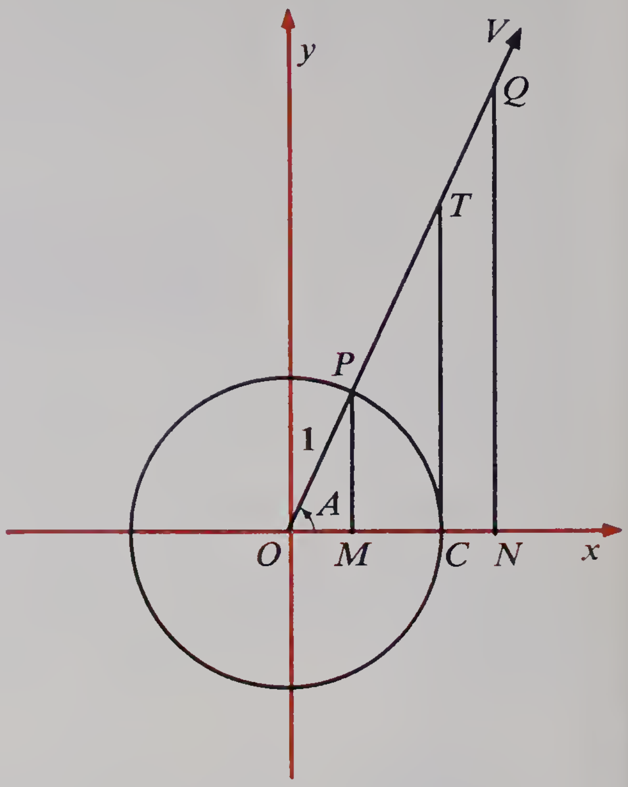


Figure 14–10

Now, by the Pythagorean Theorem, you have in right triangle OMP

$$(MP)^2 + (OM)^2 = (OP)^2.$$

But

$$MP = \sin A, \; OM = \cos A, \; OP = 1.$$

Therefore

$$(\sin A)^2 + (\cos A)^2 = 1.$$

This result which, for simplicity, may also be written

$$\sin^2 A + \cos^2 A = 1,$$

holds for every angle, without exception.

Now, using points $P(\cos A, \sin A)$ and $O(0, 0)$ to find the slope of line OV:

$$\text{slope } OV = \frac{\sin A}{\cos A}.$$

The student should be held responsible for these results and their proofs.

But you already know that

$$\text{slope } OV = \tan A.$$

Therefore

$$\tan A = \frac{\sin A}{\cos A}.$$

This result is true for every angle A, except that it does not hold when $\cos A = 0$, that is, when $A = 90°$.

See T.M. p.44(3).

Moreover, any point on line OV may be used to compute its slope. Thus you may write

$$\tan A = \frac{NQ}{ON}$$

where Q is any point whatsoever on ray OV.

In other words, the *tangent* of an angle in standard position is *ordinate over abscissa* for any point on the terminal ray.

Do not hold students responsible for reproducing this proof.

It is not hard to see that similar statements can be made about the sine and the cosine. First, for simplicity, write

$$OM = p,\ MP = m,\ ON = q,\ NQ = n.$$

Then by the Pythagorean Theorem

$$OP = \sqrt{p^2 + m^2} \quad \text{and} \quad OQ = \sqrt{q^2 + n^2}.$$

Moreover, since $\tan A$ may be written either as $\dfrac{MP}{OM} = \dfrac{m}{p}$ or as $\dfrac{NQ}{ON} = \dfrac{n}{q}$, you know that $\dfrac{m}{p} = \dfrac{n}{q}$ or $m = \dfrac{pn}{q}$.

Now by definition, $\sin A = MP$. Since $OP = 1$, this can be written

$$\sin A = MP = \frac{MP}{1} = \frac{MP}{OP} = \frac{m}{\sqrt{p^2 + m^2}}.$$

Now substitute $\dfrac{pn}{q}$ for m:
$$\sin A = \frac{\dfrac{pn}{q}}{\sqrt{p^2 + \dfrac{p^2n^2}{q^2}}}$$

For simplicity, multiply numerator and denominator by $\dfrac{q}{p}$:

$$\sin A = \frac{\dfrac{q}{p} \cdot \dfrac{pn}{q}}{\sqrt{\dfrac{q^2}{p^2}\left(p^2 + \dfrac{p^2n^2}{q^2}\right)}} = \frac{n}{\sqrt{q^2 + n^2}} = \frac{NQ}{OQ}$$

In other words, the *sine* of an angle in standard position is *ordinate over distance from the origin* for any point on the terminal ray.

Now there is an easy way to find the cosine. You know that

$$\sin A = \frac{NQ}{OQ} \text{ and that } \tan A = \frac{NQ}{ON}.$$

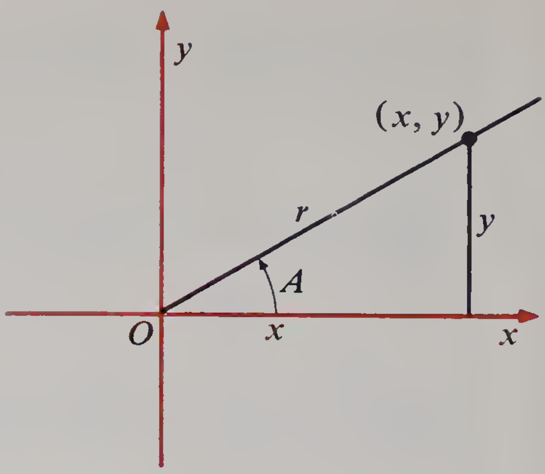
You also know that

$$\frac{\sin A}{\cos A} = \tan A.$$

Therefore
$$\cos A = \frac{\sin A}{\tan A} = \frac{\dfrac{NQ}{OQ}}{\dfrac{NQ}{ON}} = \frac{NQ}{OQ} \cdot \frac{ON}{NQ} = \frac{ON}{OQ}.$$

In other words, the *cosine* of an angle in standard position is *abscissa over distance from the origin* for any point on the terminal ray.

To summarize, in Figure 14–11



$$\sin A = \frac{y}{r} \qquad \cos A = \frac{x}{r}$$

$$\tan A = \frac{y}{x}$$

Figure 14–11

Your students should memorize these. Be sure that they know that x, y, r refer to a point on the terminal ray when the angle is in standard position.

WRITTEN EXERCISES

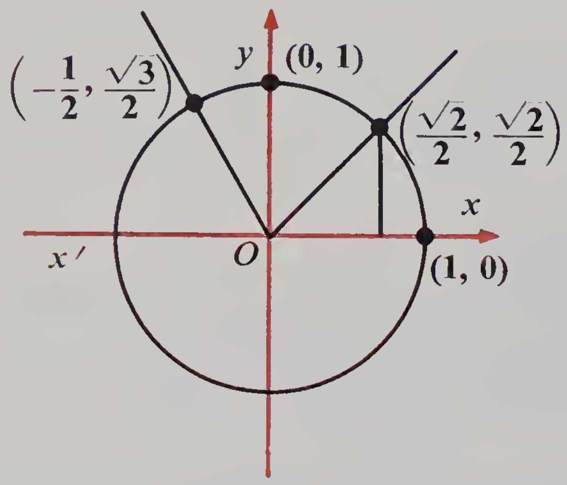
Using the data shown in the accompanying figure, find the following:

A

1. sin XOA **3.** sin XOB **5.** sin XOY **7.** cos XOX'

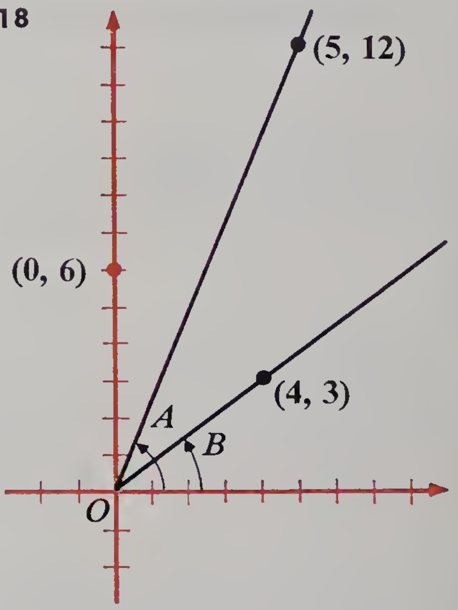
2. cos XOA **4.** cos XOB **6.** cos XOY **8.** sin XOX'

Ex. 1–8



$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
y
(0, 1)
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
x
x′
O
(1, 0)

Ex. 11–18



(5, 12)
(0, 6)
(4, 3)
A
B
O

9. If $\sin A = \frac{3}{5}$ and $\cos A = \frac{4}{5}$, (**a**) verify that $\sin^2 A + \cos^2 A = 1$; (**b**) find $\tan A$.

10. If $\cos A = \frac{12}{13}$, find $\sin A$.

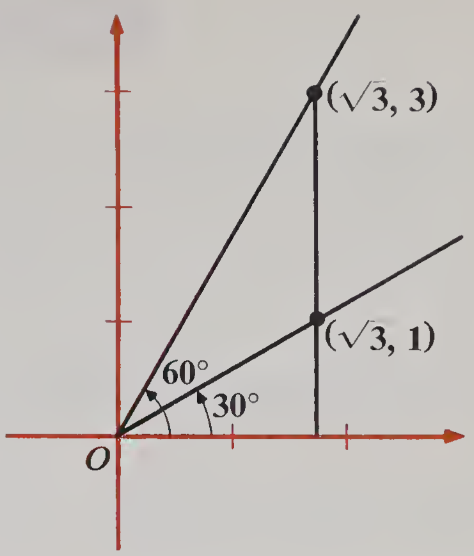
Using the data in the accompanying figure, find the following:

11. sin A **13.** tan A **15.** cos 90° **17.** cos B

12. cos A **14.** sin 90° **16.** sin B **18.** tan B

19. Given $\cos B = \frac{5}{13}$ and $\tan B = \frac{12}{5}$ verify that

$$1 + (\tan B)^2 = \left(\frac{1}{\cos B}\right)^2.$$



Using the accompanying figure, find

20. sin 30° **22.** cos 30°

21. cos 60° **23.** sin 60°

14–8 Function Values See T.M. p.44

Each of these three correspondences between angles and numbers is a *function* (see page 438). Any one of them, the sine, for instance, is clearly a set of ordered pairs, the first element of the pair being an angle and the second element a number. Moreover, it is clear directly from the definition that no angle can have more than one sine: the sine is the ordinate of one particular point, the point on the unit-circle where the terminal ray of the angle cuts it.

Since the sine is a function, you can make a table including at least some of the ordered pairs. You will find such a table on page 555.

Since the cosine and the tangent are also functions, they too can be tabulated: to save space, the tables for all three functions are printed together in one combined table. The table does not extend beyond 90°, although there are larger angles, and you saw how to work out the sine, cosine, and tangent for angles between 90° and 180°. For the use you will make of these functions, this table is sufficient.

In Table 5 (p. 555) what are the values listed for sin 90° and cos 90°? No value exists for tan 90°. Values for functions of 0° are:

$$\sin 0^\circ = 0 \quad \cos 0^\circ = 1 \quad \tan 0^\circ = 0$$

With this information, you see, as you look down the sine column, that as an angle increases from 0° to 90°, the sine increases from 0 to 1. How do the cosine and tangent change as the angle increases?

If the value of one of its trigonometric functions is given, the measure of an acute angle can be determined. The table indicates that an angle whose cosine is .7660 has a measure of 40°. Suppose $\tan A = 2.5101$, a number not listed in your table. To find the measure of $\angle A$ to the nearest degree, locate in the tangent column the entries between which 2.5101 lies: $\tan 68^\circ = 2.4751$ and $\tan 69^\circ = 2.6051$. Since $\tan A$ is closer to tan 68° than it is to tan 69°, the measure of $\angle A$, to the nearest degree, is 68°.

Don't be shaken from this position. To find tan 90° one would have to divide by 0. As this is a proscribed operation, tan 90° does not exist.

WRITTEN EXERCISES

From the table give the sines, cosines, and tangents of these angles.

A

1. 5°
2. 2°
3. 27°
4. 36°
5. 48°
6. 59°
7. 1°
8. 57°

To the nearest degree find the measure of the acute angle named.

9. $\sin A = .5446$
10. $\cos B = .5446$
11. $\tan D = .5543$
12. $\cos K = .2076$
13. $\sin C = \frac{1}{2}$
14. $\cos F = \frac{3}{4}$

See note, p.516.

14–9 Numerical Trigonometry

There are many practical problems that can be solved easily by the use of these functions. Many of these problems involve triangles. It is for this reason that the sine, cosine, and tangent are called trigonometric functions. The word *trigonometry* means triangle measure.

EXAMPLE 1. **How long a ladder is needed to reach a window in the wall of a house 25 feet above the ground, if the ladder makes an angle of 75° with the ground?**

Solution: **You see in the diagram that you must deal with the triangle *ABC*, and that what you want to find is the length of the line *AB*.**

To see how the functions about which you have just learned can help, think of coordinate axes being drawn so that the *X*-axis lies along the ground, the origin is at point *A*, and the *Y*-axis is upright, like the wall of the house.

Then angle *CAB* is in standard position, and we may write $\sin 75° = \frac{25}{l}$.

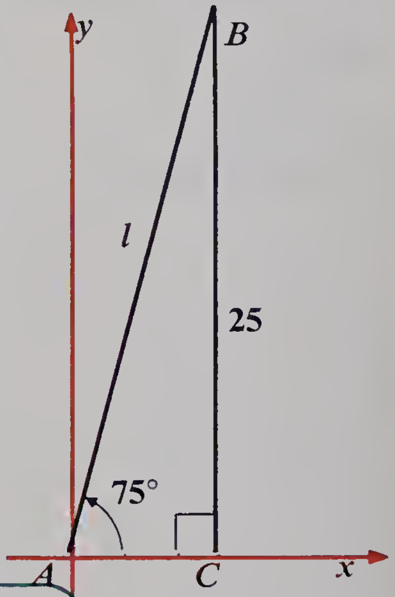
Then $l \sin 75° = 25$.

$$l = \frac{25}{\sin 75°} = \frac{25}{.9659} = 25.8.$$

You would need a 26-foot ladder, Answer.

Make sure the students look up sin 75°.

It is important that the coordinate axes be introduced into this Example to make the transition from the theoretical functions to the practical applications, and to motivate the "opposite-adjacent-hypotenuse definitions" introduced next.



In order to save yourself the trouble of drawing a set of coordinate axes in every problem, you can work out once and for all a very simple relation and remember it.

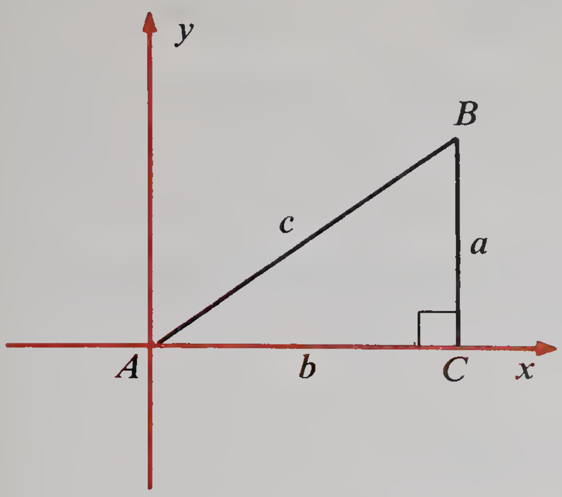


Figure 14–12

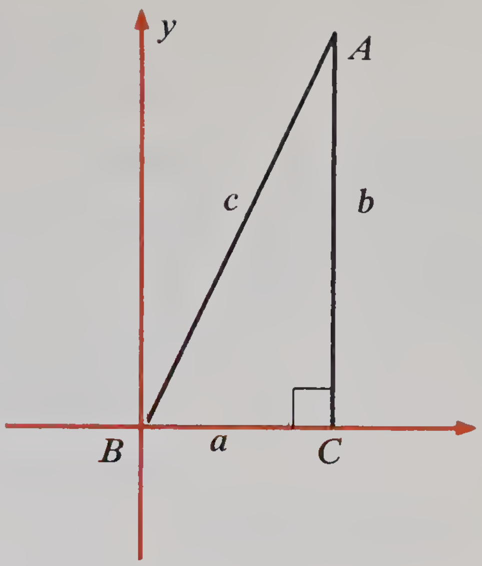


Figure 14–13

Place the right triangle ABC (in which $\angle C$ is the right angle) on a set of coordinate axes so that $\angle A$ is in standard position as in Figure 14–12. Let $AB = c$, $AC = b$, and $BC = a$. Then

$$\sin A = \frac{a}{c} \qquad \cos A = \frac{b}{c} \qquad \tan A = \frac{a}{b}$$

If $\angle B$ is in standard position as in Figure 14–13, you have

$$\sin B = \frac{b}{c} \qquad \cos B = \frac{a}{c} \qquad \tan B = \frac{b}{a}$$

Explain that these ideas are introduced for convenience in solving triangle problems. Don't let the students replace the earlier definitions and relations by them; they will be of little use, for example, in the section on vectors.

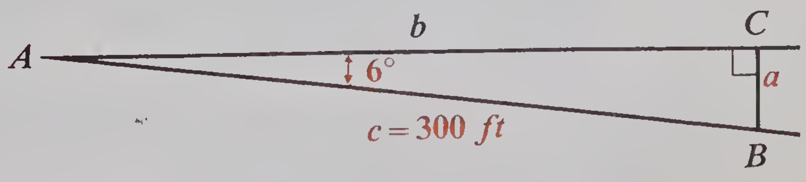
Therefore you can say that *for either acute angle of a right triangle:*

The sine is the length of the side opposite the angle divided by the length of the hypotenuse.
The cosine is the length of the side adjacent to the angle divided by the length of the hypotenuse.
The tangent is the length of the side opposite the angle divided by the length of the side adjacent to the angle.

Let us now solve another problem.

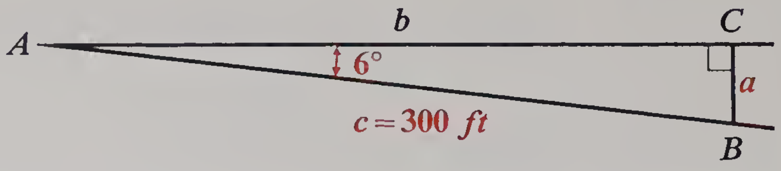
EXAMPLE 2. **If you go 300 feet into a tunnel that slopes downward at an angle of 6° from level ground, how far beneath the surface are you?**

Solution: **From the figure, you see that you want to find a, knowing angle A and c, the hypotenuse of the triangle.**



Remembering that the sine of angle A is the length of the side opposite divided by the length of the hypotenuse, you write

$$\sin 6^\circ = \frac{a}{300}.$$



This sentence is intended to emphasize the "convenience" notion referred to on p. 513.

You do not need to use coordinate axes and standard position.

From the table on p. 555, sin 6° = .1045.

Then

$$.1045 = \frac{a}{300}$$

or

$$a = (.1045)(300) = 31.35.$$

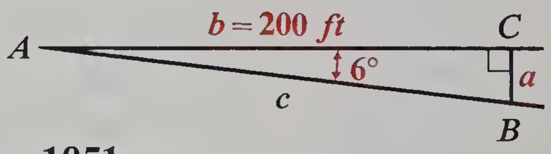
You will be approximately 31 feet beneath the surface, Answer.

EXAMPLE 3. Now suppose that it is desired to sink a ventilating shaft into the tunnel described in Example 2, 200 feet from its entrance, measured above ground. **(a)** How deep is the shaft and **(b)** how far from the mouth of the tunnel will it strike?

Solution:

a. Here you know b and A and you want to find a and c. To find a, you notice that it is the side opposite the angle and you know the side adjacent to the angle. You therefore write

$$\tan 6^\circ = \frac{a}{200}.$$



From the table, tan 6° = .1051.

Then $$a = 200 \tan 6^\circ = 21.02.$$

The length of the ventilating shaft will be 21 feet, Answer.

b. To find c, you can write

$$\cos 6^\circ = \frac{200}{c}.$$

From the table, cos 6° = .9945

$$c = \frac{200}{\cos 6^\circ} = \frac{200}{.9945} = 201.1$$

The base of the shaft is approximately 201 feet from the entrance to the tunnel, Answer.

PROBLEMS

Solve the following problems, drawing a diagram for each one. Express the distances to the nearest foot. Use the trigonometric ratios in the Appendix.

A

1. Find the height of a tree casting a forty-foot shadow when the sun's rays make a thirty-seven-degree angle with the ground.
2. Find the height of a flagpole casting a fifty-foot shadow when the sun's rays strike the ground at an angle of 40 degrees.
3. How far from a building must the base of a forty-foot ladder be placed to make a safe angle of 75 degrees with the ground?
4. A thirty-foot ladder makes a 15° angle with the wall it leans against. How high up the wall does the ladder reach?
5. A rectangle is 25 inches wide. Its diagonal makes an angle of 42 degrees with the longer side. How long is the diagonal?
6. How long a ladder is needed to reach a window 18 feet above the ground if the ladder makes an angle of 70° with the ground?
7. An airplane flies 2500 feet at a constant angle of 16° with the horizontal. Find to the nearest foot (a) the ground distance covered, (b) the change in altitude.
8. How many feet of wire will be needed to brace a pole with a wire from the top of the pole to a stake on the ground 18 feet from the foot of the pole if the taut wire makes an angle of 40° with the ground?

See T.M. p.44(4)

The term *angle of elevation*, and a related term, *angle of depression*, occur frequently in practical problems. The exact meaning of these terms is illustrated in the accompanying diagram. $\angle CAB$ is an angle of elevation; the point B is elevated with respect to the observer at A and the horizontal line AC through A. $\angle SRQ$ is called an angle of depression; the point Q is depressed with respect to the observer at R and the horizontal line SR through R.

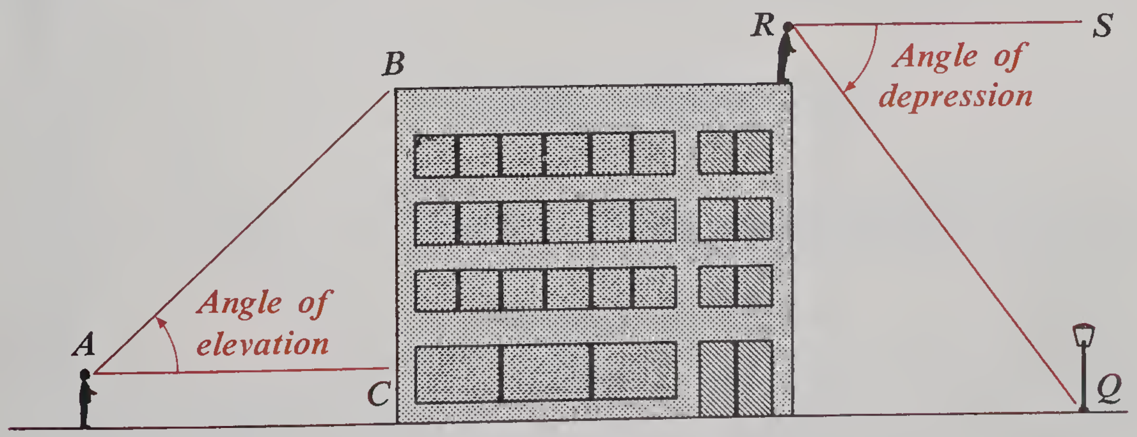


Figure 14–14

PROBLEMS

Give each answer to the nearest tenth unless otherwise specified. Use the values of the trigonometric functions in the Appendix.

A

1. A straight road up a hill is 1250 feet long and makes an angle of 18 degrees with the horizontal. Find the height of the hill.
2. A straight cable 800 feet long is run from the top of a tower to the ground. The cable makes an angle of 27 degrees with the ground. How high is the tower?
3. At 120 feet from a building, the angle of elevation to the roof is 40 degrees. Find the height of the building.
4. Find the height of a tower whose shadow is 50 feet long when the angle of elevation of its top from the end of the shadow is 60 degrees.
5. A lighthouse built at sea level is 150 feet high. From its top the angle of depression of a buoy is 25 degrees. How far is the buoy from the foot of the lighthouse?
6. From 1200 feet above an airport at sea level a pilot finds the angle of depression of a ship at sea is 14°. How far is the ship from the airport?
7. A road is inclined 9 degrees to the horizontal. How far must one walk up the road to increase one's altitude 40 feet?
8. An airplane rises at an angle of 10 degrees with the ground. How far is it from its starting point when it attains a height of 600 feet?

Sometimes a problem asks that the measure of an angle be found.

EXAMPLE 4. **A boy 5′6″ tall casts a shadow 4′ long. What is the angle at which the sun's rays strike the ground? Or, this could be phrased, What is the angle of elevation of the sun?**

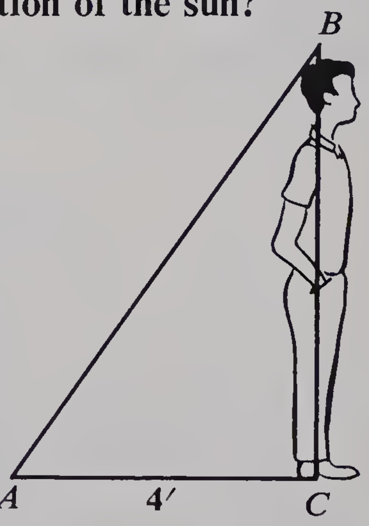
Solution: **You should see at once that**

$$\tan A = \frac{5.5}{4} = 1.3750.$$

Referring to the tables, you find

$$\tan 53^\circ = 1.3270,$$
$$\tan 54^\circ = 1.3764.$$

Avoid any discussion of interpolation. Common sense is quite sufficient to justify the statement that 1.3764 is nearer 1.3750 than is 1.3270.



To the nearest degree, therefore, the angle at which the sun's rays strike the earth is 54°, Answer.

PROBLEMS

Give answers to tenths unless otherwise specified.

A

1. Find to the nearest degree the sun's angle of elevation when a seventy-foot flagpole casts a one-hundred-foot shadow.
2. Find to the nearest degree the sun's angle of elevation when a vertical pole 6 feet high casts a shadow 10 feet long.
3. Find the angle of climb of an airplane that travels 12,500 meters through the air to rise 1500 meters.
4. A ship sails from port in a direction 155° from north. When it is 70 miles east of the port, (a) how far has it sailed? (b) How far south of the port is it?
5. Find to the nearest degree the angles of a triangle whose sides are 8′, 15′, and 17′ long.
6. How long is a funicular railway which is inclined 14° and goes up a hill 210 feet high?

B

7. Two flagpoles are 100 yards apart. The angles of elevation to their tops from the center of the ground between them are 20° and 15°, respectively. How much higher is one pole than the other?
8. From the crow's nest of a ship, an observer finds the angle of elevation of the top of a lighthouse to be 5° and the angle of depression of the foot of the lighthouse to be 8°. If the observer is 80 feet above sea level, find the height of the top of the lighthouse above sea level.

14–10 Similar Triangles

Triangles ABC and $A'B'C'$ have the same shape, but differ in size. What can be said about the lengths of their sides? Clearly, you must first know precisely what is meant by "having the same shape." This means

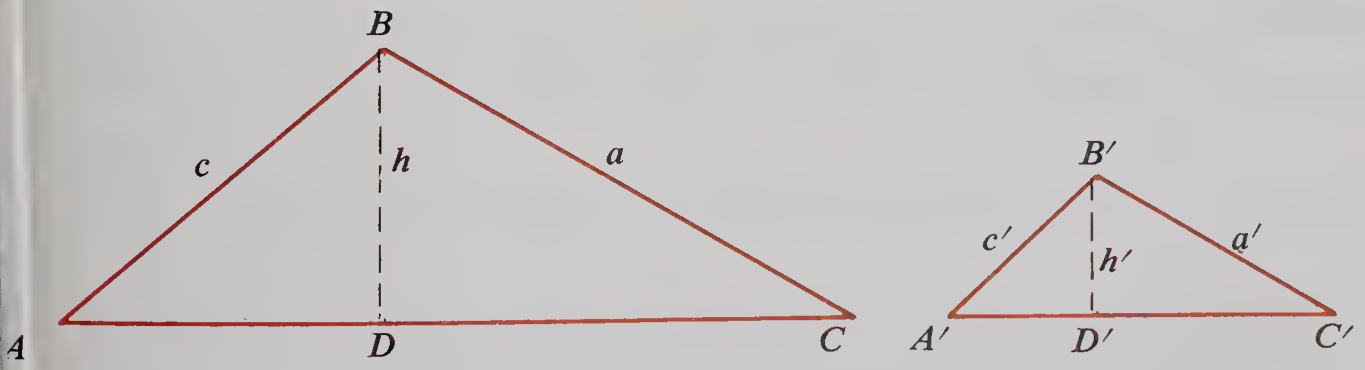


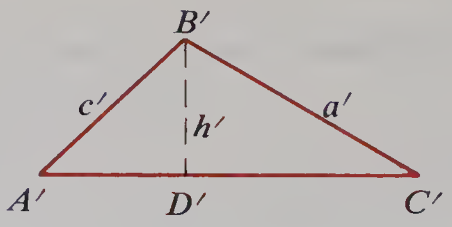
Figure 14–15

$$m\angle A = m\angle A', \quad m\angle B = m\angle B', \quad m\angle C = m\angle C'$$

In other words, here, similar triangles are defined as equiangular triangles.

The proportionality of the sides is based on trigonometry, rather than the reverse. This is because a course in geometry is not presupposed.

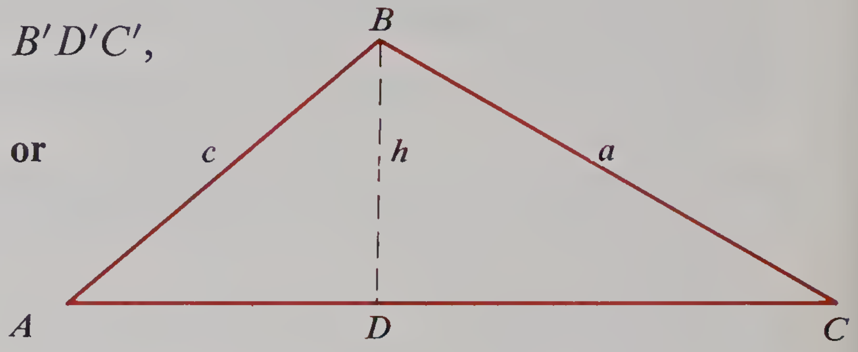
Draw $\overline{BD}$ and $\overline{B'D'}$. Call their lengths h and h' respectively. Then, since $m\angle A = m\angle A'$,



$$\sin A = \frac{h}{c} = \sin A' = \frac{h'}{c'}$$

$$\frac{h}{c} = \frac{h'}{c'} \quad \text{or} \quad \frac{h}{h'} = \frac{c}{c'}$$

Using triangles BDC and $B'D'C'$,



$$\sin C = \frac{h}{a} = \sin C' = \frac{h'}{a'}, \text{ or}$$

$$\frac{h}{a} = \frac{h'}{a'} \quad \text{or} \quad \frac{h}{h'} = \frac{a}{a'}$$

Similarly, by drawing perpendicular lines from C and C' to AB and $A'B'$ you can show that $\frac{a}{a'} = \frac{b}{b'}$.

Thus $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$, and it has been proved that

THEOREM: The sides of two similar triangles are proportional.

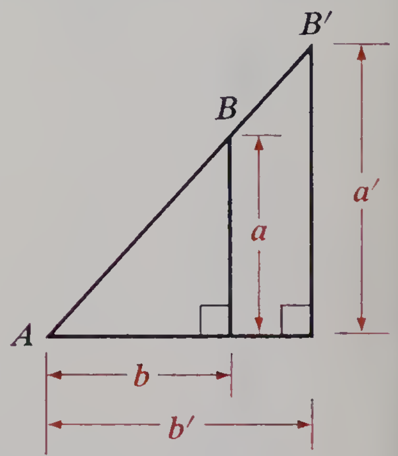
See T.M. p.45(5).

WRITTEN EXERCISES

A

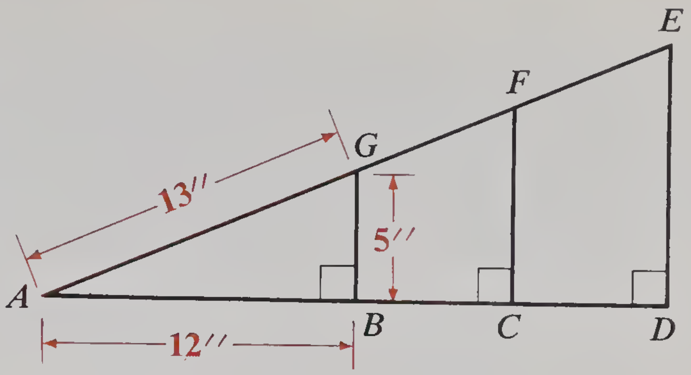
1. Two angles of a triangle measure 20° and 65°. What is the measure of the largest angle in a similar triangle?

Exercises 2–5 refer to the adjoining figure.



2. Find a' when $a = 3$ ft., $b = 5$ ft., and $b' = 15$ ft.
3. Find b' when $a = 4$ ft., $b = 2$ ft., and $a' = 30$ ft.
4. Find BB' when $AB = .39$ meter, $a = .30$ meter, and $a' = .50$ meter.
5. Find b' when $AB = 4.2$ meters, $BB' = 6.0$ meters, and $b = 3.5$ meters.
6. A triangle has sides of 12, 16, and 20 inches. The longest side of a similar triangle is 15 inches. How long are its other sides?

Exercises 7–10 refer to the figure, which has right angles at B, C, and D.



7. Find the ratio of CF to AC.

8. Find the ratio of DE to AD.

9. If at a point H on $\overline{AE}$ a line were drawn to a point J on $\overline{AD}$, so that $m\angle AJH = 90°$, what would be the ratio of JH to AJ?

10. Name two ratios of the sides of triangles in the figure, each of which equals the ratio of CF to AF.

Similar triangles can be used in practical computations also.

EXAMPLE 1. **How high is a flagpole which casts a fifty-foot shadow when a man six feet tall casts a seven-foot shadow?**

Solution:

By trigonometry, you would say

$$\tan A = \tfrac{6}{7}$$

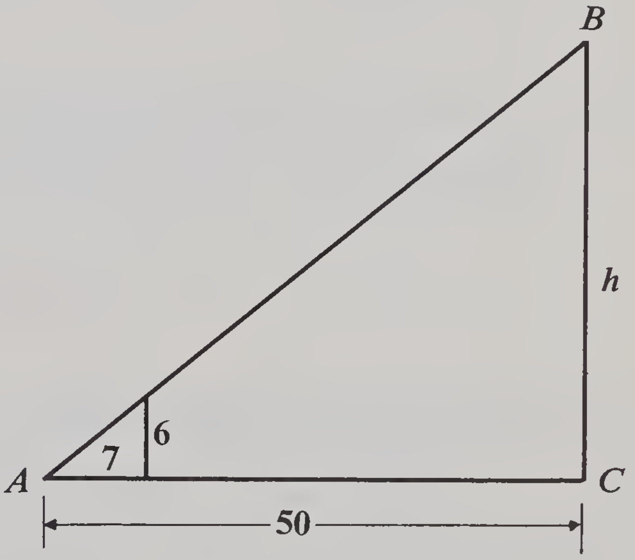
$$\tan A = \frac{h}{50}$$

$$h = 50 \tan A = 50 \cdot \tfrac{6}{7} = 43$$

By similar triangles, you say

$$\frac{h}{6} = \frac{50}{7}$$

$$h = \frac{50 \cdot 6}{7} = 43$$



Do you see that the two methods used in this Example are virtually identical?

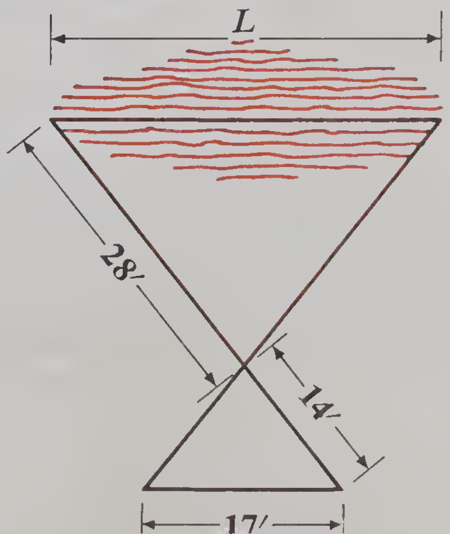
But there are problems that are much easier to do by similar triangles.

EXAMPLE 2. **Taking the distances as shown in the diagram, determine the length of the pond.**

Solution:

$$\frac{L}{17} = \frac{28}{14}$$

$$L = \frac{28}{14} \cdot 17 = 34.$$

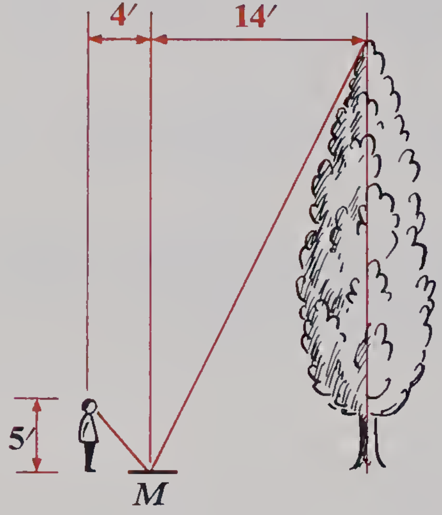


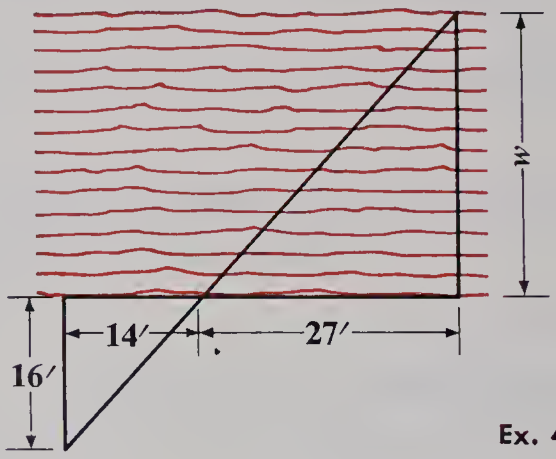
PROBLEMS

A

1. A pole six feet high casts an eight-foot shadow, and a tree nearby casts a forty-foot shadow. How high is the tree?
2. A tree has a shadow 50 feet long when a vertical yardstick casts a five-foot shadow. How high is the tree?
3. A girl places a mirror on the ground and stands where she can see the top of a tree in it. How tall is the tree as shown in the figure?

Ex. 3





Ex. 4

4. A method of finding the distance across a stream is illustrated. If two boys make the measurements noted, what is the distance w?

VECTORS

14–11 Working with Vectors

In the town where Arthur lives, Main Street is laid out in a straight line for five miles. Arthur lives on Main Street $1\frac{1}{2}$ miles from the Post Office, downtown. If you are told that Arthur started out at noon to go downtown and that he walks at the rate of 3 miles per hour, you can easily compute just where he is at any time.

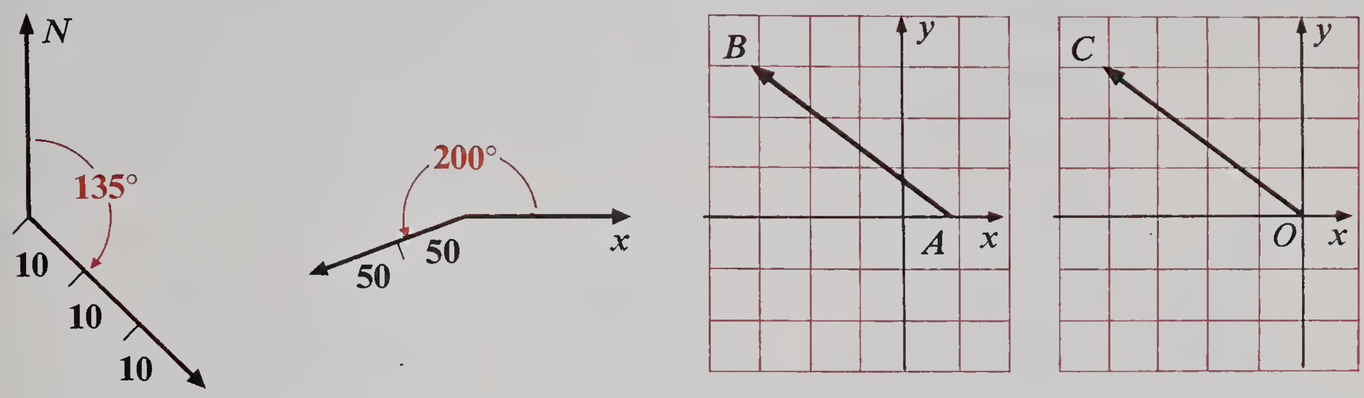
On the other hand, if you are told that an airplane left the St. Louis Airport at noon traveling at the rate of 300 miles per hour, you cannot possibly tell where it is at any time. You could say that at 1:00 P.M. it was 300 miles away, but you would not know whether it was over Arkansas, Mississippi, Tennessee, Kentucky, Indiana, Illinois, Wisconsin, Iowa, Kansas, or Oklahoma.

Arthur could go only in a straight line, and it was sufficient to know his speed. The airplane can fly in any direction, and to be able to locate it you have to know both its speed and the direction in which it

There actually is a point in each of these states 300 miles from St. Louis.

was traveling. Speed in a given direction is called velocity, which is a good example of a *vector* quantity.

A *vector* is a quantity that has both magnitude and direction. A vector quantity is represented by a directed line segment or arrow. Here are a few examples. Units of length on the arrow indicate the units of magnitude of the vector, and the direction of the arrow indicates the direction of the vector.



a *b* *c* *d*

Figure 14–16

In Figure 14–16 vectors represent:

a. A velocity of 30 m.p.h. in a southeasterly direction.
b. A force of 100 lbs. exerted to the left and 20° downward.
c. Motion in a straight line from $A(1, 0)$ to $B(-3, 3)$.
d. Motion from the origin to $C(-4, 3)$.

In order to deal adequately with vectors the concept of angle presented earlier must be somewhat extended. The notion of angle

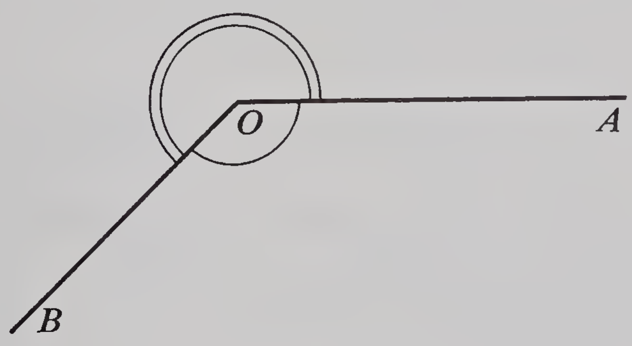
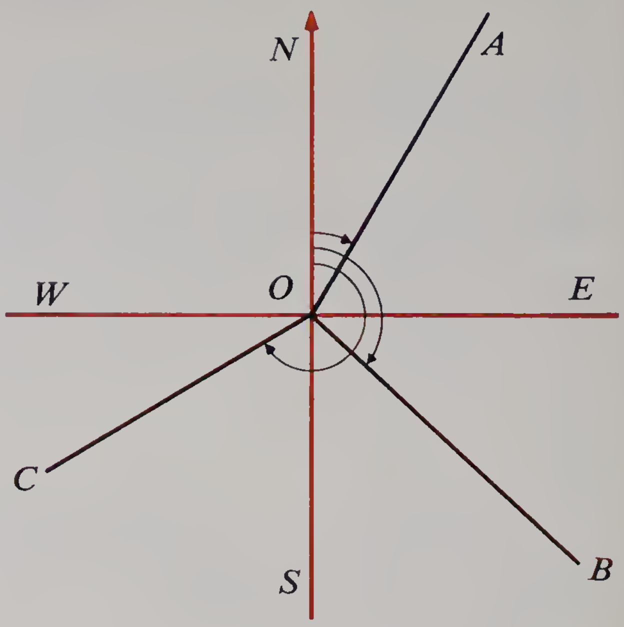


Figure 14–17

as the set of points formed by two rays proceeding from the same point does not enable you to distinguish between angle AOB less than 180°, marked with a single arc, and the angle AOB marked with a double arc. In working with vectors it is convenient to be able to make such a distinction. This is because in navigation, for instance, angles are always measured clockwise from the north. Thus, Figure 14–18 shows *headings*, or *bearings* as navigators call them, as indicated.

NOA, 30°
NOB, 135°
NOS, 180°
NOC, 240°
NOW, 270°

Similarly, mathematicians often measure angles counterclockwise from the positive x-axis. A few examples are shown.

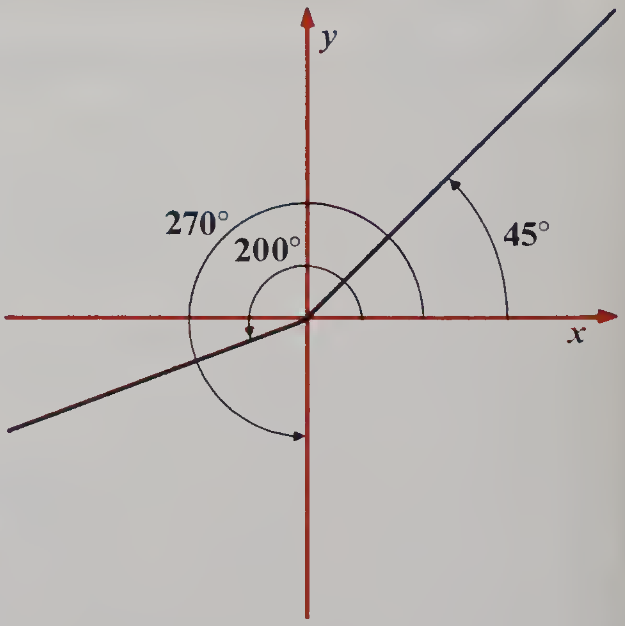


· *Figure 14–18* ·

It does not seem necessary or desirable to give a formal redefinition of angle. The only essential point is to understand that angles can now exceed 180°, and, in any particular problem, to be sure you know where the initial ray of the angle is. As stated, in most problems of pure mathematics, the initial ray is the positive x-axis; in navigation problems, it is north; in other problems it should be clear from the context. Ask the students to draw a number of large angles to ensure that they have grasped the idea.

Now that the idea of angle has been extended to include angles greater than 180°, do you see that a vector may be specified by giving its magnitude and direction as a number pair, for example, (100, 200°) or by giving its initial and terminal points, for example, $A(1, 0)$ to $B(-3, 3)$?

A short way to represent a vector is to place an arrow above the names of its end points, thus: $\overrightarrow{AB}$ represents the vector from A to B and $\overrightarrow{BA}$ represents the vector from B to A. $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are *opposite vectors* because they have the same magnitude, AB, and are collinear but



· *Figure 14–19* ·

have opposite directions; therefore, you may write $\overrightarrow{AB} = -\overrightarrow{BA}$.

Vectors like $\overrightarrow{AB}$ and $\overrightarrow{OC}$ in Figure 14–16 are *equivalent vectors;* they have the same magnitude and act in the same direction; you can see that $AB = OC$, and $\overrightarrow{AB}$ is parallel to $\overrightarrow{OC}$ with the same direction.

The vector representing the **sum** of two or more vectors is their **resultant**. The resultant of the forces exerted by a man pulling horizontally to the right with a force of 140 pounds and by a boy pushing in the same direction with a force of 100 pounds is a horizontal force of 240 pounds to the right (Figure 14–20, left). If the boy pushes to the left, the resultant is a force of 40 pounds to the right.

Here we add collinear vectors.

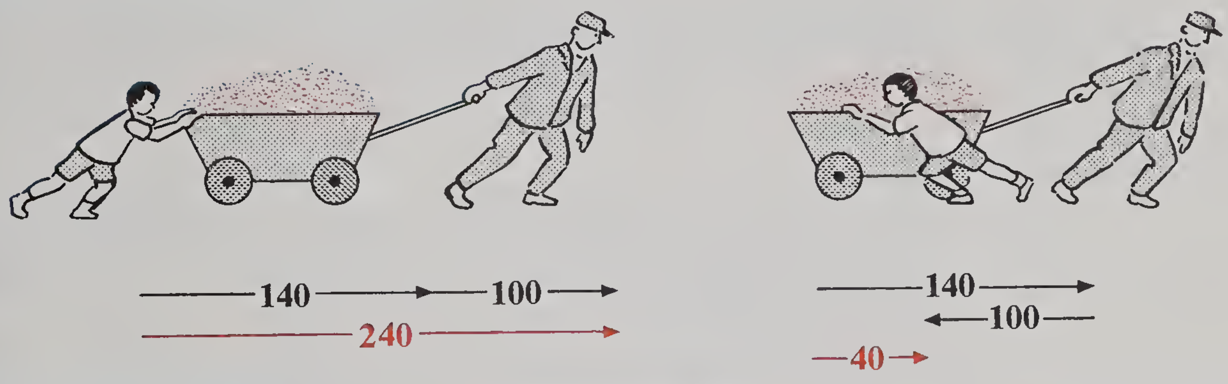
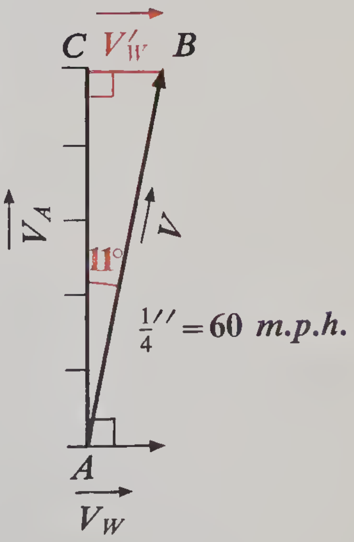


Figure 14–20

This is the triangle (or parallelogram) law for the addition of noncollinear vectors. See T.M. p.45(6)

EXAMPLE. **Determine the resultant velocity $\vec{V}$ of an airplane heading north at 300 m.p.h., $\overrightarrow{V_A}$, if it meets westerly winds blowing at 60 m.p.h., $\overrightarrow{V_W}$.**

Solution 1: **To add $\overrightarrow{V_A}$ and $\overrightarrow{V_W}$, construct to scale a right triangle having $\overrightarrow{V_A}$ as one side and a vector $\overrightarrow{V'_W}$ equivalent to $\overrightarrow{V_W}$ as the other side. The hypotenuse $\vec{V}$ is the resultant. From the scale drawing you can estimate that the plane is flying at 310 m.p.h. at a heading of 11°, Answer.**



Solution 2: **You may also solve the problem by trigonometry.**

direction of $\vec{V}$ is $\angle CAB (\angle A)$

$$\tan A = \frac{CB}{AC}$$

$$= \frac{60}{300} = .2000$$

$$A \doteq 11°$$

magnitude of $\vec{V}$ is AB

$$(AB^2) = (AC)^2 + (CB)^2$$

$$= (300)^2 + (60)^2$$

$$= 93{,}600$$

$$AB \doteq 306$$

Emphasize that a scale drawing gives only an estimate.

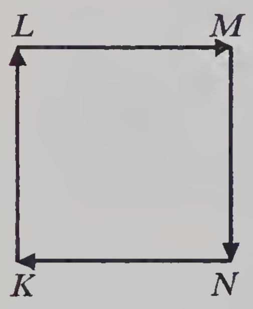
∴ The airplane is flying at approximately 306 m.p.h. at a heading of 11°, Answer.

WRITTEN EXERCISES

Does each expression describe a vector?

Ⓐ

1. Your glove size
2. A batting average of .297
3. A south wind at 6 miles per hour
4. A force of 50 pounds upward
5. A strike thrown by a pitcher
6. An elevator ride to the third floor
7. A three-mile bus ride downtown
8. A speed of 30 miles per hour



Ex. 9–12

Name a vector equivalent to each vector in square *KLMN*.

9. $\overrightarrow{KL}$ **10.** $\overrightarrow{MN}$ **11.** $\overrightarrow{LM}$ **12.** $\overrightarrow{NK}$

What vector in rectangle *RSTV* makes each statement true?

13. $\overrightarrow{VR} + \overrightarrow{RS} = \underline{\ ?\ }$

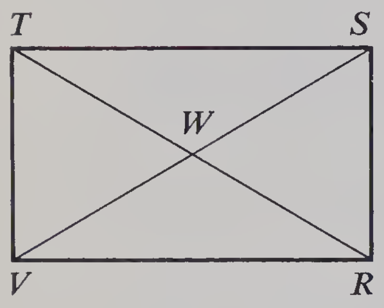
14. $\overrightarrow{ST} + \overrightarrow{TV} = \underline{\ ?\ }$

15. $\overrightarrow{SW} + \underline{\ ?\ } = \overrightarrow{SR}$

16. $\overrightarrow{RT} + \underline{\ ?\ } = \overrightarrow{RW}$

17. $\underline{\ ?\ } + \overrightarrow{TW} = \overrightarrow{SW}$

18. $\overrightarrow{VW} + \underline{\ ?\ } = \overrightarrow{VS}$



Ex. 13–18

Find each resultant by using (a) a scale drawing, (b) trigonometry.

19. Forces of 100 lb. to the right and 70 lb. up

20. Forces of 520 lb. to the left and 90 lb. up

21. Forces of 60 kg. to the right and 80 kg. down

22. Forces of 70 g. to the left and 150 g. down

PROBLEMS

Draw a diagram; then find the answer to the nearest integer by trigonometry.

Ⓐ

1. Two men stretch a wire fence around a post, making a right-angle corner. One pulls an end north, and the other, east, each with a force of 100 pounds. Describe the force acting on the post at that moment.
2. A man walks north at 5 m.p.h. across a ship sailing east at 12 m.p.h. At what rate and in what direction is he actually moving?
3. An airplane whose speed in still air is 150 m.p.h. heads west in a wind blowing south at 22 m.p.h. In what direction does it move and at what rate?
4. A plane flying at 210 m.p.h. heads west into a wind blowing north at 22 m.p.h. In what direction does the plane move, and at what speed?

5. A balloon pulls up on its anchor rope with a force of 500 pounds, while the east wind blows against it with a force of 200 pounds. Describe the actual pull and the angle the rope makes with the ground.
6. An east wind acts on an anchored ship with a force of 2600 pounds, as a tide flows north with a force of 3500 pounds. Describe the resultant force on the ship.

14–12 Resolving a Vector

Two vectors, such as $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ in Figure 14–21, whose sum is $\overrightarrow{V}$ are called *components* of $\overrightarrow{V}$. Of particular interest are the horizontal and vertical components of a vector because they refer to a rectangular coordinate system.

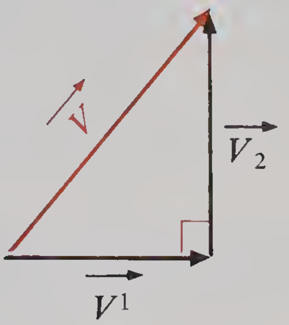
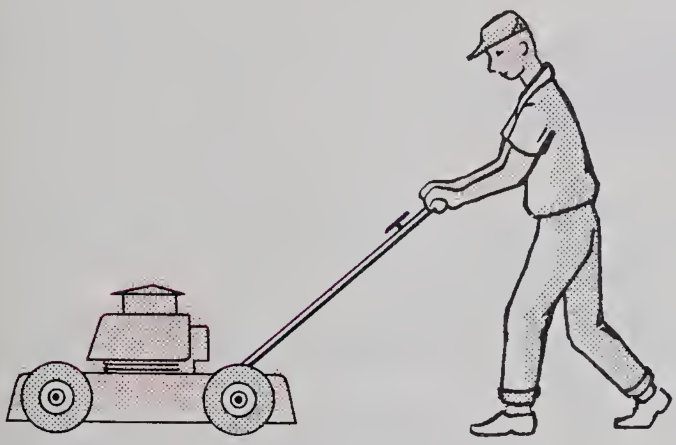
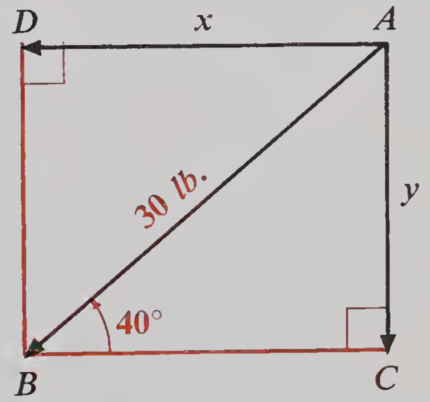


Figure 14–21

See T.M. p.45(7)

EXAMPLE 1. **A force of 30 pounds is applied to a lawn mower whose handle is 40° from the horizontal. What is the horizontal force moving the lawn mower?**





Solution: **1. Draw a right triangle and label it to represent the known forces and angles.**

2. $\cos 40° = \dfrac{CB}{30} \doteq .766$

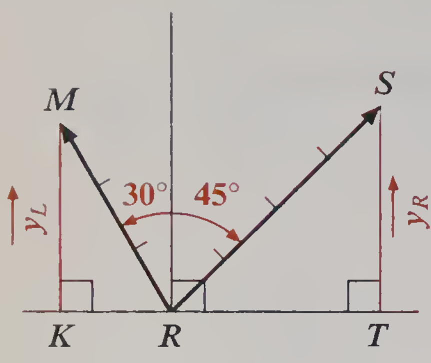
$CB \doteq 23$

∴ The horizontal force is 23 pounds, Answer.

By using sin 40°, you can find the vertical component to be 19 pounds.

Can you find the resultant of two vectors whose horizontal and vertical components are known? The component of the resultant in each direction is the sum of the components of the vectors in that direction. Now any pair of vectors can be added. See T.M. p.46(8).

EXAMPLE 2. Two boys lift a heavy rock by means of rope attached to it. One exerts a force of 30 pounds at 30° to the left of the vertical, and the other exerts a force of 40 pounds at 45° to the right of the vertical. What is the force moving the rock vertically?



Solution:

Let $\overrightarrow{y_L}$ = the vertical component of the force to the left
$\overrightarrow{y_R}$ = the vertical component of the force to the right
$\overrightarrow{y}$ = the total vertical force lifting the rock

In $\triangle RKM$: $\sin 60° = \dfrac{\overrightarrow{y_L}}{30}$ In $\triangle RTS$: $\sin 45° = \dfrac{\overrightarrow{y_R}}{40}$

Point out this use of the complementary angle.

$$\therefore \overrightarrow{y_L} = 30 \sin 60° \qquad \overrightarrow{y_R} = 40 \sin 45°$$
$$\doteq 30(.866) \qquad \doteq 40(.707)$$
$$\overrightarrow{y_L} \doteq 26 \qquad \overrightarrow{y_R} \doteq 28$$
$$\overrightarrow{y} \doteq 26 + 28 \doteq 54$$

∴ The total vertical force is approximately 54 pounds, Answer.

PROBLEMS

A

1. Bill pushes a lawn mower with a force of 45 pounds at 37° from the horizontal. How great a force moves the mower?
2. How much of a thirty-pound force applied along the handle of a carpet sweeper at 55° from the floor is effective?
3. Tom pulls the rope of a sled with a force of 15 pounds at 35° from the ground. How many pounds of force are effective?
4. A man operates a pneumatic drill at 60° from the horizontal applying a force of 150 pounds. What is the magnitude of the effective force?
5. A ship's towline makes a 20° angle with the direction of motion when a force of 1200 pounds is applied. Find the pull on the ship.
6. Wind at 52° from the direction of motion of a boat blows against the sail with a 500-pound force. What force pushes the boat forward?
7. A balloon with a lifting force of 3000 pounds is blown so that its anchor rope makes an angle of 35° with the ground. Find the horizontal force of the wind and the pull on the rope.

B

8. A boy weighing 125 pounds sits in the center of a hammock. Find the pull on each supporting rope if each is 45° from the horizontal.

Chapter Summary

Inventory of Structure and Method

1. **Geometry** deals with the properties of sets of points. **Theorems** stating properties of geometric figures are proved on the basis of axioms stating assumptions for numbers, points, lines, and planes. Once a theorem is proved, you can use it in proving other theorems.
2. **Every line segment** has a length, and every angle has a measure. In a triangle the sum of the measures of the angles is 180°.
3. The **tangent** of an angle in standard position is the slope of the terminal ray. It may be pictured by the length cut off by the terminal ray above (or below) the x-axis on the line $x = 1$.
4. The **sine** of an angle in standard position is the ordinate of the point on the terminal ray whose distance from the origin is 1. The abscissa of the point is the **cosine** of the angle.
5. In any right triangle, if you know two sides, or one side and one angle, you can find the parts you do not know by using the tangent, sine, or cosine.
6. **Similar triangles** are triangles that have the same shape, that is, all of whose angles respectively have the same measure. Sides of similar triangles are proportional.
7. The **resultant of vectors** acting in the same or opposite directions at a point may be found by adding or subtracting their magnitudes. To find the resultant of two vectors acting at right angles to each other, find the hypotenuse of a right triangle whose sides represent the vectors.
8. To find the **vertical and horizontal components** of a vector, determine the sides of a right triangle whose hypotenuse represents the vector.

Vocabulary and Spelling

collinear (*p. 494*)
distance (*p. 495*)
between (*p. 496*)
line segment (*p. 496*)
ray (*p. 498*)
angle (*p. 498*)
triangle (*p. 499*)
standard position (*p. 502*)
tangent function (*p. 504*)
degrees (*p. 504*)
sine function (*p. 507*)
cosine function (*p. 507*)
trigonometric function (*p. 511*)
similar triangles (*p. 517*)
angle of elevation (*p. 515*)
angle of depression (*p. 515*)
vector (*p. 521*)
resultant (*p. 523*)

Chapter Test

14–2 **1.** Points R, S and T are collinear and have coordinates r, s, and t respectively. If $r < s < t$, which point lies between the other two?

14–3 State whether the graph of each open sentence is a ray, a point, a line segment, a line, or none of these.

2. $y \leq 2$ **3.** $x > -1$

14–4 **4.** In $\triangle ABC$, $m\angle A = 22°$, and $m\angle B = 87°$. What is $m\angle C$?

14–5 **5.** If $m\angle B = 45°$, and angle B is placed in standard position, which one of the following describes the position of its terminal ray?

a. $x = 0$ **b.** $y = 0$ **c.** $y = |x|$ **d.** $y = x$

14–6 **6.** The terminal ray of an angle in standard position passes through $P(2, 4)$ and $Q(5, 10)$. What is the tangent of the angle?

14–7 **7.** If $\sin A = \frac{12}{13}$ and $\cos A = \frac{5}{13}$, what is $\tan A$?

14–9 Here are some excerpts from the table of trigonometric functions. It contains the functional values you will need in this test.

Angle	*Sine*	*Cosine*	*Tangent*
8°	.1392	.9903	.1405
15°	.2588	.9659	.2679

8. A ramp leading to an underground parking garage makes an angle of 8° with the horizontal, and is 85 feet long. How deep below street level is the floor of the parking garage?

14–10 **9.** A flagpole casts a shadow of 102′ when a boy 5′ tall casts a shadow of 6′. How high is the flagpole?

Two forces having magnitudes of 90 pounds and 135 pounds, respectively, act at right angles to each other.

14–11 **10.** Find the magnitude of the resultant, to the nearest pound.

14–12 **11.** A force of 145 pounds acts at an angle of 15° with the vertical. Find the horizontal and vertical components, to the nearest pound.

Chapter Review

14–1 The Structure of Geometry *Pages 493–494*

1. Geometry is concerned with the properties of sets of $\underline{\quad ? \quad}$.

14–2 Geometric Assumptions *Pages 494–497*

2. Every line contains at least $\underline{\quad ? \quad}$ points and every plane contains at least $\underline{\quad ? \quad}$ noncollinear points.
3. If $P < Q < R$, point $\underline{\quad ? \quad}$ is between points $\underline{\quad ? \quad}$ and $\underline{\quad ? \quad}$.

14–3 Rays and Angles *Pages 497–499*

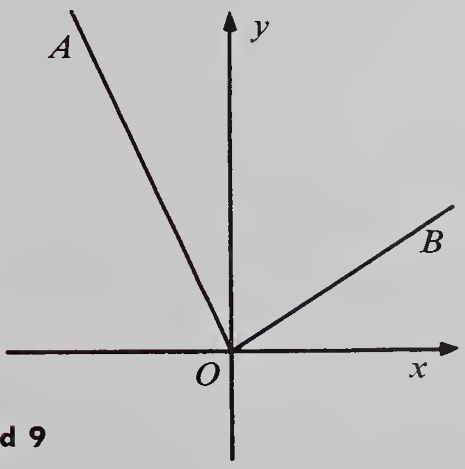
If on line l the coordinate of C is 3, of D is 5, and of K is 9,

4. Write an open sentence whose graph on l is the ray DK.
5. Angle CDK is a $\underline{\quad ? \quad}$ angle.

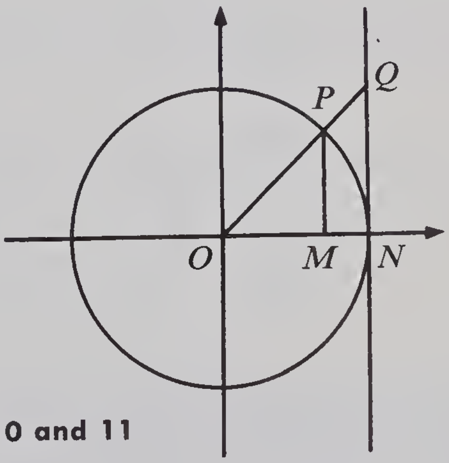
14–4 Triangles *Pages 499–500*

In triangle ABC, $m\angle C = 90°$, $m\angle A = 20°$, $AC = 3$, $BC = 4$.

6. $m\angle B = \underline{\quad ? \quad}$.
7. $AC = \underline{\quad ? \quad}$.



Exs. 8 and 9



Exs. 10 and 11

14–5 The Measurement of Angles *Pages 501–503*

8. Any angle in standard position in the accompanying figure has ray $\underline{\quad ? \quad}$ as its initial ray.

14–6 Tangent of an Angle *Pages 503–506*

9. If the slope of line OB is $\frac{2}{3}$, the tangent of $\angle XOB$ is $\underline{\quad ? \quad}$.
10. A line segment whose length is $\tan MOP$ is $\underline{\quad ? \quad}$.

14–7 Sine and Cosine of an Angle *Pages 506–511*

11. A line segment whose length is $\cos MOP$ is $\underline{\quad ? \quad}$.
12. $\sin^2 A + \cos^2 A = \underline{\quad ? \quad}$.

14–8 Function Values *Pages 511–512*

Use Table 5, Appendix, to answer Exercises 13–24.

13. sin 55° = _?_ **14.** cos _?_ = .3090

14–9 Numerical Trigonometry *Pages 512–517*

15. A wire used to brace a telephone pole is 25 feet long and makes an angle of 55 degrees with the ground. How high up the pole (to the nearest foot) does the wire reach?

16. How long (to the nearest foot) is a ladder that makes an angle of 50 degrees with the ground at a point 40 feet from the base of the building it leans against?

14–10 Similar Triangles *Pages 517–520*

17. The measures of the corresponding angles of similar triangles are _?_.

18. The sides of a triangle are 10 inches, 12 inches, and 13 inches. The corresponding sides of a similar triangle are _?_ inches, _?_ inches, and 52 inches.

19. A right triangle contains a 35° angle. Another right triangle contains a 55° angle. Are the triangles similar?

14–11 Working with Vectors *Pages 520–525*

20. A force of 100 pounds acts on an object horizontally to the right. Another force of 100 pounds acts vertically upward on it. Draw to scale the vectors and their resultant.

21. The magnitude of the resultant in Exercise 30 is _?_. It acts at an angle of _?_ with the horizontal.

22. Describe the motion of a canoe when a boy paddles north at 5 m.p.h. and the current is 2 m.p.h. to the east.

14–12 Resolving a Vector *Pages 525–526*

23. A small boy is pulling a wagon with a force of 10 pounds. The handle makes an angle of 30 degrees with the wagon. Make a scale drawing showing the vector of the force the boy applies, and its horizontal and vertical components.

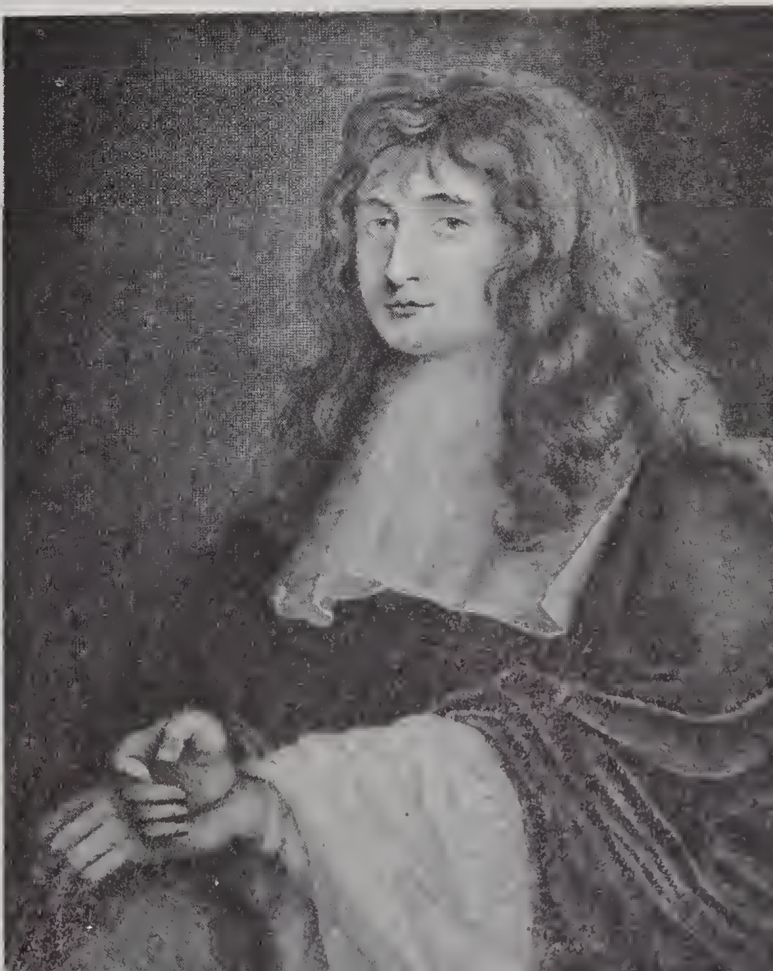
24. A force of 200 pounds is acting at an angle of 55° with the horizontal. Find the vertical and horizontal components, each to the nearest pound.

A Giant among Men

At the University of Cambridge, in the year 1705, a man knelt before Queen Anne of England to be dubbed a Knight of the Realm. Many men before him had been thus honored, but this man was vastly different from his predecessors. Their distinction had been won on the battlefields; his, in the sphere of natural philosophy and mathematics. His name was Isaac Newton.

The abilities which set Newton apart from other men were displayed at an early age. After a brief schooling, he was to learn the business of his father's farm, but he spent most of his time making mechanical models, reading, and solving problems. His mother wisely resolved to encourage his interests and, on the advice of an uncle, sent him to Cambridge in 1661. In his first five years there, Newton not only formulated the theory of gravitation but began investigating the properties of light. He also invented the calculus, a new branch of mathematics which could express the principles of gravitation as well as many discoveries yet to be made in the science of physics. No wonder that in 1669, Newton's former professor resigned his post in favor of this extraordinary student! No wonder that the Queen chose for Newton an honor never before granted a scientist!

Newton once said, " . . . I . . . have been . . . like a boy, playing on the seashore . . . now and then finding a smoother pebble, or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me. . . . If I have seen a little farther than others, it is because I have stood on the shoulders of giants." Three hundred years later we can say that Newton, for all his modesty, was one of the greatest giants and, by standing on his shoulders, scientists have begun perhaps to discover some of the ocean of truth.



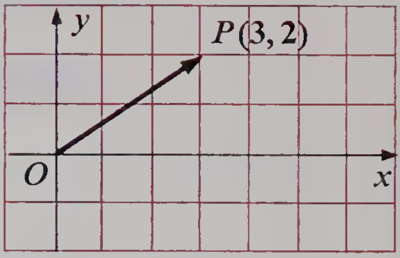
Isaac Newton as a young man. This engraving was made from a portrait of Newton painted when he was at Trinity College, Cambridge.

Extra for Experts

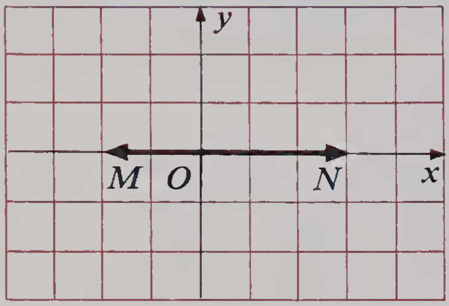
Complex Numbers

In a plane a point like $P(3, 2)$ determines a vector ($\overrightarrow{OP}$ in Figure 14–13) from the origin to the point. To pair a number with each such vector, begin with the real numbers as the partners of the horizontal vectors from O (Figure 14–14).

See T.M. p.46(9).



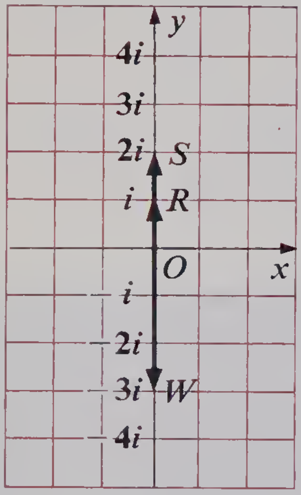
· *Figure 14–13* ·



· *Figure 14–14* ·

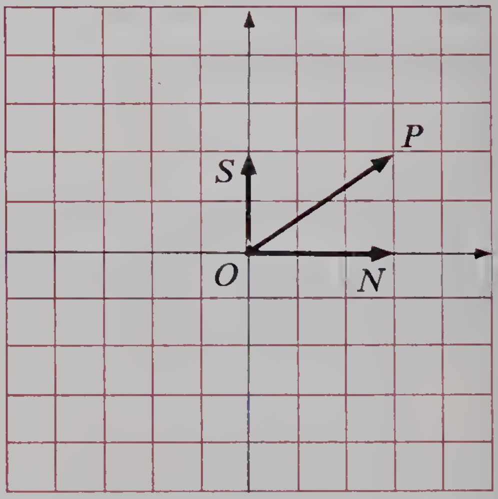
Point	Vector	Number
(3, 0)	$\overrightarrow{ON}$	3
(−2, 0)	$\overrightarrow{OM}$	−2
(0, 0)	$\overrightarrow{OO}$ (length 0)	0

Since you use all the real numbers as partners of the horizontal vectors, you need *new* numbers as partners of the vertical vectors. Use the letter i to designate the partner of $\overrightarrow{OR}$, $R(0, 1)$, and, in general, ai to name the partner of the vector to $(0, a)$ (Figure 14–15). By agreement, $0i = 0$.



Point	Vector	Number
(0, 1)	$\overrightarrow{OR}$	i
(0, 2)	$\overrightarrow{OS}$	$2i$
(0, −3)	$\overrightarrow{OW}$	$-3i$

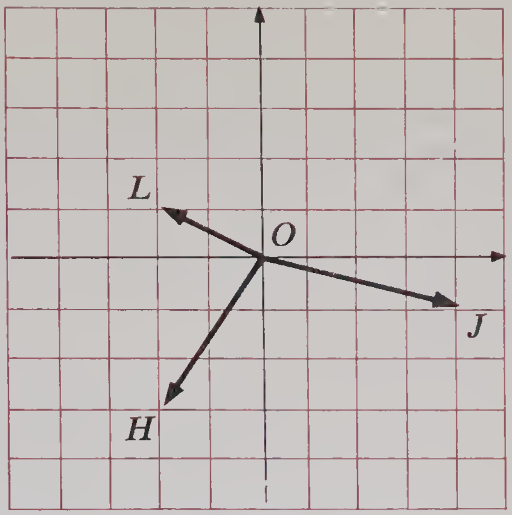
· *Figure 14–15* ·



The vector $\overrightarrow{OP}$ in Figure 14–16 has $\overrightarrow{ON}$ as its horizontal component and $\overrightarrow{OS}$ as its vertical component. Since $\overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{OS}$, pair $\overrightarrow{OP}$ with the sum of the partners of $\overrightarrow{ON}$ and $\overrightarrow{OS}$: $3 + 2i$. In general, pair the vector from O to (a, b) with the sum: $a + bi$.

· *Figure 14–16* ·

Point	Vector	Number
$(-2, 1)$	$\overrightarrow{OL}$	$-2 + i$
$(-2, -3)$	$\overrightarrow{OH}$	$-2 - 3i$
$(4, -1)$	$\overrightarrow{OJ}$	$4 - i$



Expressions of the form $a + bi$, where a and b are real numbers, represent the elements of the *set of complex numbers*. You add two complex numbers just as you add vectors whose horizontal and vertical components you know.

$$(2 + 3i) + (4 - 7i) = (2 + 4) + (3 - 7)i = 6 - 4i$$

If $a + bi$ and $c + di$ are complex numbers, then

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

Why is the sum a complex number? Because $a + c$ and $b + d$ are real, since the set of real numbers is closed under addition.

The identity element for addition is $0 + 0i$, or 0. Because

$$(a + bi) + (-a - bi) = 0 + 0i,$$

the additive inverse of $a + bi$ is $-a - bi$.

To multiply complex numbers, first define

$$i^2 = -1$$

Encourage pupils to verify that the basic laws of arithmetic hold here.

Then, multiply (as you multiply binomials) as follows:

$$\begin{aligned}(a + bi) \cdot (c + di) &= ac + i^2bd + adi + bci \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

$$\begin{aligned}\text{Thus, } (2 + 3i)(4 - 7i) &= 8 - i^2 \cdot 21 - 14i + 12i \\ &= (8 + 21) + (-14 + 12)i \\ &= 29 - 2i\end{aligned}$$

The identity element for multiplication is $1 + 0i$, or 1. To find the reciprocal of a complex number such as $3 + 2i$, proceed as follows:

$$\frac{1}{3 + 2i} = \frac{1}{3 + 2i} \cdot 1 = \frac{1}{3 + 2i} \cdot \frac{3 - 2i}{3 - 2i} = \frac{3 - 2i}{9 + 4}$$

$$\therefore \frac{1}{3 + 2i} = \frac{3}{13} - \frac{2}{13}i$$

Numbers like $3 + 2i$ and $3 - 2i$ are *conjugate complex numbers*. The product of a complex number and its conjugate is a real number:

$$(a + bi)(a - bi) = a^2 + b^2.$$

The complex number system is closed under addition, subtraction, multiplication, and division (except by 0).

EXAMPLE. **If $r = 3 + i$ and $s = 4 - 3i$, find $r + s$, $r - s$, $r \cdot s$, $r \div s$.**

Solution:

$$r + s = (3 + i) + (4 - 3i) = 7 - 2i$$

$$r - s = (3 + i) - (4 - 3i) = (3 + i) + (-4 + 3i) = -1 + 4i$$

$$r \cdot s = (3 + i) \cdot (4 - 3i) = 12 - 3(-1) - 9i + 4i = 15 - 5i$$

$$r \div s = \frac{3 + i}{4 - 3i} = \frac{3 + i}{4 - 3i} \cdot \frac{4 + 3i}{4 + 3i} = \frac{9 + 13i}{25} = \frac{9}{25} + \frac{13}{25}i$$

The complex number system has the commutative and associative properties of addition and multiplication and also the distributive property. Moreover, any polynomial equation of degree n having this system as the domain of its coefficients and variable has n roots in this system. For example, the solution set of $x^2 + 1 = 0$ is $\{i, -i\}$. These complex numbers are thus the square roots of -1: $i = \sqrt{-1}$, $-i = -\sqrt{-1}$. Similarly, $\sqrt{-4} = \sqrt{4} \cdot \sqrt{-1} = 2i$ and $\sqrt{-48} = 4i\sqrt{3}$.

This is very important. See T.M. p.46(10).

Questions

1. Express the following in a form with i as a factor.

a. $\sqrt{-64}$ **b.** $\sqrt{-27}$ **c.** $\sqrt{-\frac{81}{25}}$ **d.** $\sqrt{-.25}$

2. Name the additive inverse and the conjugate.

a. $5 + 2i$ **b.** $6 - 7i$ **c.** $-3 - \sqrt{-1}$ **d.** $7 + \sqrt{-9}$

3. Let r be the first complex number and s, the second in the following pairs. Compute $r + s$, $r - s$, $r \cdot s$, $r \div s$.

a. $3 + 7i, i$ **c.** $-i, 1 - 2i$ **e.** $9 - 3i, 7 + 6i$

b. $15 + i, 1 - i$ **d.** $i, -3 + 2i$ **f.** $-3 - 5i, 6 + i$

4. Simplify:

a. i^3 **b.** i^4 **c.** $(-3 + 7i)^2$ **d.** $(x - yi)(x + yi)$

e. $(5 + 3i) + (2 - 7i) - (3 - 8i)$ **g.** $(-i)(1 + 2i)(-3 + i)$

f. $\dfrac{3 + 3i}{8 + 6i} - \dfrac{7 + 4i}{2 - 3i}$ **h.** $\dfrac{3 - 9i}{7 + 2i} \div \dfrac{3 - 4i}{8 + 3i}$

5. Solve the following equations, if the domain of the variable is the set of complex numbers.

a. $x^2 + 25 = 0$ **c.** $x^2 - 2x + 2 = 0$ **e.** $t^2 - 6t + 34 = 0$

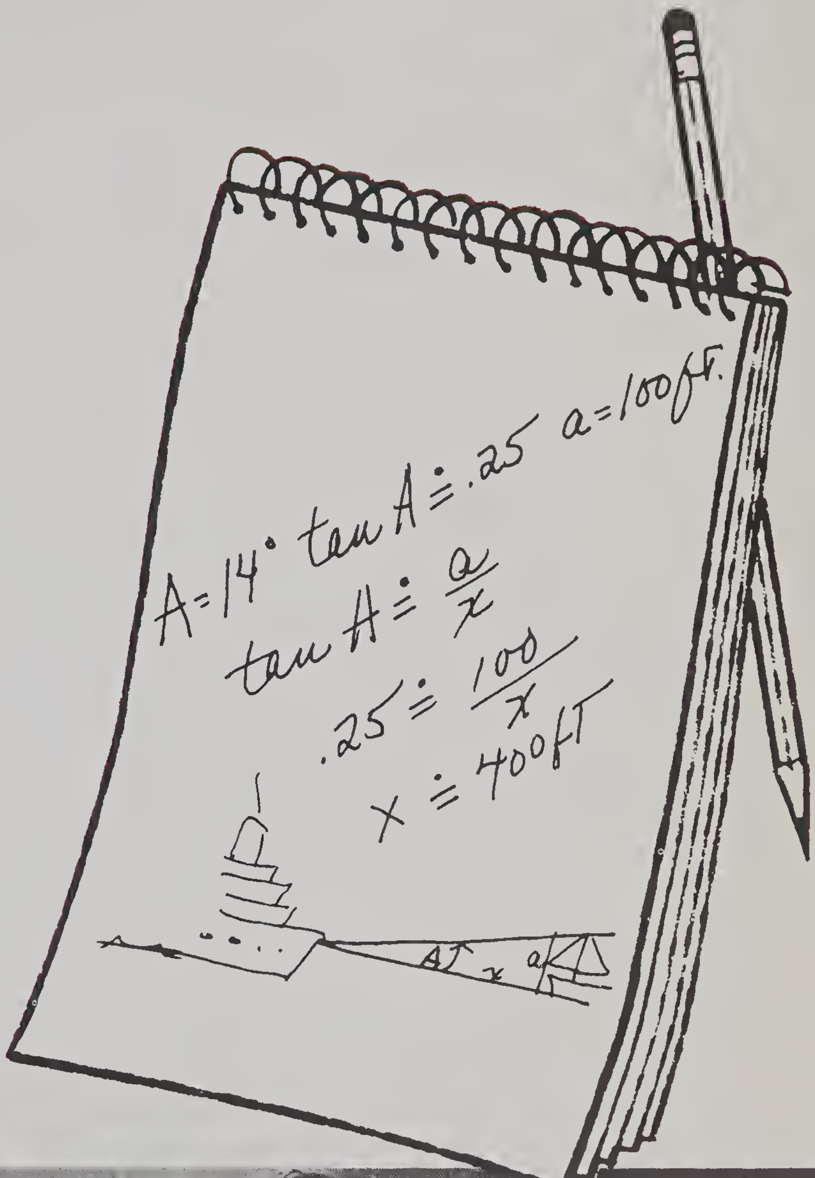
b. $y^2 + 75 = 0$ **d.** $z^2 - 4z + 13 = 0$ **f.** $t^2 + 6t + 34 = 0$

Navigators and Mathematics

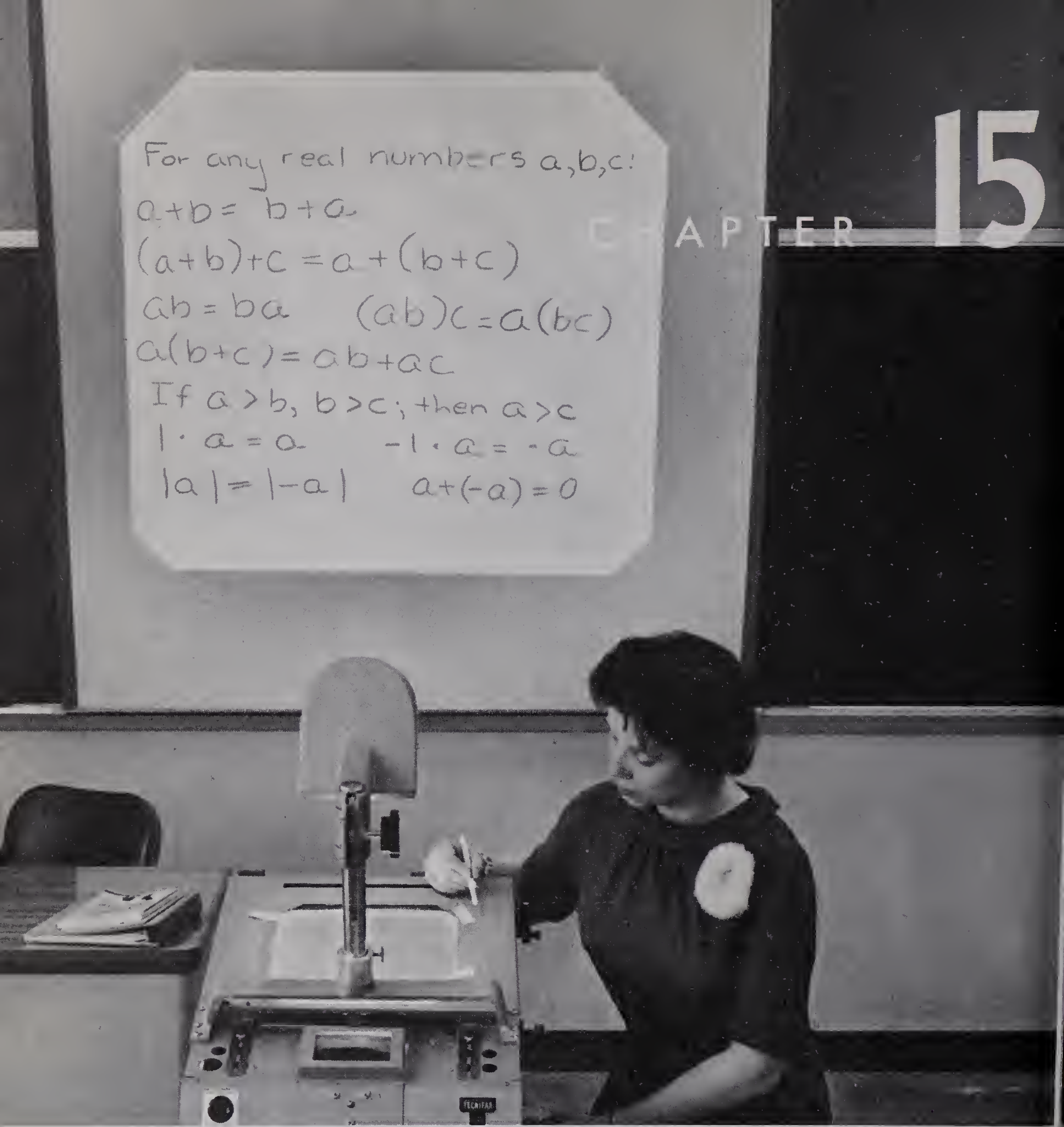
On the open sea, when the skies are clear, a navigator takes readings of his position by sighting on the stars with a sextant, an instrument which measures the angle of a star above the horizon. With these sightings and the information contained in various mathematical tables, the navigator is able to determine his ship's position, which he plots on charts in terms of longitude and latitude.

The known and exactly plotted orbits of certain satellites enable marine navigators and aeronauts to plot positions with greater accuracy than from the stars. Some of these satellites, called "lighthouses in the sky," are sighted with a sextant in the same way as stars, while others send radio signals which may be used to compute the location of a ship or plane and keep it on course.

Whether a navigator uses radar or a sextant or plots his theoretical course on a map as the civilian navigator in the photograph is doing, he needs to know algebra, geometry, and trigonometry for plotting the direction and speed of his craft. Applied mathematics may also enable a navigator to maintain a safe course when approaching or avoiding another vessel. It is possible to find the approximate distance to the other ship by measuring with a sextant the angle between its waterline and the top of its mast, as illustrated on the work pad. Knowing the height of the ship from its type, the navigator can calculate the approximate distance to it from his own vessel. And, by taking such readings at regular intervals, he can compute the other ship's speed and plot its course relative to his own.


$A = 14°$ $\tan A \doteq .25$ $a = 100$ft.
$\tan A \doteq \frac{a}{x}$
$.25 \doteq \frac{100}{x}$
$x \doteq 400$ft
A
x
a

CHAPTER 15


For any real numbers a, b, c:
a + b = b + a
(a + b) + c = a + (b + c)
ab = ba   (ab)c = a(bc)
a(b + c) = ab + ac
If a > b, b > c; then a > c
1 · a = a   -1 · a = -a
|a| = |-a| a + (-a) = 0

Comprehensive Review and Tests

When you climb a mountain trail, you pay close attention to each obstruction until at some moment you pause, look back, and realize how far you have come. Sometimes at the end of a school year, you may have this same sense of accomplishment. You have met each difficulty as it has appeared, but not until the close of the year can you evaluate your total progress.

In this chapter you have an opportunity to review your learning for the year. In the light of your total knowledge of algebra, you may now restudy topics you found difficult. Furthermore, you will be able to relearn subjects which have become hazy with time. These next few lessons provide an opportunity for you to clinch your understanding, to sharpen your knowledge, to deepen your insight, and to review the framework needed for further work in mathematics.

REVIEW YOUR ALGEBRA

15–1 Properties of Numbers: Structure

Give a reason (property, principle, definition, etc.) to justify each statement. The set of real numbers is the replacement set for each variable, with exceptions as noted.

1. $\dfrac{4a - 6}{2} = 2a - 3$

2. $729 + (93 + 71) = 729 + (71 + 93)$

3. $a - b = a + (-b)$

4. If $a = -a'$, then $a + a' = 0$.

5. $\dfrac{d}{5} = q$, if $d = 5q$.

6. If $rs = 0$ and $r \neq 0$, then $s = 0$.

7. $(a - b)(c - d) = a(c - d) - b(c - d)$

8. $(a - b)^3 = (a - b)(a - b)(a - b)$

9. If $x^2 - 6x + 3 = 0$, then $x^2 - 6x = -3$.

10. $a \div \frac{b}{c} = a \cdot \frac{c}{b}$ when $b \neq 0$ and $c \neq 0$.

11. If $m \neq n$, then $m > n$ or $m < n$.

12. If $p + q = r$, and p and q are integers, then r is an integer.

13. If $S_1 = \{-2, -1, 0\}$ and $S_2 = \{1, 2\}$, the only subset of both is $\emptyset$.

14. If $n = -n$, then $n = 0$ only.

15. If $\frac{m}{3}$ is an integer, then m is an integral multiple of 3.

16. If $x > y$ and $y > z$, then $x > z$.

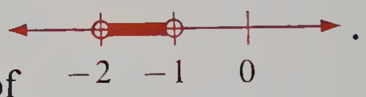
17. $x + 2 = 5$ and $y - 2 = 1$ are equivalent equations.

If a statement is always true, answer *true;* if not, answer *false.*

18. If a set is closed under an operation, any subset (except $\emptyset$) of that set is closed under that operation.

19. {points on the equator} is infinite.

20. The solution set of $-2 < x < 1$ is shown by



21. The associative property for addition in the set of real numbers is proved from other properties.

22. The solution of $3x + 5y = 7$ is a set of ordered number pairs.

23. If $ab < 0$, then $a < 0$.

24. In right triangle ABC if $a = 3$, $b = 4$, $c = 5$, then $\sin A = \frac{4}{5}$.

Express as an ordered pair the number of each statement and the letter of the reason justifying the conclusion drawn. Each variable represents a real number.

	Statements	Conclusions		Reasons
25.	$(3x - 2)(x + 3)$	$= 3x^2 + 9x - 2x - 6$	**A.**	Symmetric property of equality
26.	$3x - 2 = k + 4$ $k + 4 = 5$	$3x - 2 = 5$	**B.**	Transitive property of equality
27.	$\frac{3x - 2}{k - 1} = \frac{k - 1}{2}$	$k \neq 1$	**C.**	Substitution principle
28.	$3x - 2 = 5$	$5 = 3x - 2$	**D.**	Division by 0 is undefined
29.	$3x - 2 > x - 1$	$2 - 3x < 1 - x$	**E.**	Distributive property
30.	$3x - 2 = k + 4$ $k = x - 1$	$3x - 2 = x + 3$	**F.**	Multiplicative property of -1

15–2 Algebraic Representation

1. What is the average of the numbers a, p, and n?
2. The area of a square is $9n^2$. Represent its perimeter.
3. A rectangle is 5 units long and n units wide. Represent its diagonal.
4. From a board f feet long, a carpenter cut a piece t feet 3 inches long. Represent, in inches, the remaining piece.
5. If n articles cost c cents, how many can you buy for a quarter?
6. Charles spent c cents for n notebooks, at t cents each, and p pencils. Represent the cost of a pencil.
7. An article cost the dealer m dollars. Represent the selling price with (a) a 20% profit, and (b) a 10% loss.
8. Represent a man's possible annual salary n on a job which starts at a minimum of s dollars monthly and can go up to t dollars annually.
9. Represent the area A of a circle whose radius is between 7 and 8 centimeters.

Using $a = 3$, $b = -1$, $c = \frac{1}{2}$, $d = -2$, and $e = 0$, find these values.

10. $3ab^2$
11. $\dfrac{2ad}{c - b}$
12. $2a^2 - 4c$
13. $5ab - 2c^3d^2 + 1$
14. $\sqrt{4(d - e)^2 - 6abc}$
15. $a \times d \div 2b \div c \times d$

Round each number to one place fewer.

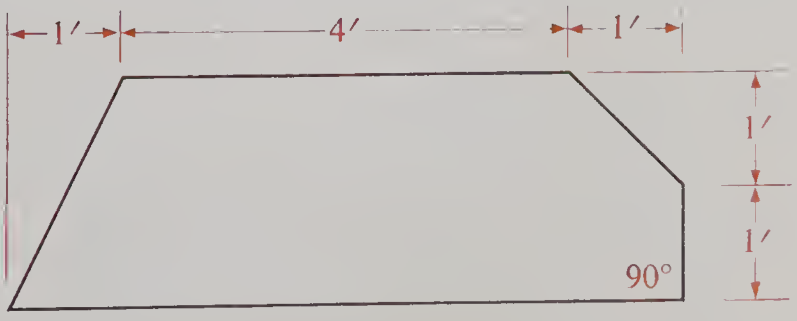
16. 58.6
17. 5.84
18. .06
19. 0.9
20. 0.95

Find to the nearest tenth.

21. $\frac{5}{13}$
22. $\sqrt{57}$
23. $\sqrt{7.1}$

Give each ratio in its simplest form.

24. 3 feet to 2 yards
25. 3000 pounds to $4\frac{1}{2}$ tons
26. 2 weeks to 26 days
27. The volume of a one-inch cube to that of a cube 3 inches on a side.
28. Find the area of the figure at the right, to the nearest tenth of a square inch.



29. Triangles ABC and DEF are similar; $\angle A = \angle D$, $\angle B = \angle E$, $AB = 6$, $BC = 7$, $DE = 12$, $EF = ?$

30. In triangle ABC, $m\angle C = 90°$, $m\angle A = 40°$, and $AB = 12$; find BC and AC if $\sin 40° \doteq .643$, $\cos 40° \doteq .766$, $\tan 40° \doteq .839$.

15–3 Fundamental Operations and Factoring

Simplify each expression.

1. $5(a + 2) - 3(a + 4)$

2. $2m^2n(3m^3n - 5n^2)$

3. $(3x - 2)(2x + 1)$

4. $(a^2 + 3ab + 9b^2)(a - 3b)$

5. $(2a - b)^5 \div (2a - b)^4$

6. $(1 - 4a)^2$

7. $(x^3 + 8) \div (x + 2)$

8. $(18ax^3 - 6a^2x^2 + 3a^3x) \div (-3ax)$

9. $(5t^3 + 13t^2 + 5t + 7) \div (t + 3)$

10. $8(1 - a) - 2[(a + 2b) - 3(2b - 1)]$

11. $3[14c - (-2 + 3c)] - 5[(2c - 1) + (c - 5)] - 3$

12. $(2a - b)(3a + c)$

13. $\dfrac{2b^2 - 18}{6} \cdot \dfrac{15}{b^2 - 3b}$

14. $\dfrac{2a^3x^5}{15b^4} \div \dfrac{8a^6x^6}{9b^4y}$

15. $\dfrac{x^2 - 1}{(x - 1)^2}$

16. $\dfrac{2a + 8}{a^2 - 3a - 28}$

17. $\dfrac{9c^2 - 1}{(3c + 1)} \div \dfrac{6c - 2}{6c^2 + 2c}$

Find the prime factors of each polynomial over the set of real numbers.

18. $3x^2 - 21x^3$

19. $1 - 9a^2$

20. $4\pi R^2 - 2\pi RH$

21. $y^2 - y - 12$

22. $x^2 - 10x + 25$

23. $P + PRT$

24. $36x^2 - 5x - 1$

25. $d^2 - .04$

26. $\frac{1}{2}b_1h + \frac{1}{2}b_2h$

27. $2h^4 - 162$

28. $n^4 - 25n^2 + 144$

29. $18r^3 + 93r^2 - 90r$

Do the indicated operations, and express in simplest form.

30. $\dfrac{a}{a + 1} + \dfrac{5}{6a + 6}$

31. $\dfrac{h^2}{h^2 - 4} - \dfrac{h}{h + 2}$

32. $\dfrac{2x + 3y}{x^2y} - \dfrac{2x - 4y}{xy^2}$

33. $\dfrac{2r + 3s}{4s} - \dfrac{7r + 5s}{6r}$

34. $\dfrac{2x+3}{x^2-1}+\dfrac{x}{1-x}$

35. $1-\dfrac{t+1}{t-1}$

36. $\dfrac{1}{a+b}-\dfrac{1}{a^2-b^2}+\dfrac{1}{a-b}$

37. $\dfrac{1+\dfrac{n}{4}}{\dfrac{n}{2}+2}$

38. $\dfrac{c^2-\frac{1}{4}}{3+\dfrac{3}{2c}}$

39. $\dfrac{1-\dfrac{r-1}{r+1}}{1-\dfrac{r+1}{r-1}}$

15–4 Radicals

Simplify each expression.

1. $\sqrt{28}$

2. $\frac{2}{3}\sqrt{180a^2}$

3. $\frac{1}{2}\sqrt{4b^{16}}$

4. $6\sqrt{\frac{1}{3}}$

5. $2\sqrt{\frac{7}{8}}$

6. $\sqrt{\frac{4}{3}}$

7. $3\sqrt{4a^2+4b^2}$

8. $\dfrac{3}{\sqrt{2}}$

9. $\dfrac{\sqrt{2}}{2\sqrt{3}}$

10. $\dfrac{20}{\sqrt{8}}$

11. $\dfrac{3\sqrt{2}+6}{\sqrt{18}}$

12. $\dfrac{a}{\sqrt{2a}}$

13. $\dfrac{1}{\sqrt{2}+1}$

14. $\dfrac{6}{3-\sqrt{5}}$

Combine the expressions.

15. $\sqrt{32}-\sqrt{8}$

16. $3\sqrt{50}-4\sqrt{98}$

17. $4\sqrt{27}+\sqrt{9}-2\sqrt{48}$

18. $\frac{1}{2}\sqrt{12}-3\sqrt{75}+\sqrt{18}$

19. $\frac{3}{2}\sqrt{\frac{1}{6}}-3\sqrt{24}+4\sqrt{\frac{3}{2}}$

20. $10\sqrt{\frac{4}{5}}-\frac{4}{3}\sqrt{45}$

Perform the indicated operations, and simplify the results.

21. $(\sqrt{6}-2)(\sqrt{3})$

22. $(4\sqrt{6})(3\sqrt{2}-2\sqrt{3})$

23. $(3\sqrt{2}+\sqrt{3})(3\sqrt{2}-\sqrt{3})$

24. $(2\sqrt{30})(6\sqrt{\frac{4}{5}})$

Find to the nearest tenth.

25. $\sqrt{29}$

26. $\sqrt{425}$

27. $\sqrt{37.8}$

28. $\sqrt{8.24}$

29. By substitution show that $2\sqrt{3}$ is a root of $x^3-2x^2-12x+24=0$.

30. If $\dfrac{\sqrt{6}}{\sqrt{6}+1}-\dfrac{\sqrt{6}}{\sqrt{6}-1}=k\sqrt{6}$, find k.

Express as a rational fraction in lowest terms.

31. $1.393939\ldots$

32. $.5\overline{7}$

15–5 Equations

Find the solution set of each equation.

1. $3a + 7 = 5a - 11$

2. $2c = 3(2c - 1)$

3. $\frac{2f}{3} + \frac{3f}{4} + \frac{17}{6} = 0$

4. $\frac{g}{2} - \frac{g + 2}{3} = 1$

5. $\frac{1}{2p} - \frac{1}{6p} = \frac{1}{2}$

6. $4(2d + 1) - 2(2d - 1) = 0$

7. $\frac{2}{2t + 3} + 2 = \frac{10}{3}$

8. $2[3(2k + 1) - (k - 3)] = 5 - 3(2k - 5)$

9. $\frac{1 - v}{6v - 3} - \frac{2}{3} = \frac{1}{2v - 1}$

10. $\frac{10}{w^2 - 25} + \frac{w}{w - 5} = \frac{30 + w}{w + 5}$

11. $\frac{2y}{2y - 1} - \frac{5}{1 - 2y} = 3$

12. $\frac{6}{x + 3} + \frac{2}{3 - x} = 0$

13. $\frac{1}{1 - c^2} + \frac{1}{c - 1} = 0$

14. $4.8(x - 2) - 3(.9x + 1) = 0$

Solve for x.

15. $2ax = 6b$ **16.** $mx + n^2 = m^2 - nx$ **17.** $\frac{2}{n} - \frac{2}{m} = \frac{1}{x}$

Find the solution set, expressing irrational results to the nearest tenth.

18. $x^2 + 64 = 16x$

19. $3x^2 = 75$

20. $h^2 - 3h = 0$

21. $3t^2 - 9t + 1 = 0$

22. $\sqrt{2y} + 1 = 9$

23. $\sqrt{9x^2 - 1} = 3x$

24. $\sqrt{s} + \frac{2}{\sqrt{s}} = \frac{s + 2}{\sqrt{s}}$

25. $\sqrt{11x + 20} + x = 2$

Solve for both variables.

26. $x - y = 1$
$3x - 2y = 0$

27. $2x - y = -6$
$6x + 4y = 3$

28. $10t + u = 8(t + u) + 7$
$u = t - 6$

29. $\frac{m}{2} + 3n = -15$
$\frac{m}{4} - 5n = 38$

30. $.3r + .2s = 9.5$
$.2r + .5s = 15.5$

Solve by graphing.

31. $x + y = 1$
$3x + y = 9$

32. $2x - y = -5$
$2x + 3y = -12$

33. $y = |x| + x$
$x + 4y = 2$

34. For what value of k will $kx - 3y = 1$ pass through $(4, -3)$?

35. For what value of k will $2x + 3ky = k$ pass through $(-2, 3)$?

36. Find the x-intercept of $2x - 3y = 6$.

37. Find the y-intercept of $4x - 3y = 12$.

In Exercises 38–43, find an equation of the line

38. having the slope -4 and passing through $(5, \frac{1}{2})$.

39. having the slope $\frac{3}{2}$ and passing through $(-1, -6)$.

40. parallel to the line $3x - y = 2$ and passing through $(2, 4)$.

41. parallel to the line $2x + 3y = 3$ and having the y-intercept 3.

42. passing through points $(-1, 10)$ and $(1, -2)$.

43. passing through points $(1, 6)$ and $(-1, -9)$.

44. Find k so that the slope of the graph of $6x + ky = k$ will be 2.

45. Graph points (x, y) satisfying both $x + 2y = k$ and $2x - y = k$.

15–6 Functions and Variation

Solve each equation for the indicated variable.

1. $v^2 = 2gs$ for s

2. $R = \frac{4}{9}(F - 32)$ for F

3. $v = \sqrt{2gh}$ for h

4. $A = \frac{1}{2}h(B + b)$ for B

In Items 5–10, write a formula for the function shown by each table.

5.

x	1	3	4
y	0	2	3

6.

x	−1	1	2
y	−1	5	8

7.

x	1	2	4	5
y	1	3	7	9

8.

n	1	2	4	5
c	2	7	17	22

9.

t	2	4	6	8
s	−1	0	1	2

10.

x	1	2	3	5
y	1	−2	−5	−11

11. Graph $d = 25t$ if $0 < d \leq 110$. What is the domain of this function?

12. A room's dimensions are l, w, and h. It has two $a \times b$ windows and a $c \times d$ door. Write an equation for the wall surface S of the room.

13. In the formula $V = \frac{4}{3}\pi R^3$, find V when $R = 3\frac{1}{2}$, and $\pi \doteq \frac{22}{7}$.

Write as proportions.

14. x varies directly as y

15. m varies inversely as n^2

16. n varies directly as c and inversely as d.

17. A varies jointly as h and b

18. $rs = k$

19. $A = kB$

Write as equations using the constant k.

20. h varies directly as t

21. V varies inversely as P

22. $\dfrac{x_1}{x_2} = \dfrac{y_2}{y_1}$

23. $A_1s_2^2 = A_2s_1^2$

24. If f is proportional to w, and $f = 81$ when $w = 18$, find w when $f = 90$.

25. If s varies as t^2, and $s = 64$ when $t = 2$, find t when $s = 100$.

26. If x is proportional to y, and y varies directly as z, find z when $x = 8$, if $x = 6$ when $z = 9$.

27. If r varies inversely as s, and $r = 6$ when $s = \frac{2}{3}$, find s when $r = 9$.

28. F varies inversely as d^2; $d = 6$ when $F = 28$; find F if $d = 9$.

29. A six-inch pulley is belted to a fourteen-inch pulley which is running at 420 r.p.m. How fast is the smaller pulley running?

30. A gear containing 46 teeth is meshed with a gear having 48 teeth. If the latter runs at 600 r.p.m., how fast does the first run?

31. If a machine digs a trench 6 feet deep and 32 yards long in 3 hours, how fast would it dig one 7 feet deep and 20 yards long?

32. **a.** If 4.6 land miles m equal 4 nautical miles n, find k in $m = kn$.
b. How many nautical miles are in 207 land miles?

15–7 Inequalities

Graph the solution sets.

1. $2v + 5 \geq 6v - 7$

2. $5 < 2n - 3 \leq n + 2$

3. $|2t + 7| > 3$

4. $|2k - 5| > 5$

5. $x^2 + 2x > 3$

6. $$\begin{aligned} x + 2y &= 4 \\ 2x - y &< 3 \end{aligned}$$

7. $$\begin{aligned} x + 2y &> 4 \\ x - y &\geq 1 \end{aligned}$$

8. $$\begin{aligned} 3x &> 2y - 5 \\ y &= \frac{x + 3}{2} \end{aligned}$$

15–8 Problems

1. How large is a portion of 100 which is 4 more than 3 times the rest?

2. If the smaller of two numbers in the ratio $\frac{4}{5}$ is increased by 4, and the larger, decreased by 4, their ratio is $\frac{13}{14}$. Find the original numbers.

3. Find the sides of a right triangle which are consecutive even numbers.

4. Find two numbers in the ratio $\frac{3}{1}$ if the sum of their reciprocals is 2.

5. The sum of two numbers is 38. When the larger is divided by the smaller, the quotient is 3 and the remainder, 2. Find the numbers.

6. If 7 is added to both numerator and denominator of a fraction, the result equals $\frac{4}{5}$. If 7 is subtracted from both numerator and denominator, the result equals $\frac{2}{3}$. Find the original fraction.

7. Dan is 3 times as old as Ethelynn. In 4 years, he will be twice as old as she is then. How old are they?

8. Mrs. Parker has a yard 30 feet by 36 feet in which a walk borders a garden measuring 720 square feet. How wide is the walk?

9. One angle of a triangle is 7° less than $\frac{1}{3}$ of the second angle, and the third is $1\frac{1}{2}$ times as large as the second. How large is the smallest?

10. What angle has a supplement 10° less than 3 times its complement?

11. When a vertical ten-foot pole casts a six-foot shadow, how tall is a tree which casts a fifteen-foot shadow?

12. The sides of a triangle are 8, 10, and 12. Find the perimeter of a similar triangle whose longest side is 54.

13. Bert, sitting 6 feet from the middle of a seesaw, balances Ned, sitting 7 feet from the middle. How heavy is Ned if Bert weighs 140 pounds?

14. If 245 grams of potassium chlorate produce 96 grams of oxygen, how many grams, to the nearest tenth, of oxygen do 100 grams produce?

15. A house plan shows a living room as $1\frac{7}{8}$ inches by $1\frac{3}{8}$ inches. If the scale reads 1 inch = 10 feet, what are the dimensions of the room?

16. Rod *A*, with 100 equal parts, is the same length as Rod *B*, with 80 equal divisions. What height on Rod *B* corresponds to a height of 60 on *A*?

17. If a line $2\frac{1}{2}$ inches long represents 100 feet, what distance is represented by a line $1\frac{7}{8}$ inches long?

18. Two trains leave a town at the same time and travel in opposite directions. One goes 18 m.p.h. faster than the other. If they are 205 miles apart after $2\frac{1}{2}$ hours, how fast is each train traveling?

19. Sam cycles to Dick's house at 12 m.p.h. and returns by bus at 36 m.p.h. If the round trip is 1 hour, how far from Dick does Sam live?

20. Ben's boat goes 5 m.p.h. in still water. How far does he travel in a three-hour round trip on a river flowing at 1 mile an hour?

21. Starting at the same time, Mark cycles north, and Carl rides south. Mark rides 3 m.p.h. faster than Carl. If in one hour the boys are 15 miles apart, how fast is each?

22. The total income of two sums invested at 5% and $3\frac{1}{2}$% is \$65. Were the rates interchanged, the income would be \$71. Find each sum.

23. Make a bar graph of one day's fruit sales, by pounds: apples, 43; peaches, 31; pears, 18; cherries, 16; and grapes, 25.

24. Make a circle graph of this budget: rent, 20%; food, 40%; clothes, 10%; insurance, 10%; housewares, 5%; other, 10%; savings, 5%.

25. Mrs. Adams buys 2 yards of silk and 8 yards of cotton for \$8.30. Mrs. Sherman pays \$7.85 for 3 yards of the silk and $5\frac{1}{3}$ yards of the cotton. What is the cost per yard of each?

26. The total income of two parts of \$1000 invested at 7% and 2% is \$32. What amount is invested at each rate?

27. How many pounds of nuts worth 44¢ a pound added to 10 pounds worth 80¢ a pound produce a mixture worth 59¢ a pound?

28. A child's bank opens when it contains \$10. If 52 quarters and dimes open it, how many coins of each kind are there?

29. How many milliliters of water added to 12 milliliters of an 80% acid solution will make the result a 60% solution?

30. How many pints of pure disinfectant added to 85 pints of a 5% disinfectant solution will result in a 15% solution?

31. Find the number which is 1 less than 9 times the sum of its two digits, and whose units digit is 6 less than its tens digit.

32. How long will it take two machines together to do a job which one does in 12 hours and the other, in 18 hours?

33. Some boys buy a \$96 tent. When two boys cannot pay, each of the others pays \$4 more. What was the original number of boys?

34. If the first of three consecutive even numbers is divided by 4, the second by 6, and the third by 8, the sum of the quotients equals 29. Find the numbers.

35. Find the diagonal of a cube whose edge measures 4 inches.

36. Fence posts are placed at equal intervals around a field whose perimeter is 720 feet. There would be 10 fewer posts if they were 1 foot farther apart. How many posts are there?

37. Make a broken-line graph of this day's temperature variation.

Hour	9	11	1	3	5	7
Temperature	−4	0	5	6	0	−5

38. The tens digit of a number is 2 less than the units digit. Three times the square of the tens digit increased by the sum of its two digits equals the number, itself. Find the number.

39. Find a number which is 28 times the sum of its three digits, whose units and hundreds digits are equal, and whose tens digit is one more than the sum of the other two.

40. In triangle ABC, $m\angle A$ is 20° more than twice that of $\angle B$. If $m\angle C$ is at least 1°, find the largest possible measure of angle A.

41. Mr. House buys flagstones to cover a patio of 160 square feet. If the patio may be between 8 and 10 feet wide, what limitations are there on its length, assuming that Mr. House uses all the flagstones?

42. A master mechanic works at least twice as fast, but not as much as three times as fast, as his apprentice. If together they do a job in 6 hours, how fast can the master do the job alone?

15–9 Indirect Measurement; Vectors

Find each trigonometric function.

1. sin 49° **2.** tan 75° **3.** cos 31°

Find each angle to the nearest degree.

4. $\sin x = .8250$

5. $\tan y = 1.2$

6. $\cos A = .2275$

7. $\tan B = \frac{5}{12}$

Make a sketch, and solve.

8. In triangle ABC, $m\angle C = 90°$, $m\angle A = 52°$, and $AB = 100$. Find CB to the nearest tenth.

9. In triangle ABC, $m\angle C = 90°$, $m\angle A = 27°$, and $AB = 100$. Find AC to the nearest tenth.

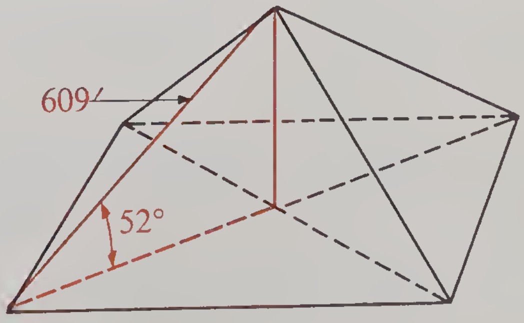
10. In triangle ABC, $m\angle C = 90°$, $CB = 4$, and $AC = 5$. Find $m\angle A$ to the nearest degree.

11. In triangle ABC, angle A measures 39° and angle B measures 73°. If the altitude to side AB measures 10 inches, find

- **a.** AB to the nearest inch;
- **b.** the area of the triangle to the nearest square inch; and
- **c.** the perimeter of the triangle to the nearest inch.

12. A pilot begins his descent 4300 feet from the runway, at an altitude of 750 feet. Find the angle of descent, to the nearest degree.

13. As shown, the Great Pyramid has an edge of 609 feet making a 52° angle with the base. How high, to the nearest foot, is the pyramid?



14. A twenty-four-foot ladder leaning against a wall is 8 feet from it at the ground. Find the angle that the ladder makes with the ground, to the nearest degree.

15. Find, to the nearest inch, the hypotenuse of a right triangle if an acute angle measures 52° and the side opposite this angle is 20 inches.

16. From the top of a lighthouse 120 feet high the angle of depression of a ship is 8°. How far, to the nearest foot, is the ship from the lighthouse?

17. Tex and Slim pull a steer so that their ropes are at right angles. Tex pulls with 180 pounds of force, and Slim, with 150 pounds. Find the magnitude of the resultant, and the angle it makes with the larger force.

18. When a weight of 50 pounds is hung on a rope, supported at both ends, the tension in one part of the rope is 3 times that in the other, and the parts of the rope make a right angle at the point of suspension. Find the tension in each part of the rope, to the nearest pound.

19. In a west wind, an airplane headed north actually flies at 370 knots in a direction 4° from north. Find the wind's speed, to the nearest knot.

20. A ferry heads south at 12 miles an hour, but the river flowing east at 5 miles an hour changes its course. Find how fast the ferry moves and the direction of motion.

ALGEBRAIC PRINCIPLES

This section contains two tests which should help you discover how well you understand the principles of algebra.

15–10 A True-False Test

Determine whether or not each statement is true. Unless otherwise stated, the set of all real numbers is the domain of each variable.

1. The factor common to $(x - 1)^2$ and $(x^2 - 1)$ is $(x - 1)$.

2. $\dfrac{x - y}{m} = \dfrac{x}{m} - \dfrac{y}{m}, m \neq 0$

4. $a^{3m} \div a^{3m} = a^m, a \neq 0, a \neq 1$

3. $\dfrac{a}{m - n} = \dfrac{-a}{n - m}, m \neq n$

5. $\dfrac{2x - y}{2m} = \dfrac{x - y}{m}, m \neq 0, y \neq 0$

6. $(a + b)^2 = a^2 + b^2$

7. $2^m \cdot 2^n = (4)^{m+n}$

8. $\dfrac{a^3}{a^3} = 0$

9. $a^2 - 1$ is a polynomial

10. A quadratic equation always has 2 real roots.

11. The graph of $3x - y = 6$ passes through the point $(4, -6)$.

12. $x + y = 2$ and $2x + 2y = 6$ have a common solution.

13. The graphs of $3x = 2y$ and $2x = 3y$ meet at the origin.

14. When each side of a square is doubled, the area is quadrupled.

15. When each side of an equilateral triangle is doubled, its perimeter is 6 times as long as at first.

16. If the radius of a circle is multiplied by 4, the circumference becomes 8 times as long.

17. As the complement of an angle increases, so does its supplement.

18. If n men do a job in $6h$ hours, $2n$ men can do the job in $3h$ hours.

19. If $.3m = .06$, then $m = .002$.

20. If $\dfrac{2\sqrt{3}}{m} = \dfrac{m}{3\sqrt{3}}$, the positive value of m is $3\sqrt{2}$.

21. {rational fractions} is finite.

22. {bacteria in one liter of a culture} is finite at a given instant.

23. If the perimeter of a triangle is 18, and one side is twice another, the only possible integral lengths of the sides are 4, 6, and 8.

24. The difference between the supplement and the complement of any angle is 90 degrees.

25. The sum of the complements of two angles of a triangle equals the third angle.

26. The graph of $x + ay = b$, with $a \neq 0$, has a slope equal to a.

27. Several lines with equal slope may be parallel or coincident.

28. Straight lines whose slope equals 2 are represented by $y = 2x + b$.

29. If a train does 45 m.p.h. from A to B, but only 30 m.p.h. from B to A, its average rate for the round trip is 37.5 m.p.h.

30. In triangle ABC, if $\sin A = \cos B$, then angle C measures 90°.

31. If $a \neq 0$, the reciprocal of the reciprocal of a is a.

32. A straight-line graph represents direct variation between the variables.

33. The graph of $x^2 - 4x + 3 = y$ has two x-intercepts.

34. If an open sentence has no roots, it is not an equation.

35. If $n \geq 5$ and $n \leq 5$, then $n = 5$.

36. $\{-1, 0, 1\}$ is closed under addition.

37. If a peddler has n articles of the 40 he had yesterday, the replacement set for n is $\{1, 2, 3, \ldots, 40\}$.

38. An irrational number can be expressed as a decimal numeral if you consider enough places.

39. To show that $a \cdot 3b = 3ab$, you must use the associative, commutative, and distributive properties as reasons.

40. If a point moves so that the sum of its distances from two perpendicular axes is always 5, its path is described by $x + y = 5$.

15–11 A Completion Test

Supply the expression that completes each statement correctly.

1. If the perimeter of a rectangle is 36 and one side is 6, the other side is _?_.

2. If the quotient of two numbers is -1, their sum is _?_.

3. The weight w of a wire varies directly as its length s and as the square of its diameter d. If k is the constant, $w =$ _?_.

4. The heat H you get from an electric heater varies inversely as the square of your distance d from the heater. The formula expressing this relationship is $\frac{H_1}{H_2} =$ _?_.

5. If the middle of three consecutive integers is m, the first number is _?_ and the third is _?_.

6. If the abscissa of a point on the graph of $2x - 3y = 7$ is 2, its ordinate is _?_.

7. Cement, sand, and gravel in the ratio of 2 to 3 to 5 are used in a concrete. For 25 pounds of mixture, you need _?_ pounds of cement.

8. If $k = 5 - 3 \cdot 2$, the value of k is _?_.

9. If $-m = 8$, the absolute value of m is _?_.

10. If the area of a square is 28 square inches, each side, to the nearest tenth of an inch, is _?_ inches.

11. If $\frac{(3 - m)}{8} = \frac{(m + 7)}{12}$, then $m =$ _?_.

12. If the sides of a rectangle are $2\sqrt{5} - 3$ and $2\sqrt{5} + 3$, the area is _?_.

13. If Tom averages 82% on three tests, he needs a mark of _?_ on the fourth test to raise his average to 85%.

14. If $(2\sqrt{6})(3\sqrt{8}) = x\sqrt{3}$, then $x =$ _?_.

15. If a man travels 4 hours at 5 m.p.h. and then travels h hours at r m.p.h., his average rate is _?_ m.p.h.

16. If $m = \frac{1}{a - 1}$, then as a increases from 2 on, m _?_.

17. If $m = \frac{4}{4 - a}$, then as a increases from 1 to 3, m _?_.

18. If y varies inversely as x, and $y = 8$ when $x = 6$, then $y =$ _?_ when $x = 12$.

19. Supplies lasting 16 persons 60 days last 20 persons _?_ days.

20. The largest possible diameter, expressed as an integer, of a circular table top which can be carried through a doorway 7 by 4 feet is _?_.

21. If the line $y = ax + b$ passes through the origin, ___?___ equals zero.

22. If the line $ay = 3x - 4$ passes through the point (0, 2), a equals ___?___.

23. If $\frac{4x^2 + y^2}{4x^2} = 2$, the positive value of $\frac{y}{x}$ is ___?___.

24. Four squares in line form a rectangle whose diagonal is 200 centimeters. The diagonal of one of the squares is ___?___ centimeters.

25. If $r \in \{-3, -2, -1, 0, 1, 2, 3\}$ and $2r + 3 \leq 3r + 1$, the solution set of the inequality is ___?___.

26. If $c - d = 7$ and $cd = 5$, then $c^2 + d^2 =$ ___?___.

27. If r, s, and t are directly proportional to 4, 7, and 15, in that order, and $2r + s = 45$, then $t =$ ___?___.

28. If $(h + k)^2 = 20$ and $hk = 1$, the positive value of $h - k$ is ___?___.

29. The distance between the points $(-2, 0)$ and $(4, 8)$ is ___?___ units.

30. The lines $x + 2y = 5$ and $4x + ky = 5$ are parallel when $k =$ ___?___.

31. In solving $n^2 - 5n = 2$ by completing the square, you add ___?___ to each member of the equation.

32. To transform $\frac{x}{3 + x}$ into a fraction whose denominator is $x^2 - 9$, you multiply numerator and denominator by ___?___.

33. The graph of $ax + by = c$ is horizontal when ___?___ $= 0$ and ___?___ $\neq 0$.

34. Expressed as a rational fraction in lowest terms, .636363... = ___?___.

35. The set of ordered pairs $(x, 3x - 4)$ for all real numbers x is a special kind of relation, called a ___?___.

36. In the formula $A = \pi r^2$, if r is doubled, A is multiplied by ___?___.

37. The formula expressing y in terms of x for the corresponding values shown in the chart is ___?___.

x	0	1	2	3
y	-1	1	3	5

38. If in $\triangle ABC$, $m\angle C = 90°$ and $\sin A = .5$, then $AC:BC =$ ___?___.

39. If the set of integers is the replacement set of x, then the solution set of $4 < \frac{x - 1}{2} < 6$ is ___?___.

40. If the square of a positive number is 3, its cube is ___?___.

APPENDIX

TABLE 1

FORMULAS

Circle	$A = \pi r^2$, $C = 2\pi r$	Cube	$V = s^3$
Parallelogram	$A = bh$	Rectangular Box	$V = lwh$
Right Triangle	$A = \frac{1}{2}bh$, $c^2 = a^2 + b^2$	Cylinder	$V = \pi r^2 h$
Square	$A = s^2$	Pyramid	$V = \frac{1}{3}Bh$
Trapezoid	$A = \frac{1}{2}h(b + b')$	Cone	$V = \frac{1}{3}\pi r^2 h$
Triangle	$A = \frac{1}{2}bh$	Sphere	$V = \frac{4}{3}\pi r^3$
Sphere	$A = 4\pi r^2$		

TABLE 2

AMERICAN SYSTEM OF WEIGHTS AND MEASURES

LENGTH

12 inches = 1 foot
3 feet = 1 yard
$5\frac{1}{2}$ yards = 1 rod
5280 feet = 1 land mile
6076 feet = 1 nautical mile

AREA

144 square inches = 1 square foot
9 square feet = 1 square yard
160 square rods = 1 acre
640 acres = 1 square mile

VOLUME

1728 cubic inches = 1 cubic foot
27 cubic feet = 1 cubic yard

WEIGHT

16 ounces = 1 pound
2000 pounds = 1 ton
2240 pounds = 1 long ton

CAPACITY

Dry Measure

2 pints = 1 quart
8 quarts = 1 peck
4 pecks = 1 bushel

Liquid Measure

16 fluid ounces = 1 pint
2 pints = 1 quart
4 quarts = 1 gallon
231 cubic inches = 1 gallon

METRIC SYSTEM OF WEIGHTS AND MEASURES

LENGTH	10 millimeters (mm)	=	1 centimeter (cm).	=	.3937	inch
	100 centimeters	=	1 meter (m)	=	39.37	inches
	1000 meters	=	1 kilometer (km)	=	.6	mile
CAPACITY	1000 milliliters (ml)	=	1 liter (l)	=	1.1	quart
	1000 liters (l)	=	1 kiloliter (kl)	=	264.2	gallons
WEIGHT	1000 milligrams (mg)	=	1 gram (g)	=	.035	ounce
	1000 grams	=	1 kilogram (kg)	=	2.2	pounds

TABLE 3

SQUARES OF INTEGERS FROM 1 TO 100

Number	*Square*	*Number*	*Square*	*Number*	*Square*	*Number*	*Square*
1	1	**26**	676	**51**	2601	**76**	5776
2	4	**27**	729	**52**	2704	**77**	5929
3	9	**28**	784	**53**	2809	**78**	6084
4	16	**29**	841	**54**	2916	**79**	6241
5	25	**30**	900	**55**	3025	**80**	6400
6	36	**31**	961	**56**	3136	**81**	6561
7	49	**32**	1024	**57**	3249	**82**	6724
8	64	**33**	1089	**58**	3364	**83**	6889
9	81	**34**	1156	**59**	3481	**84**	7056
10	100	**35**	1225	**60**	3600	**85**	7225
11	121	**36**	1296	**61**	3721	**86**	7396
12	144	**37**	1369	**62**	3844	**87**	7569
13	169	**38**	1444	**63**	3969	**88**	7744
14	196	**39**	1521	**64**	4096	**89**	7921
15	225	**40**	1600	**65**	4225	**90**	8100
16	256	**41**	1681	**66**	4356	**91**	8281
17	289	**42**	1764	**67**	4489	**92**	8464
18	324	**43**	1849	**68**	4624	**93**	8649
19	361	**44**	1936	**69**	4761	**94**	8836
20	400	**45**	2025	**70**	4900	**95**	9025
21	441	**46**	2116	**71**	5041	**96**	9216
22	484	**47**	2209	**72**	5184	**97**	9409
23	529	**48**	2304	**73**	5329	**98**	9604
24	576	**49**	2401	**74**	5476	**99**	9801
25	625	**50**	2500	**75**	5625	**100**	10,000

TABLE 4

SQUARE ROOTS OF INTEGERS FROM 1 TO 100

Number	Square Root	Number	Square Root	Number	Square Root	Number	Square Root
1	1.000	**26**	5.099	**51**	7.141	**76**	8.718
2	1.414	**27**	5.196	**52**	7.211	**77**	8.775
3	1.732	**28**	5.292	**53**	7.280	**78**	8.832
4	2.000	**29**	5.385	**54**	7.348	**79**	8.888
5	2.236	**30**	5.477	**55**	7.416	**80**	8.944
6	2.449	**31**	5.568	**56**	7.483	**81**	9.000
7	2.646	**32**	5.657	**57**	7.550	**82**	9.055
8	2.828	**33**	5.745	**58**	7.616	**83**	9.110
9	3.000	**34**	5.831	**59**	7.681	**84**	9.165
10	3.162	**35**	5.916	**60**	7.746	**85**	9.220
11	3.317	**36**	6.000	**61**	7.810	**86**	9.274
12	3.464	**37**	6.083	**62**	7.874	**87**	9.327
13	3.606	**38**	6.164	**63**	7.937	**88**	9.381
14	3.742	**39**	6.245	**64**	8.000	**89**	9.434
15	3.873	**40**	6.325	**65**	8.062	**90**	9.487
16	4.000	**41**	6.403	**66**	8.124	**91**	9.539
17	4.123	**42**	6.481	**67**	8.185	**92**	9.592
18	4.243	**43**	6.557	**68**	8.246	**93**	9.644
19	4.359	**44**	6.633	**69**	8.307	**94**	9.695
20	4.472	**45**	6.708	**70**	8.367	**95**	9.747
21	4.583	**46**	6.782	**71**	8.426	**96**	9.798
22	4.690	**47**	6.856	**72**	8.485	**97**	9.849
23	4.796	**48**	6.928	**73**	8.544	**98**	9.899
24	4.899	**49**	7.000	**74**	8.602	**99**	9.950
25	5.000	**50**	7.071	**75**	8.660	**100**	10.000

SINES, COSINES, AND TANGENTS OF ANGLES FROM 1 TO 90 DEGREES

• TABLE 5

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175	**46°**	.7193	.6947	1.0355
2°	.0349	.9994	.0349	**47°**	.7314	.6820	1.0724
3°	.0523	.9986	.0524	**48°**	.7431	.6691	1.1106
4°	.0698	.9976	.0699	**49°**	.7547	.6561	1.1504
5°	.0872	.9962	.0875	**50°**	.7660	.6428	1.1918
6°	.1045	.9945	.1051	**51°**	.7771	.6293	1.2349
7°	.1219	.9925	.1228	**52°**	.7880	.6157	1.2799
8°	.1392	.9903	.1405	**53°**	.7986	.6018	1.3270
9°	.1564	.9877	.1584	**54°**	.8090	.5878	1.3764
10°	.1736	.9848	.1763	**55°**	.8192	.5736	1.4281
11°	.1908	.9816	.1944	**56°**	.8290	.5592	1.4826
12°	.2079	.9781	.2126	**57°**	.8387	.5446	1.5399
13°	.2250	.9744	.2309	**58°**	.8480	.5299	1.6003
14°	.2419	.9703	.2493	**59°**	.8572	.5150	1.6643
15°	.2588	.9659	.2679	**60°**	.8660	.5000	1.7321
16°	.2756	.9613	.2867	**61°**	.8746	.4848	1.8040
17°	.2924	.9563	.3057	**62°**	.8829	.4695	1.8807
18°	.3090	.9511	.3249	**63°**	.8910	.4540	1.9626
19°	.3256	.9455	.3443	**64°**	.8988	.4384	2.0503
20°	.3420	.9397	.3640	**65°**	.9063	.4226	2.1445
21°	.3584	.9336	.3839	**66°**	.9135	.4067	2.2460
22°	.3746	.9272	.4040	**67°**	.9205	.3907	2.3559
23°	.3907	.9205	.4245	**68°**	.9272	.3746	2.4751
24°	.4067	.9135	.4452	**69°**	.9336	.3584	2.6051
25°	.4226	.9063	.4663	**70°**	.9397	.3420	2.7475
26°	.4384	.8988	.4877	**71°**	.9455	.3256	2.9042
27°	.4540	.8910	.5095	**72°**	.9511	.3090	3.0777
28°	.4695	.8829	.5317	**73°**	.9563	.2924	3.2709
29°	.4848	.8746	.5543	**74°**	.9613	.2756	3.4874
30°	.5000	.8660	.5774	**75°**	.9659	.2588	3.7321
31°	.5150	.8572	.6009	**76°**	.9703	.2419	4.0108
32°	.5299	.8480	.6249	**77°**	.9744	.2250	4.3315
33°	.5446	.8387	.6494	**78°**	.9781	.2079	4.7046
34°	.5592	.8290	.6745	**79°**	.9816	.1908	5.1446
35°	.5736	.8192	.7002	**80°**	.9848	.1736	5.6713
36°	.5878	.8090	.7265	**81°**	.9877	.1564	6.3138
37°	.6018	.7986	.7536	**82°**	.9903	.1392	7.1154
38°	.6157	.7880	.7813	**83°**	.9925	.1219	8.1443
39°	.6293	.7771	.8098	**84°**	.9945	.1045	9.5144
40°	.6428	.7660	.8391	**85°**	.9962	.0872	11.4301
41°	.6561	.7547	.8693	**86°**	.9976	.0698	14.3007
42°	.6691	.7431	.9004	**87°**	.9986	.0523	19.0811
43°	.6820	.7314	.9325	**88°**	.9994	.0349	28.6363
44°	.6947	.7193	.9657	**89°**	.9998	.0175	57.2900
45°	.7071	.7071	1.0000	**90°**	1.0000	.0000	

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LIST OF SYMBOLS

		PAGE
$\|a\|$	absolute value of *a*	123
$\rightarrow$	because *or* meaning	120
$\wedge$	conjunction	105
$\ldots$	continue unendingly	13
	or and so on through	401
$\vee$	disjunction	105
$\in$	is an element of	11
$\notin$	is not an element of	11
$\neq$	does not equal	7
$\doteq$	equals approximately	402
$>$	is greater than	7
$\geq$	is greater than or equal to	45
i	$\sqrt{-1}$	532

		PAGE
$\cap$	intersection	30
$<$	is less than	7
$\leq$	is less than or equal to	45
$\overline{AB}$	line segment *AB*	496
$m\angle$	measure of an angle	501
$^{-}2$	negative 2	112
$\sqrt[n]{}$	*n*th root	403
$^{+}2$	positive 2	112
a'	*a*-prime	518
$\{\ \}$	set	11
$\cup$	union	60
$\overrightarrow{AB}$	vector *AB*	522

Answers for
Modern Algebra

STRUCTURE AND METHOD

·BOOK ONE

1965 Impression

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ANSWERS

Chapter 1. Symbols and Sets

Pages 3–5 Written Exercises A 1. R: $\frac{3}{4}$; S: $1\frac{1}{2}$; T: $1\frac{3}{4}$ **2.** R: $1\frac{2}{3}$; S: 2; T: $2\frac{1}{3}$ **3.** R: $\frac{2}{5}$; S: $\frac{4}{5}$; T: $1\frac{2}{5}$ **4.** R: $\frac{1}{2}$; S: $\frac{5}{6}$; T: $1\frac{1}{3}$ **5.** R: $\frac{7}{8}$; S: $1\frac{1}{4}$; T: $1\frac{5}{8}$ **6.** R: $\frac{1}{5}$; S: $\frac{7}{10}$; T: $1\frac{3}{5}$ **7.** 5 **8.** 0 **9.** 2 **10.** 5 **11.** 4 **12.** $\frac{7}{2}$ **13.** $\frac{7}{2}$ **14.** $\frac{4}{3}$ **15.** 8 **16.** 0 **17.** $\frac{5}{2}$ **18.** $\frac{8}{3}$ **B 19.** $\frac{35}{6}$ **20.** $\frac{14}{5}$ **21.** 5 **22.** 2 **23.** 3, 4 **24.** 7, 8 **25.** 3, 4, 5, 6, 7 **26.** 0 **C 27.** 3, 6 **28.** None **29.** $\frac{7}{2}$ **30.** $\frac{4}{3}$

Pages 6–7 Written Exercises A 1. F **2.** F **3.** T **4.** T **5.** F **6.** F **7.** T **8.** T **9.** F **10.** T **B 11.** 2 **12.** 0 **13.** $2\frac{7}{12}$ **14.** $2\frac{1}{3}$ **15.** .2 **16.** .7 **17.** .6 **18.** $\frac{1}{4}$

Pages 9–10 Written Exercises A 1. = **2.** < **3.** = **4.** < **5.** = **6.** = **7.** > **8.** = **9.** > **10.** = **11.** > **12.** = **13.** < **14.** > **15.** > **16.** > **17.** > **18.** > **19.** < **20.** = **21.** < **22.** > **23.** < **24.** = **25.** 1 **26.** Any num. other than 7 **27.** 5 **28.** 0 **29.** 0 **30.** 2 **31.** 7 **32.** 0 **33.** Any num. <5 **34.** Any num. **35.** 5 **36.** Any num. > 4 **37.** Any num. other than 12 **38.** Any num. ≥ 8 and <16 **39.** Any num. > 0 **40.** 8 **B 41.** 1 **42.** 12 **43.** Any num. other than 0 **44.** $\frac{1}{10}$ **45.** Any num. > 1 and <3 **46.** Any num. $>\frac{1}{3}$ and $<\frac{2}{3}$ **47.** Any num. >0 and $<.1$ **48.** Any num. $>.4$ and $<.5$ **49.** Any num. $>\frac{62}{76}$ and $<\frac{82}{76}$ **50.** Any num. $>\frac{200}{109}$ and $<\frac{200}{107}$ **C 51.** Any num. >1.53327 and <1.54327 **52.** Any num. $> \frac{14867}{23864}$ and $< \frac{14857}{23764}$ **53.** Any num. $>\sqrt{6}$ and $<\sqrt{7}$ **54.** Any num.

Pages 14–15 Written Exercises A 1. {8, 9}, finite **2.** {0, 1, 2, 3, 4, 5}, finite **3.** {i, e}, finite **4.** ∅, finite **5.** ∅, finite **6.** {1, 3, 5, 7, 9}, finite **7.** {10, 12, 14, 16, 18}, finite **8.** {Feb.}, finite **9.** {1, 2, 3, . . . , 1775}, finite **10.** {202, 204, 206, . . . , 998}, finite **B 11.** {3, 6, 9, 12, 15}, finite **12.** {25, 50, 75, . . .}, inf. **13.** {1964, 1968, 1972, . . .}, inf. **14.** {40}, finite **15.** {$\frac{1}{2}$}, finite **16.** {1, 3, 5, . . .}, inf. **17.** {2, 4, 6, . . .}, inf. **18.** ∅, finite **C 19.** {1, 3, 5}, finite **20.** {0, 1, 4, 9, 16}, finite **21.** ∅, finite **22.** ∅, finite **23.** f **24.** d **25.** h **26.** j **27.** a **28.** g **29.** b **30.** i **31.** c **32.** k **33.** e **34.** l

Page 19 Written Exercises A 1. {3}, {15}, {10} **2.** {3, 15}, {3, 10}, {10, 15} **3.** {3, 15, 10} **4.** ∅ **5.** {10}; {15}; {10, 15} **6.** ∅ **7.** {10} **8.** {3}, {15}, {3, 15} **9.** {the nos. of arith.} **10.** {the nos. of arith.} **11.** {all fractions} **12.** {all men} **13.** {all cities of Alaska} **14.** {the islands of Hawaii}

Page 22 Written Exercises A 1. 0 **2.** 25 **3.** 510 **4.** 510 **5.** 2 **6.** 8 **7.** 18 **8.** 2 **9.** 360 **10.** 360 **11.** $\frac{40}{3}$ **12.** 1 **B 13.** 15 **14.** $\frac{10}{39}$ **15.** $\frac{1}{6}$ **16.** 2 **17.** 51 **18.** $\frac{1}{3}$ **19.** 56 **20.** $11\frac{5}{6}$ **C 21.** 1000 **22.** 725,500 **23.** $\frac{64}{13}$ **24.** $\frac{100}{3}$

Page 24 Written Exercises A 1. 5 **2.** 3 **3.** 23 **4.** 51 **5.** 9 **6.** 9 **7.** 6 **8.** 15 **9.** 1 **10.** 5 **11.** 9 **12.** 5 **13.** 23 **14.** 23 **15.** 7 **16.** 7 **B 17.** 1 **18.** 1 **19.** 51 **20.** 7 **21.** $\frac{1}{2}$ **22.** 3 **23.** $\frac{1}{2}$ **24.** 90 **25.** 150 **26.** 1 **27.** 591.3 **28.** 73.3 **29.** 10 **C 30.** 30 **31.** $\frac{1}{2}$ **32.** $\frac{397}{28}$ **33.** 5 **34.** 54

Pages 27–28 Chapter Test 1. $\frac{3}{4}$ **2.** $\frac{5}{4}$ **3.** $\frac{3}{2}$ **4.** 4 **5.** $\frac{7}{2}$ **6.** $\frac{8}{3}$ **7.** T **8.** F **9.** T **10.** F **11.** Any num. other than 2 **12.** Any num. >20 **13.** Any num. >2 and <4 **14.** Any num. $>\frac{2700}{245}$ and $<\frac{2700}{235}$ **15.** {6, 12, 18, 24} **16.** {100, 101, 102, . . . , 999} **17.** Inf. **18.** Finite **19.** ∅ **20–22.** no. lines **23.** {4, 8} **24.** 1 **25.** 13 **26.** 1

Pages 28–30 Chapter Review 1. number **2.** zero **3.** magnitude **4.** distance **5.** $\frac{3}{4}$ **6.** $\frac{9}{4}$ **7.** E **8.** numerical, numeral **9.** equal **10.** = **11.** ≠ **12.** greater **13.** $5 < 7$ *or* $7 > 5$ **14.** 1, 9 **15.** 5 **16.** set **17.** member *or* element **18.** ∈ **19.** not **20.** roster **21.** rule **22.** 12, 15, 18 **23.** infinite **24.** empty **25.** finite **26.** empty *or* null **27.** 3 **28.** no. line **29.** subset **30.** {20} **31.** brackets, braces, bar **32.** one **33.** 14 **34.** $\frac{1}{4}$ **35.** inclusion, divisions, multiplications, additions, subtractions **36.** 6 **37.** 27 **38.** 19 **39.** 44.45 **40.** 1

Chapter 2. Variables and Open Sentences

Pages 37–38 Written Exercises A **1.** 2; 12 **2.** 2; 11 **3.** 1; 12 **4.** 1; 1296 **5.** 1; 1296 **6.** 1; 0 **7.** 2; 60 **8.** 5; 2 **9.** 2; 9 **B** **10.** 2; 15 **11.** 1; 0 **12.** 1; 225 **13.** 1; $\frac{5}{7}$ **14.** 1; $\frac{8}{35}$ **15.** 1; 9 **16.** 1; 1 **17.** 1; 11 **18.** 1; 2 **19.** 1; 1

Pages 38–39 Problems A **1.** .375 in **2.** $\frac{5}{6}$ **3.** 68°F **4.** 33.926 ft **B** **5.** 26 sq in **6.** 28 **7.** 25°C **8.** 110 ft/sec **9.** 10 **10.** 1 **11.** 150° **12.** .0138 ft **C** **13.** $553\frac{1}{7}$ sq in **14.** 1575 (sq in)^2 **15.** 214 sq ft **16.** 2090 amp **17.** .750 in **18.** $\frac{15}{41}$ amp **19.** 140 **20.** 34.0125 ft

Pages 42–43 Written Exercises A **1.** b^2 **2.** c^3 **3.** a^3 **4.** d^2 **5.** $7n^3$ **6.** $14m^3$ **7.** n^3 **8.** $8n^3$ **9.** E^4 **10.** F^3 **11.** R^5 **12.** s^6 **13.** $5y^3$ **14.** $8z^2$ **15.** $\frac{1}{2}g^2$ **16.** $\frac{1}{4}h^5$ **17.** $(2P)^2$ **18.** $(8i)^2$ **19.** $(xy)^3$ **20.** $(ab)^3$ **21.** $(a - 1)^3$ **22.** $(1 - a)^3$ **23.** $(r + 2)^3$ **24.** $(t + 7)^2$ **B** **25.** $\frac{1}{4}$ **26.** $\frac{1}{16}$ **27.** 36 **28.** 200 **29.** 9 **30.** 4 **31.** 35 **32.** 6 **33.** 58 **34.** 84 **35.** 0 **36.** 9 **C** **37.** 38 **38.** 10 **39.** 36 **40.** 12 **41.** 11 **42.** 18 **43.** 30 **44.** 1 **45.** 0 **46.** $\frac{17}{2}$ **47.** $\frac{17}{5}$ **48.** 3439 **49.** 139 **50.** 0 **51.** $\frac{1}{81}$

Pages 43–44 Problems A **1.** 225 sq cm **2.** 3375 cu cm **3.** 22,860,000,000,000,000,000 g(m/sec)^2 **4.** 2.25 amp^{2}ohm **5.** 4281.25 ft **B** **6.** 57,876.48 cu in **7.** 1,065,312,000°K **8.** 125,000 g(cm/sec)^2 **9.** 58.875 cu in **10.** $\frac{3}{2500}$ candle power/sq ft **11.** 600 ft lb/sec^2 **C** **12.** 3.32 ft/sq mil **13.** .006003 (g/cm)^2 **14.** 285,600 amp^2(ohm)(sec) **15.** $\frac{2.0125}{\pi^2}$ ft

Pages 47–49 Written Exercises A **1.** $\{4\}$ **2.** $\{4, 6, 8\}$ **3.** $\{0\}$ **4.** $\{2, 6, 8\}$ **5.** $\{2, 4, 6\}$ **6.** $\{2\}$ **7.** $\{8, 9\}$ **8.** $\{9, 10\}$ **9.** a. $x \in \{10\}, y \in \{7\}$ b. $x \in \{2\}, y \in \{0\}$ **10.** a. $\emptyset$ b. $\emptyset$ **11.** $\{3\}$ **12.** $\{1\}$ **13.** $\{11\}$ **14.** $\{3\}$ **15.** $\{0\}$ **16.** $\{0\}$ **17–22.** no. lines **23.** $\emptyset$ **24.** $\emptyset$ **25–28.** no. lines **29.** $\{8\}$ **30.** $\{5\}$ **31–34.** no. lines **35.** $\{7\}$ **36.** $\{1\}$ **37–40.** no. lines **B** **41.** True for all values given **42.** True for all values given **43.** False for all values given **44.** True only for $d = 4, e = 0$; $d = 4, e = 1$; $d = 7, e = 0$; $d = 7, e = 1$ **45.** True for all values given **46.** True only for $x = 0, y = 3, z = 5$; $x = 0, y = 3, z = 3$ **47.** True only for $t = 1, v = 1$ **48.** False for all values given **49.** True only for $m = 3, n = 1$; $m = 9, n = 1$ **50.** True only for $y = 1$ **C** **51.** True only for $t = 4, s = 4$; $t = 8, s = 4$ **52.** True only for $h = 4, d = 6$; $h = 4, d = 8$; $h = 8, d = 8$ **53.** a. F b. F **54.** a. T b. T **55.** a. T b. F **56.** a. F b. T

Page 54 Written Exercises A **1.** $x + 5$ **2.** $d + c$ **3.** $2x + 4$ **4.** $5y + 2x$ **5.** $15 - n$ **6.** $3x - 2$ **7.** $z - 1$ **8.** $x - d$ **9.** $2x - 5$ **10.** $5k - 5$ **11.** $(3x)y$ **12.** $6a$ **13.** $\frac{r + 6}{3}$ **14.** $\frac{1}{3}(r - 6)$ *or* $\frac{r - 6}{3}$ **15.** $\frac{1}{2}(x + y)$ *or* $\frac{x + y}{2}$ **16.** $\frac{a - b}{c}$ **17.** $5(2 + y)$ **18.** $a(a - b)$ **19.** $x(x + 2) + 5$ **20.** $2u - \frac{v}{2}$ **B** **21.** $\frac{5}{4x + 3k}$ **22.** $(a + b) - ab$ **23.** $\frac{1}{2(a + b)}$ **24.** $\frac{mn}{3(m - n)}$ **25.** $\frac{3(r - s)}{2(r + s)}$

Pages 54–55 Problems A **1.** $x, 2x$ **2.** $l, 3l$ **3.** $90w$ **4.** $5i$ **5.** $3w$ **6.** $3n$ **7.** $m, 3m$ **8.** $10s$ **9.** $400 - q$ **10.** $h + 10{,}000$ **11.** (a) w; (b) $2w + 6$ **12.** $3n - 17$ **B** **13.** (a) s; (b) $2s - 48$; (c) $(2s - 48) + s + s$ **14.** $57 - w$ **15.** $s + \frac{4}{5}s = 108$ **16.** $(3c + 7c) + 5$ **17.** $5c + 2(c + 28)$ **18.** (a) $2x$; (b) $2x + 3$

Pages 58–60 Problems A **1.** 28 **2.** 34 **3.** 10 **4.** 4 **5.** \$2600 **6.** \$450 **7.** 56 yr **8.** 30,585 **9.** $12\frac{1}{2}$ in **10.** 17 ft **11.** \$1287.45 **12.** \$688.14 **13.** 98 lb **14.** 59.7° **15.** \$107.50 **16.** 92 **B** **17.** 50 mi **18.** 120 **19.** 14 **20.** 86 **21.** 29 **22.** 1350 **23.** 180 **24.** 111 **C** **25.** 14 ft, 24 ft **26.** 40 mi **27.** 14 **28.** 110 lb **29.** 17 **30.** 62 ft, 25 ft, 43 ft

Page 64 Chapter Test **1.** 24 **2.** 5 **3.** 17 **4.** $\frac{80}{9}$ **5.** 54 **6.** 36 **7.** $\{1, a\}$ **8.** $\{1, 2, 3, 4, 6, 12, m, n\}$ **9.** $\{\frac{1}{3}, p, q\}$ **10.** $8q$ **11.** $\frac{1}{5}$ **12.** Num. coeff: 3; base: $(x - 1)$; exp: 2 **13.** $h \cdot h \cdot h$ *or* h^3 **14.** 6 **15.** 9, 12 **16.** F **17.** F **18.** $\{1\}$ **19.** $\{6, 7, \ldots, 10\}$ **20–21.** no. lines **24.** $52s - 12e$ **25.** 20 ft

Pages 65–67 Chapter Review **1.** symbol **2.** replacement, domain **3.** 99 **4.** product, quotient **5.** coefficient **6.** n **7.** s, s **8.** exponent **9.** c^5 **10.** 3, a, 2 **11.** 1 **12.** 3375; 45 **13.** n **14.** domain *or* replacement set **15.** left **16.** variable **17.** inequality **18.** True only for $y = 4$ **19.** True for all values given **20.** True only for $y = 0$; $y = 1$; $y = 2$; $y = 4$ **21.** False for all values given **22.** solution **23.** root **24.** root **25.** (b) **26.** $x > 2$ **27.** Three times a no. x; $x \geq 0$ **28.** Three greater than the quotient of a no. x and 3; $x \geq 0$ **29.** The product of a no. x diminished by 1 and 3; $x \geq 1$ **30.** $a - 5$ **31.** $2b + 3$ **32.** $3c - 3$ **33.** $\frac{d + e}{3}$ **34.** $3i$ **35.** $100 - 5n, n \in$

$\{1, 2, 3, \ldots, 20\}$ **36.** symbol *or* variable **37.** check **38.** $18n = 198$; $\{11\}$ **39.** $3p = 1440$; $\{480\}$ **40.** $300 \leq 25a \leq 500$ *or* $12 \leq a \leq 20$

Chapter 3. Axioms, Equations, and Problem Solving

Pages 72–73 Written Exercises 1. Closed for $+$, $-$, $\times$; not closed for $\div$: $0 \div 0$ undefined **2.** Closed for $\div$, $\times$; not closed for $+$, $-$: $1 + 1$ and $1 - 1$ not in set **3.** Not closed for $+$, $-$, $\div$, $\times$: $2 + 2$, $2 - 2$, $2 \div 2$, 2×2 not in set **4.** Not closed for $+$, $-$, $\div$, $\times$: $3 + 3$, $3 - 3$, $3 \div 3$, 3×3 not in set **5.** Not closed for $+$, $-$, $\div$, $\times$: $2 + 3$, $1 - 3$, $3 \div 2$, 2×3 not in set **6.** Closed for $\times$; not closed for $+$, $-$, $\div$: $1 + 1$, $0 - 1$ not in set, $1 \div 0$ undefined **7.** Not closed for $+$, $-$, $\div$, $\times$: $1 + 2$, $1 - 2$, $1 \div 2$, 2×2 not in set **8.** Closed for $+$, $\times$; not closed for $-$, $\div$: $15 - 20$, $25 \div 10$ not in set **9.** Not closed for $+$, $-$, $\div$, $\times$: $1 + 1$, $\frac{3}{2} - \frac{9}{5}$, $\frac{1}{2} \div \frac{1}{8}$, $\frac{3}{2} \times \frac{3}{2}$ not in set **10.** Closed for $\times$; not closed for $+$, $-$, $\div$: $\frac{1}{2} + \frac{1}{4}$, $\frac{1}{4} - \frac{1}{16}$, $\frac{1}{2} \div \frac{1}{4}$ not in set **11.** Closed for $+$, $\div$, $\times$; not closed for $-$: $2 - 2$ not in set **12.** Not closed for $+$, $-$, $\div$, $\times$: $\frac{1}{4} + \frac{3}{4}$, $\frac{7}{3} - \frac{4}{3}$, $\frac{1}{8} \div \frac{1}{16}$, $\frac{2}{3} \times \frac{3}{2}$ not in set

Page 79 Written Exercises A 1. $56x$ **2.** $85y$ **3.** $65a$ **4.** $23b$ **5.** $30a$ **6.** $7x$ **7.** $12x + 6$ **8.** $15n + 3$ **9.** $3s + 4r$ **10.** $4x + 2y$ **11.** $2a + 3b + 4c - d$ **12.** $6u - 5v - 3w - 4$ **13.** $10n + 21$ **14.** $26z + 10$ **15.** $7b$ **16.** $4x + 30$ **17.** $16a + 4b + 1$ **18.** $s + 5t + 2$ **19.** $6a + 11ab$ **20.** $12x + 5xy$ **B 21.** $10a + 10b$ **22.** $23m + 16n$ **23.** $36 + 12a$ **24.** $5a + 4$ **25.** $10d + 17$ **26.** $62e + 6$ **27.** $48w + 4v$ **28.** $26r + 40t$ **29.** $5a + 6b + 1$ **30.** $4r + 19s + 1$ **C 31.** $9a + 28$ **32.** $38b + 7$ **33.** $16r + 30s + 6$ **34.** $20m + 18n + 55$ **35.** 1850 **36.** 1900 **37.** 15 **38.** 7790 **39.** 0 **40.** 0 **41.** 75 **42.** 570 **43.** 326,326 **44.** 141 **45.** 290 **46.** 0

Page 83 Written Exercises A 1. {49} **2.** {28} **3.** {78} **4.** {0} **5.** {0} **6.** {0} **7.** {94} **8.** {110} **9.** {93} **10.** {149} **11.** {17} **12.** {51} **13.** {4.6} **14.** {.4} **15.** {9.98} **16.** {4.47} **17.** {3} **18.** {2} **19.** $\{2\frac{1}{3}\}$ **20.** $\{1\frac{3}{5}\}$ **21.** {755} **22.** {.96} **23.** {1} **24.** $\{5\frac{3}{5}\}$ **25.** $\{3\frac{2}{3}\}$ **26.** {6.45} **27.** $\{2\frac{3}{8}\}$ **28.** {4} **29.** {76.5} **30.** {0}

Pages 84–85 Written Exercises A 1. {114} **2.** {216} **3.** {11} **4.** {13} **5.** {2} **6.** {1} **7.** {18} **8.** {3.42} **9.** {20} **10.** {9} **11.** $\{1\frac{1}{3}\}$ **12.** $\{1\frac{1}{4}\}$ **13.** {4} **14.** {2.4} **15.** $\{\frac{1}{9}\}$ **16.** $\{\frac{1}{16}\}$ **17.** $\{13\frac{1}{3}\}$ **18.** {100} **B 19.** {0} **20.** {0} **21.** {.15} **22.** {.006} **23.** $\{2\frac{38}{45}\}$ **24.** $\{2\frac{3}{32}\}$ **25.** {34} **26.** {11.25} **27.** {9} **28.** {16} **29.** {7} **30.** {14} **C 31.** {6} **32.** {30} **33.** {4} **34.** {5} **35.** {6.75} **36.** {3.03} **37.** a. Reflex. prop. b. Given c. Subst. princ. **38–39.** Same as **37** **40.** a. Reflex prop. b. Given c. Subst. princ. d. Given e. Subst. princ. **41–42.** Same as **40**

Pages 88–89 Written Exercises A 1. {3} **2.** {14} **3.** {1} **4.** {1} **5.** {20} **6.** {25} **7.** $\{\frac{1}{2}\}$ **8.** $\{2\frac{14}{17}\}$ **9.** {0} **10.** {0} **11.** {4} **12.** {2} **13.** {6} **14.** {3} **15.** {3} **16.** $\{3\frac{1}{6}\}$ **17.** $\{2\frac{1}{2}\}$ **18.** {10} **19.** {19} **20.** {33} **21.** {9} **22.** {9} **23.** {2} **24.** {20} **25.** {6.8} **26.** {8.7} **27.** $\{\frac{1}{2}\}$ **28.** $\{2\frac{2}{3}\}$ **B 29.** {11} **30.** {40} **31.** $\{1\frac{1}{2}\}$ **32.** $\{1\frac{3}{5}\}$ **33.** {5} **34.** {4} **35.** {0} **36.** {1} **37.** {39.6} **38.** {16} **39.** {2} **40.** {2} **C 41.** {6} **42.** {24} **43.** {28} **44.** {0} **45.** {1} **46.** $\{\frac{3}{4}\}$ **47.** {the nos. of arith.} **48.** {the nos. of arith.} **49.** $\emptyset$ **50.** $\emptyset$

Pages 89–91 Problems A 1. 35 **2.** 27 **3.** 12 **4.** 63 **5.** 36 ft **6.** 20 ft **7.** 89¢ **8.** Pat, 92; Mary, 65 **9.** 82 **10.** 26 **11.** 5.3 **12.** 11.2 **13.** 38 **14.** 36 **15.** 23 **16.** 83 **17.** Emma, 11 yr; Bob, 22 yr; Kent, 27 yr **18.** Width, 37 ft; length, 74 ft **19.** Width, 24 ft; length, 72 ft **20.** Width, 9 ft; length, 27 ft **21.** 112 g **22.** 70¢, 35¢ **23.** Skirt, \$6; jacket, \$9 **24.** (a) 2 lb; (b) 1.6 lb **25.** (a) 165; (b) 330 **26.** 480 lb; 240 lb; 120 lb **B 27.** A, 2000; B, 4000; C, 16,000 **28.** 37.5 kg **29.** 60 in; 30 in; 59 in **30.** A, 1845; B, 1230; C, 1945 **31.** 10 nickels; 40 dimes; 20 pennies **32.** 20

Pages 93–94 Written Exercises A 1. {9} **2.** {13} **3.** {2} **4.** {4} **5.** $\{1\frac{1}{3}\}$ **6.** {4} **7.** {2} **8.** {5} **9.** {4} **10.** {1} **11.** {6} **12.** {3} **13.** $\{1\frac{4}{5}\}$ **14.** $\{5\frac{3}{5}\}$ **15.** {3} **16.** {1} **17.** $\{3\frac{3}{5}\}$ **18.** {2} **B 19.** {3} **20.** $\{1\frac{1}{4}\}$ **21.** $\{2\frac{3}{4}\}$ **22.** $\{\frac{1}{2}\}$ **23.** {0} **24.** {5} **25.** {1} **26.** {6} **27.** {3} **28.** {1} **29.** $\{1\frac{7}{11}\}$ **30.** $\{\frac{5}{9}\}$ **31.** {4} **32.** {249} **C 33.** {2} **34.** {2} **35.** {9} **36.** {40} **37.** $\emptyset$ **38.** {0} **39.** $\{\frac{1}{3}\}$ **40.** {1} **41.** {7} **42.** {10} **43.** {13} **44.** {the nos. of arith.}

Pages 94–95 Problems A 1. 1 **2.** 5 **3.** 6 **4.** $3\frac{1}{2}$ lb **5.** \$120; \$40; \$120 **6.** 14; 32 **7.** 18 yr; 9 yr **8.** 30; 43 **9.** 16 **10.** Standard, \$.15; deluxe, \$1.15 **11.** $13\frac{1}{2}$ ft; $11\frac{1}{2}$ ft **12.** Width, 17 in; length, 25 in **13.** 45 4¢ stamps; 5 7¢ stamps **14.** 14 in **15.** 44 **16.** 135 **B 17.** 6 **18.** Juneau, 30°; Omaha, 60° **19.** \$2 **20.** Onions, 4¢; potatoes, 8¢; apples, 12¢ **21.** 9 yr; 8 yr; 24 yr

Pages 98–100 Chapter Test **1.** Reflex. **2.** Symmetric **3.** No, 2 × 2 not in set **4.** Yes **5.** (a) Comm. for add.; (b) Assoc. for add.; **6.** (a) Comm. for mult.; (b) Assoc. for mult. **7.** (a) Distrib.; (b) Mult. prop. of 1; (c) Comm. for add.; (d) Distrib.; (e) Distrib. **8.** $2x$ **9.** 390 **10.** {164} **11.** {24} **12.** {4} **13.** {32} **14.** Yes **15.** \$34.71 **16.** {1} **17.** Pencil, \$.99; pen, \$1.79 **18.** {3} **19.** 12 in

Pages 100–102 Chapter Review **1.** reflexive **2.** symmetric **3.** transitive **4.** number of arithmetic **5.** element **6.** addition, multiplication **7.** No; 3 ÷ 6 not in set **8.** No; 4 − 6 not in set **9.** commutative, addition **10.** nm, commutative **11.** commutative **12.** associative **13.** associative **14.** associative **15.** distributive, subtraction **16.** distributive **17.** multiplicative **18.** multiplicative, 0 **19.** 0 **20.** 0, divisor **21.** (a) Distrib.; (b) Subst. princ., comm. for mult.; (c) Mult. prop. of 0; (d) Assoc. for mult.; (e) Subst. princ.; (f) Mult. prop. of 1; (g) Add. identity **22.** 1 **23.** $8k + 9$ **24.** proved **25.** equivalent **26.** member **27.** {6} **28.** {0} **29.** substituting **30.** member **31.** conditions **32.** divide, 5 **33.** addition; multiplication **34.** equivalent **35.** $\{\frac{1}{4}\}$ **36.** {27} **37.** {2} **38.** $\{\frac{1}{3}\}$ **39.** $6n$, $7n$ **40.** 4 **41.** additions, subtractions **42.** {4} **43.** $\{\frac{2}{5}\}$ **44.** 37 yr; 13 yr **45.** one **46.** {1} **47.** $\{\frac{7}{3}\}$ **48.** 16 in; 6 in **49.** 75 at 15¢ each; 125 at 10¢ each

Pages 102–104 Cum. Review **1.** Inf. **2.** Finite **3.** Finite **4.** Inf. **5.** {Kansas, Kentucky} **6.** {nos. between 0 and 48 which are each 1 less than a multiple of 6} **7.** {the no. 1 and the whole nos. between 1 and 82 which are powers of 3} **8–10.** no. lines **11.** 2 **12.** 13.8 **13.** {0, 1, 2, 3, 6, 9, 10, 12, 20} **14.** {3, 9, 12} **15.** {2} **16.** Transitive *or* subst. princ. **17.** Distrib. **18.** Symmetric **19.** Comm. for mult. **20.** (a) Distrib. (b) Comm. for add.; (c) Assoc. for add.; (d) Distrib.; (e) Subst. princ.; (f) Mult. prop. for 0; (g) Additive identity **21.** $4n + 21$ **22.** $d - 4$ **23–24.** no. lines **25.** Closed **26.** Not closed; 3 − 4 not in set **27.** 30 **28.** $3\frac{2}{3}$ **29.** 1 **30.** 2 **31.** $2n - 3$ **32.** $\dfrac{a + b}{5}$ **33.** $s + 5$ **34.** $50 - x$ **35.** $(R + r)^2$ **36.** $2(R^2 + r^2)$ **37.** {3} **38.** ∅ **39.** {1, 2, 3} **40.** {2, 3} **41.** {0, 1} **42.** {0, 1, 2, 3} **43.** a. $t + 5$; b. $t \in \{0, 1, 2, 3, \ldots\}$ **44.** $27 - 3a$ **45.** {16} **46.** $\{7\frac{1}{2}\}$ **47.** {8} **48.** {1} **49.** {4} **50.** ∅ **51.** 51 **52.** 11, 9 **53.** \$1.49

Chapter 4. The Negative Numbers

Page 113 Written Exercises **1–8.** no. line **9.** $^{+}10$ **10.** $^{+}5$ **11.** $^{-}3$ **12.** $^{-}5$

Pages 114–115 Written Exercises **A** **1.** $\{^{-}4, ^{-}6\}$ **2.** $\{^{-}4, ^{-}1, 0, ^{+}2, ^{+}4\}$ **3.** $\{^{-}1, 0, ^{+}2, ^{+}4\}$ **4.** $\{^{-}4, ^{-}6\}$ **5.** $\{^{-}6, ^{-}4, ^{-}1, ^{+}2, ^{+}4\}$ **6.** $\{0, ^{+}2\}$ **7.** $\{^{-}6, ^{-}4\}$ **8.** $\{^{+}2, ^{+}4\}$ **9.** $\{^{-}6, ^{-}4, ^{-}1, ^{+}2, ^{+}4\}$ **10.** ∅ **11.** $\{0, ^{+}2, ^{+}4\}$ **12.** $\{^{-}6, ^{-}4, ^{-}1\}$ **B** **13–35.** no. lines

Pages 118–119 Written Exercises **A** **1.** $^{+}80$ **2.** $^{+}41$ **3.** $^{-}89$ **4.** $^{-}120$ **5.** $^{+}1°$ **6.** $^{-}10°$ **7.** 0 **8.** 0 **9.** $^{-}23$ **10.** $^{-}50$ **11.** 0 **12.** 0 **13.** $^{-}7$ **14.** $^{+}5$ **15–28.** Same answers as **1–14.** **B** **29.** $\{^{+}8\}$ **30.** $\{^{+}3\}$ **31.** $\{^{-}3\}$ **32.** $\{^{-}5\}$ **33.** $\{^{-}13\}$ **34.** $\{^{+}14\}$ **35.** $\{^{-}9\}$ **36.** $\{^{+}18\}$ **37.** $\{^{-}109\}$ **38.** $\{^{-}4\}$ **39.** {0} **40.** {0} **41.** $\{^{+}6\}$ **42.** $\{^{+}5\}$ **43.** $\{^{+}5\}$ **44.** $\{^{+}1\}$ **45.** {0} **46.** $\{^{+}3\}$

Pages 119–120 Problems **1.** $^{+}245$ mph **2.** $^{+}6000$ ft **3.** (a) $^{-}7$, $^{-}3$, $^{-}5$; (b) $^{+}4$ fathoms, $^{-}2$ fathoms **4.** $^{-}39{,}000$ ft **5.** $^{-}7\frac{1}{2}°$ **6.** $^{-}\$3.58$ **7.** $^{-}15$ *or* 15 floors down **8.** $^{+}\$3\frac{7}{8}$ **9.** $^{+}\$2\frac{3}{8}$ **10.** $^{-}\$\frac{3}{4}$ **11.** $^{-}\$1\frac{1}{8}$ **12.** $^{-}89$ M

Pages 122–123 Written Exercises **A** **1.** −2 **2.** 9 **3.** −9 **4.** −5 **5.** −5.7 **6.** 4.1 **7.** 4 **8.** 2 **9.** −4.4 **10.** −6.4 **11.** $5\frac{3}{8}$ **12.** $-\frac{9}{10}$ **B** **13.** {−2} **14.** {3} **15.** {0} **16.** {−1} **17.** {2} **18.** {−3} **19.** {3} **20.** {−2} **21.** {0, 1, 2, 3} **22.** {−3, −2} **23.** {2, 3} **24.** {1, 2, 3} **25.** {−2, −1} **26.** ∅ **27.** {−2, −1, 0, 1, 2} **28.** $\frac{103}{60}$ **29.** $-\frac{103}{60}$ **30.** $\frac{21}{20}$ **31.** $\frac{21}{20}$ **32.** $\frac{7}{60}$ **33.** $\frac{17}{60}$ **34.** $\frac{35}{12}$ **35.** $\frac{5}{12}$ **36.** $\dfrac{-89}{30}$ **37.** $-\frac{89}{30}$ **38.** $\frac{7}{60}$ **39.** $\frac{17}{60}$

Page 124 Written Exercises **A** **1.** {0} **2.** {−4, 4} **3.** {−5, 5} **4.** ∅ **5–8.** no. lines **9.** ∅ **10.** {0} **11.** ∅ **12.** ∅ **13.** {the nonneg. nos.} **14.** {nos. ≤0} **15.** {all nos. other than 0} **16.** {the directed nos.} **B** **17.** 23.8 **18.** −4.4 **19.** −16.2 **20.** 27.2 **21.** 202.4 **22.** 299.2 **23.** 11.2 **24.** 24.9 **25.** 23.8

Page 127 Written Exercises **A** **1.** 4 **2.** 9 **3.** $54\frac{5}{12}$ **4.** $-69\frac{1}{3}$ **5.** −9 **6.** 32 **7.** 1 **8.** −2 **9.** −36 **10.** 0 **11.** 0 **12.** 5 **13.** −6 **14.** 35 **15.** $3\frac{5}{6}$ **16.** −2 **17.** 2 **18.** −11.5 **B** **19.** Comm. prop. for add.; assoc. prop. for add.; prop. of opposites; prop. of 0 **20.** Comm. prop. for add.; assoc. prop. for add.; prop. of opposites; prop. of 0

21. Comm. prop. for add.; assoc. prop. for add. **22.** Assoc. prop. for add.; assoc. prop. for add.; prop. of opposites; prop. of 0; prop. of opposites **23.** Comm. prop. for add.; assoc. prop. for add. **24.** Comm. prop. for add.; assoc. prop. for add.; prop. of opposites; prop. of 0 **25.** Assoc. prop. for add.; prop. of opposites; prop. of 0; prop. of opposites **26.** Prop. of opposite of sum; prop. of opposites **27.** −2 **28.** −3 **29.** −5 **30.** 1 **31.** −26 **32.** −1 **33.** −11 **34.** 0 **35.** $\frac{1}{3}$

Page 128 Problems **1.** \$34.30 **2.** \$8.75 **3.** −\$19.32 **4.** \$264 **5.** 275 ft **6.** −1 yd **7.** {−5, −4, −3, −1, 0, 1, 2, 3, 4}; not closed under add. **8.** −5 lb **9.** −27.5°F

Pages 131–132 Written Exercises **A** **1.** 59 **2.** 3 **3.** 0 **4.** −100 **5.** −3 **6.** −4 **7** −8 **8.** 23 **9.** 1 **10.** −.2 **11.** .9 **12.** −5 **13.** $\frac{3}{4}$ **14.** $-\frac{2}{3}$ **15.** 686 **16.** −154 **17.** y **18.** $-v$ **19.** 16 **20.** −22 **B** **21.** −9 **22.** −25 **23.** −4 **24.** −8 **25.** −2 **26.** $-16\pi + 10$ **27.** −24 **28.** −36 **29.** 65 **30.** 1 **31.** Meaning of subt.; prop. of opposite of sum **32.** Prop. of opposite of sum; prop. of opposite of sum **33.** Not closed **34.** Closed **35.** Not closed **36.** Not closed

Page 132 Problems **A** **1.** 32 ft **2.** −8° **3.** −13° **4.** −274 ft **5.** 27 yr **6.** 75 yr **7.** \$5.63 **8.** 425 **9.** (current year + 146) yr **10.** −66,812 ft **11.** \$326.72 **12.** 64°

Pages 137–138 Written Exercises **A** **1.** −84 **2.** 21 **3.** −4 **4.** −45 **5.** 0 **6.** 0 **7.** −40 **8.** −20 **9.** −1200 **10.** 0 **11.** 0 **12.** −52 **13.** −72 **14.** 148 **15.** −5 **16.** 54 **17.** −1.38 **18.** 7.63 **19.** −.1 **20.** −.6 **21.** $-3a + 5$ **22.** $-r + 2$ **23.** $-7x - 7$ **24.** $-15y + 22$ **25.** $-2nt + 2$ **26.** $-3hk - 3$ **27.** $-3r + 6s$ **28.** $15a - 15b$ **29.** $-2n - 3$ **30.** $\frac{2}{3}y - \frac{1}{2}$ **31.** $2a$ **32.** $4u - 1$ **33.** $-4k^3$ **34.** $-4xyz$ **35.** $1.2d^2$ **36.** $-9r - 9s$ **37.** $-2x^2 + 2x$ **38.** $-7y - 2y^2$ **39.** $2m - 3$ **40.** $u^3 + u^2$ **B** **41.** $-7r - 11s$ **42.** $2a + 12b$ **43.** $5p - 11q$ **44.** $31v - 3t$ **45.** $6a$ **46.** $-16 + 6a$ **47.** $-27n + 5$ **48.** $26h - 20$ **49.** −.73 **50.** −.56 **51.** −1.4 **52.** 0 **53.** .9 **54.** −10.4 **55.** −64 **56.** −1.34892 **57.** .15 **C** **58.** 195.112 **59.** −23.12 **60.** −.18 **61.** 25.92 **62.** 4.356 **63.** 81.801819

Pages 141–142 Written Exercises **A** **1.** $-\frac{1}{7}$ **2.** .05 **3.** 36 **4.** $-\frac{1}{9}$ **5.** −.06 **6.** 64 **7.** −72 **8.** −4.9 **9.** $-9b$ **10.** −32 **11.** −.36 **12.** $-49c$ **13.** −1.3 **14.** −.14 **15.** .015 **16.** −2.3 **17.** −.19 **18.** .022 **B** **19.** −10 **20.** −10 **21.** −4 **22.** 3 **23.** 75 **24.** −4 **25.** $\frac{1}{6}$ **26.** $-\frac{1}{60}$ **27.** $-\frac{9}{8}$ **28.** $-\frac{4}{45}$ **29.** 0 **30.** 0 **C** **31.** 0 **32.** 0 **33.** $-\frac{1}{2}$ **34.** $\frac{3}{8}$ **35.** −89 **36.** 2

Pages 143–144 Problems **1.** \$2023$\frac{1}{3}$ **2.** 8007$\frac{1}{7}$ M **3.** −5860 ft **4.** 17$\frac{3}{70}$ g **5.** $-1\frac{6}{7}°$ **6.** $\frac{7}{8}$ **7.** 4.187 **8.** 1649.4

Pages 145–146 Chapter Test **1.** a. −3; b. 0 **2.** 3 **3.** no. lines **4.** $2.97 \leq d \leq 3.03$; $d \in$ {directed nos. between 2.97 and 3.03 inclusive} **5.** −38 **6.** −45 ft **7.** a. {−3}; b. {−3, −1} **8.** 2, −2 **9.** −62 **10.** −35 **11.** 10 **12.** −13.7° **13.** 826 **14.** $-4a$ **15.** 45 **16.** $5ab^4 - 40b^5 - 3a^3b + 9a^2b^2$ **17.** a. {−.12}; b. $\{\frac{3}{4}\}$ **18.** a. −8; b. $\frac{4}{7}$ **19.** $-\frac{3}{2}x^9y^8$ **20.** $1\frac{1}{2}°$

Pages 146–150 Chapter Review **1.** right **2.** minus **3.** positive, negative **4.** −4.5 **5.** directed, signed **6.** C **7.** D **8.** Q **9.** M **10.** greater **11.** right **12.** negative **13.** less **14.** −2 **15.** 2 **16.** 0 **17.** −3 **18.** {1, 4} **19.** {−2, −4, −5} **20.** {−2, 0} **21.** right **22.** left **23.** −7 **24.** −16 **25.** 0 **26.** {9} **27.** {0} **28.** negative, same **29.** 0 **30.** 0 **31.** 0, additive inverse **32.** opposites **33.** {−1} **34.** {0, 1, 4} **35.** {−4, −2, −1, 0, 1} **36.** distance **37.** magnitude **38.** $|-3|$, 3 **39.** −3.2 **40.** 12 **41.** $-3\frac{1}{5}$ **42.** .085 **43.** {4, −4} **44.** ∅ **45.** {−5, −4, −2, 0, 1, 4} **46.** {−5, −4, −2, 4} **47.** positive **48.** negative **49.** a. difference; b. negative; c. negative **50.** −30 **51.** 0 **52.** 212 **53.** −114 **54.** $n + 33$ **55.** $38 - t$ **56.** opposite *or* inverse **57.** 5 **58.** −21 **59.** −38 **60.** −16 **61.** −39 **62.** $r + 109$ **63.** $-s + 15$ **64.** −7° **65.** 48 yr **66.** positive **67.** negative **68.** opposite **69.** −266 **70.** −266 **71.** −13 **72.** $\frac{h}{4} + 8$ **73.** $-3 + \frac{1}{6}k$ **74.** $10a - 31b$ **75.** $31m - 47mn$ **76.** 0 **77.** 22,360 **78.** 90 **79.** 80 **80.** positive **81.** negative **82.** divisor **83.** 0 **84.** $\frac{2}{3}$ **85.** 1 **86.** {−9} **87.** $\{\frac{1}{16}\}$ **88.** $\{-\frac{2}{3}\}$ **89.** 1 **90.** 0, multiplicative inverse **91.** reciprocal **92.** 1 **93.** $-\frac{1}{2}$ **94.** $\frac{5}{11}$ **95.** $\frac{3}{2}$ **96.** $\frac{a - b}{2}$ **97.** No, $2 \div \frac{1}{2}$ not in set **98.** Yes, except for 0 **99.** deviation; directed **100.** 3

Chapter 5. Equations, Inequalities, and Problem Solving

Page 159 Written Exercises **1.** $\{7\}$ **2.** $\{6\}$ **3.** $\{16\}$ **4.** $\{50\}$ **5.** $\{-40\}$ **6.** $\{-\frac{38}{7}\}$ **7.** $\{-60\}$ **8.** $\{-42\}$ **9.** $\{\frac{9}{2}\}$ **10.** $\{\frac{15}{2}\}$ **11.** $\{-\frac{17}{2}\}$ **12.** $\{-\frac{2}{3}\}$ **13.** $\{4\}, rs \neq 0$ **14.** $\{1\}, mn \neq 0$ **15.** $\{a\}$ **16.** $\{5a\}$ **17.** $\{-3\}$ **18.** $\{-8\}$ **19.** $\{.012\}$ *or* $\{\frac{3}{250}\}$ **20.** $\{.1\}$ *or* $\{\frac{1}{10}\}$ **21.** $\left\{\frac{3l}{2}\right\}$ **22.** $\left\{\frac{-5h}{4}\right\}$ **23.** $\left\{\frac{y-a}{2}\right\}$ **24.** $\{3a-2c\}$ **25.** $\left\{\frac{p-a}{-pr}\right\}, pr \neq 0$ *or* $\left\{\frac{a-p}{pr}\right\}, pr \neq 0$ **26.** $\left\{\frac{p-a}{-pt}\right\}, pt \neq 0$ *or* $\left\{\frac{a-p}{pt}\right\}, pt \neq 0$ **27.** $\left\{\frac{7b}{2}\right\}$ **28.** $\left\{\frac{-3m}{2}\right\}$ **29.** $\{p-a-c\}$ **30.** $\{s-o-p\}$ **31.** $\{Pt\}, t \neq 0$ **32.** $\{pA\}, A \neq 0$ **33.** $\left\{\frac{2A}{h}\right\}, h \neq 0$ **34.** $\left\{\frac{3V}{B}\right\}, B \neq 0$ **35.** $\left\{\frac{A}{4r^2}\right\}, r \neq 0$ **36.** $\left\{\frac{C}{2\pi}\right\}$ **37.** $\left\{\frac{3V}{\pi r^2}\right\}, r \neq 0$ **38.** $\left\{\frac{3V}{4\pi}\right\}$ **39.** $\left\{\frac{3V}{B}\right\}, B \neq 0$ **40.** $\{4\}$ **41.** $\{-9\}$ **42.** $\{3\}$ **43.** $\{1\}$ **44.** $\{1\}$ **45.** $\{\frac{1}{2}\}$ **46.** $\{-3\}$ **47.** $\{16\}$ **48.** $\{\frac{1}{3}\}$ **49.** $\{7\}$ **50.** $\{-3\}$ **51.** $\{-4\}$

Page 163 Written Exercises **A** **1.** $a \geq 2$ **2.** $b \leq 4$ **3.** $s > -3$ **4.** $t < -6$ **5.** $z \leq -28$ **6.** $w \geq -24$ **7.** $-4 \leq t$ **8.** $9 \geq n$ **9.** $p > -3$ **10.** $q < -4$ **11.** $x < -5$ **12.** $y > 2$ **13.** $\frac{9}{2} > w$ **14.** $v > -\frac{5}{2}$ **15.** $x \geq 1$ **16.** $y \geq 6$ **17.** $1 < t$ **18.** $1 > w$ **19.** $r \geq \frac{1}{3}$ **20.** $v \leq -\frac{3}{4}$ **21.** $0 < m$ **22.** $0 > z$ **B** **23.** $t \leq -10$ **24.** $g \leq -1$ **25.** {directed nos.} **26.** {directed nos.} **27.** $\{0\}$ **28.** $\{0\}$ **29.** $\emptyset$ **30.** $\emptyset$ **31.** {directed nos.} **32.** {directed nos.}

Pages 165–166 Written Exercises **A** **1.** $-7 \leq x \leq -4$ **2.** $-4 < y \leq 2$ **3.** $-\frac{7}{3} \leq a < 4$ **4.** $-\frac{1}{2} \leq b \leq 6$ **5.** {directed nos.} **6.** $b > 1$ or $b < 1$ **7.** $0 < m \leq 2$ **8.** $\emptyset$ **9.** $r > 1$ **10.** $-4 < v \leq 1$ **11.** $w < 1$ or $w > 5$ **12.** $k < 3$ or $k > 5$ **13.** $r \geq 2$ or $r \leq -2$ **14.** $u \geq 7$ or $u \leq -7$ **B** **15.** $v < 3$ **16.** $p \leq -2$ **17.** $x > 1$ or $x < -2$ **18.** $y > -3$ or $y < -4$ **19.** $2 < r < 3$ **20.** $-1 < s < 1$

Pages 168–169 Problems **A** **1.** {7, 12} **2.** {13} **3.** $\{\$30 \leq t \leq \$40\}$ **4.** {22, 14} **5.** $\{\$0 < A \leq \$2.75\}$ **6.** {100 ft, 225 ft} **7.** {22 ft, 72 ft} **8.** {13} **9.** {29 mph} **10.** {65 coins} **11.** {1028} **12.** {16 ft} **13.** {2 ft} **14.** {13½ ft} **15.** {5 ft} **16.** {875 rev.} **17.** {3 in, 6 in, 4 in} **18.** each side ∈ {2½, 3, 3½ . . . , 19½ ft}; base ∈ {1, 2, 3 . . . , 35 ft} **19.** {130 lb} **20.** {$16,800} **21.** (a) {8000 bu}; (b) {4000 bu} **22.** {216, 240}

Pages 171–172 Problems **A** **1.** {28, 29} **2.** {37, 38} **3.** {17, 19, 21} **4.** {35, 37, 39} **5.** {22, 24, 26, 28} **6.** {38, 40, 42, 44} **7.** {63, 64, 65} **8.** {29, 30, 31, 32} **9.** {12, 14, 16, 18 in} **10.** {79, 80, 81 ft} **11.** $\{-6, -4\}$ **12.** $\{-9, -7\}$ **B** **13.** $\{-2, -1, 0, 1\}$ **14.** $\{-4, -2, 0, 2\}$ **15.** {3, 5} **16.** {4, 6, 8} **17.** {7, 8} **18.** {10, 11, 12}

Pages 175–176 Written Exercises **1.** 90° **2.** 70° **3.** 60° **4.** 30° **5.** 20° **6.** 28° **7.** $(180-3n)°$ **8.** $(180-n)°$ **9.** $(180-m-n)°$ **10.** $(90-x)°$ **11.** $(150-2n)°$ **12.** $(200-3n)°$ **13.** −150° **14.** −75° **15.** 0° **16.** −90° **17.** 15° **18.** 0° **19.** 20 **20.** $m = 10$ **21.** 30° **22.** 155° **23.** 250° **24.** 125° **25.** 220° **26.** 330°

Page 177 Problems **A** **1.** {39°} **2.** {31°, 59°} **3.** {55°} **4.** {27°, 63°} **5.** {36°, 144°} **6.** {30°, 150°} **7.** {35°, 70° 75°} **8.** {25°, 75°, 80°} **9.** {54°, 54°, 72°} **10.** {29°, 29°, 122°} **11.** {35°, 58°, 87°} **12.** {39°, 57°, 84°} **13.** {−10°} **14.** {−12°}

Pages 180–182 Problems **A** **1.** {2 hr} **2.** {3 hr} **3.** {50 mph} **4.** {11 mph} **5.** {1:48 P.M.} **6.** {10:00 A.M.} **7.** {28 mph} **8.** {36 mph} **9.** {18 mi} **10.** {5 mi} **11.** {4 mi} **12.** {8 mi} **13.** {6 hr} **14.** {14 hr} **15.** {3 hr} **16.** {36 hr} **17.** {6 min} **18.** {25 ft/sec} **B** **19.** {60 mi} **20.** {26 mph} **21.** {5 mi} **22.** {¾ hr}

Pages 183–184 Problems **A** **1.** {400 @ 25¢, 600 @ 20¢} **2.** {107 @ 38¢, 241 @ 25¢} **3.** {60 lb @ $1.65, 40 lb @ $1.90} **4.** {10 lb @ $.90, 10 lb @ $1.50} **5.** {12 lb} **6.** {42 adults} **7.** {13 nickels, 35 pennies} **8.** {62 @ 1¢, 81 @ 3¢} **B** **9.** {6 lb walnuts, 12 lb peanuts, 3 lb almonds} **10.** {10 @ 6¢, 30 @ 3¢, 60 @ 1½¢ 60 @ 1¢} **11.** $\frac{40}{3}$ not an integer $\therefore \emptyset$ **12.** {1 pencil, 9 erasers, 90 clips} **C** **13.** $\{6\frac{2}{3}$ g$\}$ **14.** $\{\frac{10}{3}$ liter 50% soln., $\frac{20}{3}$ liter 75% soln.$\}$

Pages 187–188 Chapter Test **1.** $\{-60\}$ **2.** $\{-5\}$ **3.** $\{a(b+c), a \neq 0\}$ **4.** $\left\{\frac{p-2w}{2}\right\}$ **5.** a. $x < -5$; b. $-3 < v$ **6.** $x > -2$ **7.** $t \leq -2$ **8.** (Optional) a. $y \leq -1$ or $y \geq 3$ b. $m > 5$ or $m < -1$ **9.** {20 yd, 16 yd} **10.** {100} **11.** {25, 27, 29} **12.** {36°, 54°} **13.** {42°, 54°, 84°} **14.** {2 hr} **15.** {550 adults, 150 students}

Pages 188–191 Chapter Review **1.** directed **2.** equivalent **3.** solution set **4.** add. **5.** subt. **6.** div. **7.** mult. **8.** (a) subt. transf.; (b) div. transf.; (c) subst. princ. **9.** (a) add. transf.; (b) mult. transf.; (c) div. transf. **10.** $\{-\frac{1}{10}\}$ **11.** $\{-\frac{3}{2}\}$ **12.** $\{0\}$ **13.** $\{-1\}$ **14.** $\left\{\frac{d}{c}\right\}, c \neq 0$ **15.** $\{-p\}$ **16.** $\left\{\frac{d}{t}\right\}, rt \neq 0$ **17.** $\{2m - b\}$ **18.** $\{a - p\}$ **19.** $\left\{\frac{5F - 160}{9}\right\}$ **20.** $\{ac\}, a \neq 0$ **21.** $\{-2a\}$ **22.** $<$ **23.** left **24.** $>$ **25.** $<$ **26.** $<$ **27.** $<$ **28.** $>$ **29.** $=$ **30.** $<$ **31.** $<$ **32.** $>, <$ **33.** $>$ **34.** $-9x$ **35.** reverse **36.** $\leq$ **37.** $n < 1$ **38.** $m > -\frac{1}{2}$ **39.** $x > -3, x < 2$ **40.** 4, -4 **41.** $x \geq 2$ or $x \leq -\frac{2}{3}$ **42.** -4, 4 **43.** $x > \frac{5}{3}$ or $x < -1$ **44.** empty **45.** directed **46.** {178 for A, 356 for B, 466 for C} **47.** $\{25\}$ **48.** $\{-2, 0, 2, 4, 6\}$ **49.** $\{757, 758, 759\}$ **50.** $\{13, 15, 17, 19, 21\}$ **51.** 90 **52.** $\{57°, 33°\}$ **53.** $\{30°, 150°\}$ **54.** $\{32°\}$ **55.** {2.5 mi} **56.** {20 mi} **57.** $\{25 \leq R_B \leq 30, 50 \leq R_T \leq 60\}$ **58.** $\{63\frac{1}{3}$ mi$\}$ **59.** {40 pt} **60.** {20 lb} **61.** $\{73\}$

Chapter 6. Working with Polynomials

Pages 199–200 Written Exercises **A** **1.** $-3n - 4h$ **2.** $-7r - 2.1s$ **3.** $\frac{3}{2}x^3 - \frac{2}{3}y^3$ **4.** $\frac{5}{4}m^2 + \frac{7}{8}n^2$ **5.** $\frac{2}{3}m - \frac{2}{3}n - \frac{5}{6}$ **6.** $\frac{1}{2}x - \frac{3}{2}y + \frac{3}{4}$ **7.** $x - 2y$ **8.** 0 **9.** $-3yz + 10$ **10.** $-13ab + 4$ **11.** $10x - 10$ **12.** $13z + 3$ **13.** $.7xy + 8.8$ **14.** $.9rs + 13.3$ **15.** 1.3 **16.** $9n - .3t + .2$ **17.** $12h^2 - 4.3dh + .4d^2$ **18.** $-2xy - .1y^2$ **19.** $8x^3 - 60x^2 + 150x - 125$ **20.** $2y^3 - 7y^2 + 5y - 1$ **21.** Correct **22.** Incorrect; $5y - 73$ **23.** Incorrect; -4 **24.** Incorrect; $11x - 9$ **25–29.** Answers are correct as given in text. **B** **30.** $12x^3 + 4x^2 - 3$ **31.** $5y^2 - 10y + 2$ **32.** $t^5 + 2t^4 - t^3 - t^2 + 3t + 2$ **33.** $-s^4 - s^3 + 10s + 15$

Pages 202–203 Written Exercises **A** **1.** $9 - 8x$ **2.** $2x + 3y$ **3.** $x^2 - x + 15$ **4.** $b^2 - 5b - 6$ **5.** $-.1x^2 + 3x + .1$ **6.** $-3 + 3a + a^2$ **7.** $x^2 - x - 3$ **8.** $a^2 + 1$ **9.** $-.7x^2 + 16x - 1.6$ **10.** $\{13\}$ **11.** $\{5\}$ **12.** $\{5\}$ **13.** $\{8\}$ **14.** $\{-3\}$ **15.** $\{-4\}$ **16.** $\{-14\}$ **17.** $\{-8\}$ **18.** $\{6\}$ **19.** $\{8\}$ **20.** $\{11\}$ **21.** $\{3\}$ **22.** $\{1\}$ **23.** $\{\frac{23}{21}\}$ **24.** $\{-6\}$ **25.** $\{-4\}$ **26.** $\{2\}$ **27.** $\{\frac{1}{3}\}$ **28.** $\{30\}$ **29.** $\{30\}$ **B** **30.** $\{2\}$ **31.** $\{-2\}$ **32.** $\{1\}$ **33.** $\{-\frac{10}{7}\}$ **34.** $\{0\}$ **35.** $\{3\}$ **C** **36.** $\{-1\}$ **37.** $\{\frac{1}{2}\}$ **38.** $\{-\frac{3}{2}\}$ **39.** $\{-\frac{1}{6}\}$

Page 204 Written Exercises **A** **1.** $-15r^6s^3$ **2.** $-48x^3y^5$ **3.** $27a^3$ **4.** $60a^2b^2$ **5.** $-30xyz$ **6.** $216n^3$ **7.** $72x^2y^2$ **8.** $-14\,abc$ **9.** $\frac{1}{4}g^2t^2$ **10.** $-48r^3s^3t$ **11.** $\frac{1}{16}t^2x^2$ **12.** $-r^3s^2$ **13.** $-a^2b^3$ **14.** r^4t^3 **15.** $12x^6y^6$ **16.** $10a^9b^6$ **17.** $.0064r^4s^{15}$ **18.** $7.29z^4v^9$ **19.** $c^9d^5f^6$ **20.** $w^5x^{12}y^6$ **21.** $-a^5b^4$ **22.** $-h^4k^7$ **23.** $-.72d^3m^4$ **24.** $-1.28b^3c^4$ **B** **25.** $62a^2b^2$ **26.** $-34p^4q^2$ **27.** 0 **28.** 0 **29.** $91c^6d^{10}e$ **30.** $-8m^7n^7p^5$ **C** **31.** $5x^4 + 4x^3$ **32.** $r^4 - 2r^3$ **33.** $10^{m+n} + 10^{m+r}$ **34.** $a^{m+n} + 2a^mb^m + b^{m+n}$

Page 206 Written Exercises **A** **1.** $-8m^6$ **2.** $-27s^6$ **3.** $-48x^3y^2$ **4.** $-150m^2n^3$ **5.** $27b^4$ **6.** $216r^4$ **7.** $64m^4n^3$ **8.** $375x^3y^4$ **9.** $20a^5b^2$ **10.** $-63b^5c^2$ **11.** $.2x^5y^6$ **12.** $-.09a^5b^6$ **13.** $.0144b^{10}d^6$ **14.** k^9p^6 **15.** $81s^6t^6v^8$ **B** **16.** $-11x^2y^2$ **17.** 0 **18.** $a^3 - 32a^8$ **19.** $-4b^3$ **20.** $1.1t^4z^5$ **21.** $.74m^6k^7$ **C** **22.** $x^5y^2 + 7x^3y^4$ **23.** $c^2d + 3c^6d^4$ **24.** $2.8k^3m^3 - 3.2k^2m$ **25.** $.7a^5b^3 - .7a^4b^5$

Pages 207–208 Written Exercises **A** **1.** $-5x^2 + 15x - 35$ **2.** $-6a^2 + 12a - 6$ **3.** $a^3 - 2a^2b + ab^2$ **4.** $x^3 - 2x^2y + xy^2$ **5.** $-ab^2 + b^3 + b^2c$ **6.** $-r^3 - r^2s + r^2t$ **7.** $-5x^4 + x^3 - 2x^2$ **8.** $-7y^2 + 3y^3 + y^5$ **9.** $-4v^4 + 6v^5 + 8v^6$ **10.** $-3x^5 + 18x^4 - 27x^3$ **11.** $15x^2y - 6x^3y^5 + 9x^4y^4 - 3x^2y^6$ **12.** $15rs^2 + 20r^6s^3 - 5r^4s^6 - 35r^2s^9$ **13.** $-2a^7b^3 + 2a^6b^4 - 4a^5b^6 + 2a^3b^7$ **14.** $-3c^7d^5 - 12c^6d^7 + 3c^4d^8 + 6c^2d^9$ **15.** $\{3\}$ **16.** $\{14\}$ **17.** $\{-4\}$ **18.** $\{-\frac{62}{39}\}$ **19.** $\{-4\}$ **20.** $\{-1\}$ **21.** $\{\frac{1}{2}\}$ **22.** $\{10\}$ **B** **23.** $\{-\frac{7}{2}\}$ **24.** $\{-16\}$ **25.** $\{-1\}$ **26.** $\{2\}$ **27.** $\{\frac{9}{4}\}$

Pages 208–209 Problems **1.** $d = (490 + 4x)$ mi **2.** 200 mph **3.** 3.5 hr **4.** 7 hr **5.** 30 ft, 50 ft **6.** 120 ft **7.** \$4.50, \$3.50 **8.** 6.1 mi/min; 10.7 mi/min **9.** 550, 660 **10.** 5 hr, 7 hr

Page 210 Written Exercises **A** **1.** $x^2 + 9x + 14$ **2.** $a^2 + 15a + 56$ **3.** $y^2 - 4y - 45$ **4.** $n^2 - 5n - 36$ **5.** $x^2 - 13x + 42$ **6.** $y^2 - 11y + 24$ **7.** $10x^2 + 17x + 3$ **8.** $56n^2 + 95n + 36$ **9.** $35x^2 - x - 6$ **10.** $60w^2 + w - 10$ **11.** $12y^2 - 34y + 24$ **12.** $42x^2 - 83x + 40$ **13.** $a^2 + 2ab + b^2$ **14.** $x^2 - 2xy + y^2$ **15.** $x^2 - y^2$ **16.** $c^2 - d^2$ **17.** $m^2 - 10m + 25$ **18.** $x^2 + 18x + 81$ **19.** $m^2 + 6mn + 9n^2$ **20.** $m^2 - 8mn + 16n^2$ **21.** $r^2 - 9s^2$ **22.** $9a^2 - b^2$ **23.** $2a^2 + 9.7ab - 1.5b^2$ **24.** $2a^2 + 5.5ab - 1.5b^2$ **B** **25.** $m^3 - 3.3m^2 + 1.3m - .12$ **26.** $2n^3 - 1.5n^2 - 1.82n + .3$ **27.** $a^3 - b^3$ **28.** $a^3 + b^3$ **29.** $a^4 - 7a^2 - 8$ **30.** $y^4 + 35y^2 - 200$

31. $x^2y^2 - 1.5xy + .56$ **32.** $a^2b^2 + .2ab - .03$ **C** **33.** $3x^4 + 5x^3 + 3x^2 + 3x + 2$ **34.** $5y^4 - 16y^3 - 4y^2 - 7y - 6$ **35.** $2a^4 + 1.6a^3 + 8.2a^2 + 3.97a - .4$ **36.** $.4b^4 - .2b^3 + 1.2b^2 + 5.5b + 1.5$ **37.** $8a^2 - 8ab - 2ac - 16b^2 - 32bc - 15c^2$ **38.** $2m^2p + 3mp - 3mp^2 - m^3p + m^2p^2$ **39.** $x^5 - 1$ **40.** $y^5 + 1$ **41.** Distrib. prop.; distrib. prop. **42.** Distrib. prop.; distrib. prop. **43.** Distrib. prop.; distrib. prop.; comm. prop. for mult.; prop. of opposites; prop. of 0 **44.** Distrib. prop.; distrib. prop.; comm. prop. for mult.; prop. of 1; distrib. prop.

Pages 212–213 Problems **A** **1.** 6 in **2.** 21 in **3.** 60 ft **4.** 27 ft **5.** 44 ft; 20 ft **6.** 94 ft; 50 ft **7.** 225 ft; 300 ft **8.** 2000 ft; 2500 ft **B** **9.** 6 in **10.** 4 ft **11.** 100 in **12.** 12 in **13.** 20 ft; 22 ft; 18 ft **14.** 2 in; 4 in; 2 in

Page 214 Written Exercises **A** **1.** $a^2 + 2ab + b^2$ **2.** $a^2 - 2ab + b^2$ **3.** $a^3 - 3a^2b + 3ab^2 - b^3$ **4.** $a^3 + 3a^2b + 3ab^2 + b^3$ **5.** $9x^2 + 12x + 4$ **6.** $25x^2 + 20x + 4$ **7.** $49y^2 + 42yz + 9z^2$ **8.** $4m^2 + 24mn + 36n^2$ **9.** $r^2 - 16rs + 64s^2$ **10.** $c^2 - 8cd + 16d^2$ **11.** $27x^3 - 27x^2 + 9x - 1$ **12.** $8y^3 - 12y^2 + 6y - 1$ **B** **13.** $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$ **14.** $a^2 - 2ab + 2ac + b^2 - 2bc + c^2$ **15.** $.04x^4 + .4x^3 + .88x^2 - .6x + .09$ **16.** $.09y^4 - .6y^3 + 1.12y^2 - .4y + .04$ **17.** $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ **18.** $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ **19.** $8x^6 + 12x^5 - 18x^4 - 23x^3 + 18x^2 + 12x - 8$ **20.** $27y^6 - 27y^5 + 63y^4 - 37y^3 + 42y^2 - 12y + 8$ **21.** $.512x^6 - 1.728x^5 + 1.944x^4 - .729x^3$

Pages 214–215 Problems **A** **1.** 12 in; 15 in **2.** 50 in × 50 in **3.** 51; 52 **4.** 47; 48 **5.** 5 in **6.** 20 in **7.** 13; 14 **8.** 16; 17 **B** **9.** 15 ft; 20 ft **10.** 16 ft; 21 ft **11.** (a) 84 ft; (b) \$862.40 **12.** (a) 2 in; 1 in; (b) 60π cu in; 15π cu in; (c) $16\frac{2}{3}\pi$ lb; $4\frac{1}{6}\pi$ lb

Page 218 Written Exercises **A** **1.** $8ab$ **2.** $\frac{8b^2}{c^2}$ **3.** $\frac{8d}{e}$ **4.** $\frac{12x}{y}$ **5.** $-\frac{9z}{y^2}$ **6.** $-6r^2s$ **7.** $-\frac{12s}{t}$ **8.** $-\frac{12m}{n}$ **9.** $\frac{b^4}{7a^{10}}$ **10.** $\frac{h^2}{3g}$ **11.** $-\frac{3y^6}{50x^6}$ **12.** $-\frac{3s^3}{40r}$ **13.** $-\frac{5}{21bc^2}$ **14.** $-\frac{1}{16mn^2p^3}$ **15.** $\frac{3ab}{2c}$ **16.** $\frac{2yz}{3x}$ **17.** $-\frac{p^4}{10r^2}$ **18.** $\frac{10w^2}{v}$ **19.** $-\frac{8}{x^9z^5}$ **20.** d^3 **B** **21.** $-\frac{3a^{x-1}b^{y-1}}{2}$, if x and y are integers >1 **22.** $-\frac{x^{a-b}}{3y^{a-b}}$, if $a > b$; $-\frac{y^{b-a}}{3x^{b-a}}$; if $b > a$ **23.** $4b^{m-n}$; if $m > n$; $\frac{4}{b^{n-m}}$; if $n > m$ **24.** $-\frac{51a^{mn-m}}{13b^{mn-n}}$, if $mn > m$ and $mn > n$; i.e., if $n > 1$ and $m > 1$

Page 219 Written Exercises **1.** $3 - 9x^3$ **2.** $2y^2$ **3.** $35e^6 - 1$, if $e \neq 0$ **4.** $42s^4 + 1$, if $s \neq 0$ **5.** -8 **6.** $\frac{3}{4}$ **7.** x^2 **8.** $2 - y^2$, if $y \neq 0$ **9.** $3x - 6$, if $(3x - 5) \neq 0$ **10.** $5 - y$, if $(y - 4) \neq 0$

Pages 220–221 Written Exercises **A** **1.** $4x + 5y$ **2.** $x + 2y$ **3.** 5 **4.** 22 **5.** $5 + \frac{b}{a}$ **6.** $\frac{8m}{n} - 1$ **7.** $r - 7$ **8.** $2s + 4$ **9.** $\frac{4c^2}{3} + c$ **10.** $\frac{3b^2}{2} + b$ **11.** $4a^2 - 2$ **12.** $c^2 + 2$ **13.** $2x^2 + 1$ **14.** $3y - 1$ **15.** $8n^2 - 4n + 5$ **16.** $10r^2 + 2r - 7$ **17.** $-8x^2 + 4x + 2$ **18.** $-2n^3 + 3n^2 + 4n$ **19.** $10m^2n^2 - 6mn + 1$ **20.** $-2x^2y^2 + xy - 3$ **21.** $x^3 - 3x^2 + 9x - 2$ **22.** $5a^3 - 4a^2 - 8a - 2$ **23.** $8ab + 2b + 100a$ **24.** $8t + 2s + 100st$ **B** **25.** $-4a^3 + \frac{3}{4}a^2b - 3ab^2$ **26.** $-\frac{h^2}{3} + \frac{2hk}{3} + \frac{k^2}{6}$ **27.** $\frac{2x^2y}{5} - \frac{3x}{5} + \frac{1}{5y}$ **28.** $\frac{4}{7t^2} + \frac{6}{7t} - \frac{2r}{7t}$ **29.** $-\frac{3m^2}{5n^2} + \frac{4m}{5n} - 1$ **30.** $-\frac{r}{3s} - \frac{2}{3} + \frac{s}{r}$ **C** **31.** 2.5 **32.** .3 **33.** $\{-5\}$ **34.** $\{-3\}$

Page 223 Written Exercises **A** **1.** $x + 3$ **2.** $x + 2$ **3.** $x + 4$ **4.** $x + 4$ **5.** $y - 7$ **6.** $x - 8$ **7.** $r - 6 + \frac{-1}{r + 1}$ **8.** $n - 8 + \frac{-1}{n + 1}$ **9.** $n - 2$ **10.** $z - 8$ **11.** $4x + 7$ **12.** $5x - 9$ **13.** $a - 9 + \frac{162}{a + 9}$ **14.** $x - 3 + \frac{18}{x + 3}$ **15.** $y + 3z$ **16.** $w + 6x$ **17.** $3x - 5y + \frac{10y^2}{x - 3y}$ **18.** $5x - 3y + \frac{10y^2}{x - 2y}$ **19.** $3y - 2$ **20.** $5n - 4$ **21.** $4x^2 + 2xy + y^2$ **22.** $a^2 - 6ab + 36b^2$ **23.** $x^2 - 7xy + 49y^2$ **24.** $x + 7 + \frac{3}{2x - 3}$ **25.** $6x - 7 + \frac{3}{2x + 3}$ **B** **26.** $3x^2 + 2x + 5$ **27.** $2x^2 + 3x - 2$ **28.** $3x^2 + 8x + 11 + \frac{22}{2x - 5}$ **29.** $n^2 - n + 9 + \frac{13}{2n - 3}$ **30.** $x^2 - 2x + 3$ **31.** $y^2 + 2y - 1$ **32.** $x^2 - x + 1$ **33.** $y^2 + y + 1$ **34.** No **35.** No **36.** 12 **37.** 14

Page 226 Chapter Test **1.** $69x + 45$ **2.** $-a^2 + a - 13$ **3.** $27 + (-76) = -49$ **4.** $18 + 7 = 25$ **5.** $7x - 10y - 4z$ **6.** $2x^2 + 2y^2$ **7.** $\{1\}$ **8.** $\{-3\}$ **9.** $-3m^3n^3t^5$ **10.** $-6r^6s^6$ **11.** $-64m^6n^9$ **12.** $72x^5y^6z^{15}$ **13.** $-60x^2 + 88xy$ **14.** $-6t^2 - 2t^3 + 3t^5$ **15.** $c^2 + 6c - 16$ **16.** $16a^2 - 42ay + 20y^2$ **17.** 21 ft **18.** 16 **19.** $\frac{16h^2}{k^2}$ **20.** a. 1; b. 3 **21.** $4a^2 - 1$ **22.** $-\frac{4}{c^2d} + \frac{2b}{cd} + \frac{1}{4b}$ **23.** $4n - 9$ **24.** $2x^2 - 3x + 1 + \frac{15}{5x - 3}$

Pages 227–229 Chapter Review **1.** monomials **2.** trinomial **3.** 5 **4.** third **5.** $3n^3 + n^2 - 8n - 3$ **6.** $-2mn^3 + m^2n^2 - 5m^3n$ **7.** 0, 1 **8.** Correct answer: $-a - 4b - 3$ **9.** Correct in text **10.** opposite (In ex. 11–14, values used in checking will vary.) **11.** $2t^2 - 1$ **12.** $2 - 4c + 2c^2$ **13.** $-h^2 + 2h - 1$ **14.** $-a - 3b - 4$ **15.** $\{-3\}$ **16.** $\{0\}$ **17.** 2, 18 **18.** c^4 **19.** $3a^{11}b^9c^7$ **20.** 2; 3; b^3; $8a^3b^3$ **21.** b^n **22.** $-9p^{16}r^{23}s^5$ **23.** $2a$; $3b$ **24.** $-9y^2z^3 + 6y^3z^2$ **25.** $\{\frac{1}{2}\}$ **26.** $\{-7\}$ **27.** $2n^2 - 5n$ **28.** $2a, -3, 8a^2 - 2a - 15$ **29.** $15x^2 + 41x + 14$ **30.** $21a^2 - 16ax + 3x^2$ **31.** $2a^2 + 14a - 36$ **32.** $3s$; $3s^2$ **33.** $3s^2 + 2s - 8$ **34.** $3s^2 = 3s^2 + 2s - 8$ **35.** 4 in; 12 in **36.** 40 ft; 70 ft **37.** $a^2 + 2a + 1$ **38.** $x^6 - 15x^5 + 72x^4 - 95x^3 - 72x^2 - 15x - 1$ **39.** 5 cm **40.** $-\frac{2n^3}{5m^3}$ **41.** $\frac{1}{2st^2}$ **42.** 1 **43.** 1 **44.** $-\frac{5a}{3} + 1$ **45.** $-\frac{1}{10mn^2} - \frac{1}{5m^2n^2} + \frac{1}{2m^2n}$ **46.** increasing, decreasing **47.** $2a - 7$ **48.** $4c^2 - 4c + 1$ **49.** $2a^2 - 5ab - 3b^2$ **50.** $3r^2 + 4rs - s^2 + \frac{-s^3}{3r - 2s}$

Pages 230–232 Cum. Review **1.** Proper subset *or* subset **2.** no. line **3.** 20 **4.** Closed; multiplication **5.** Distributive **6.** Equivalent, same **7.** Associative (for add.) **8.** $\{-2, 0, 1, 2\}$, since $x \neq 1$ **9.** $2n^3$ **10.** $n + n^2$ **11.** $n - \frac{1}{n}$ **12.** $a^2 + b^2$ **13.** $-4m^3$ **14.** $\{-80\}$ **15.** $\{6\}$ **16.** $\{5\}$ **17.** $\{-3\}$ **18.** $-\frac{3}{2} \leq x < 4$ **19.** $r \geq 4$ or $r \leq -1$ **20.** $m \geq -4$ **21.** -27 **22.** factor **23.** -1 **24.** $b - 2a$ **25.** $2a - b$ **26.** $-\frac{1}{p}$ **27.** $5b - 4a$ **28.** $-8r^6s^9$ **29.** $2a^3$ **30.** $2b^3$ **31.** $a^6 - b^6$ **32.** $3n^2 - 8n - 3$ **33.** $-6a^2x^3 + 30a^3x^2 - 4a^4x$ **34.** $4m^5 - 12m^2n^4 - 5m^4n^3 + 10m^3n^4$ **35.** $1 - \frac{1}{2x} + \frac{1}{3y}$ **36.** $3x^3 - 17x^2 + 16x - 4$ **37.** $2x - 3$ **38.** $8r^3 - 36r^2s + 54rs^2 - 27s^3$ **39.** $2x^2 + 3x - 4 + \frac{-10}{3x - 5}$ **40.** $P = 8a + 2b + 2c$ **41.** $A = bc + 2ab + 2ac$ **42.** $P = 2\pi n$ **43.** $A = 4n^2 - \pi n^2$ **44.** $T = \frac{3lwh}{94}$ **45.** $k = 4$ **46.** $C = 4\frac{4}{9}°$ **47.** $\{1\}, m \neq 0$ **48.** $b = \frac{lwd}{2.5}$ **49.** 600 mph, 625 mph **50.** 46, 29 **51.** 68, 204; 88 **52.** 9 nickels; 14 dimes **53.** 10 ft × 20 ft **54.** \$1.25; \$2.00

Chapter 7. Special Products and Factoring

Page 240 Written Exercises **A** **1.** $2 \cdot 3 \cdot 5 \cdot 7$ **2.** $2 \cdot 7 \cdot 13$ **3.** $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ **4.** $3 \cdot 5 \cdot 7 \cdot 11$ **5.** $2^2 \cdot 3 \cdot 5^3$ **6.** $2^4 \cdot 5^3$ **7.** prime **8.** prime **9.** 1, 5, 7, 35 **10.** 1, 7, 13, 91 **11.** 1, 3, 7, 9, 21, 63 **12.** 1, 2, 4, 13, 26, 52 **13.** 1, 2, 4, 5, 10, 20, 25, 50, 100 **14.** 1, 3, 7, 9, 21, 49, 63, 147, 441 **15.** 1, 2, 3, 6, 11, 22, 33, 66, 121, 242, 363, 726 **16.** 1, 2, 5, 10, 13, 26, 65, 130, 169, 338, 845, 1690 **17.** 18 **18.** 21 **19.** 36 **20.** 350 **21.** xy **22.** $3ab$ **23.** $5abc^2$ **24.** $7d^2e$ **25.** $3mn^2p$ **26.** $-3r^3s$ **27.** $-19p^3d^2q^4$ **28.** $-13u^2v$ **29.** $-4mj$ **30.** $-5m^3$ **B** **31.** $(x)^5$ **32.** $(y)^7$ **33.** $(b^2)^2$ **34.** $(h^2)^3$ **35.** $(5r)^3$ **36.** $(3v)^3$ **37.** $(3rp)^1$ **38.** $(7rq^3)^2$ **39.** $(2x^2y^2)^1$ **40.** $(3u^3v^3)^2$ **41.** $(5a^2b^3)^2$ **42.** $(4r^3s^2)^3$ **43.** $(2pq^4)^4$ **44.** $(7m^3n)^3$ **45.** $(2c^5d^3e)^3$

Pages 242–243 Written Exercises **A** **1.** $3(x^2 + 4y^2)$ **2.** $7(a^2 + 2b^2)$ **3.** $6x(3x - 2)$ **4.** $9r(3r - 10)$ **5.** $x(x + 7)$ **6.** $n(n + 13)$ **7.** $3x^2(1 - 7x)$ **8.** $5b^2(1 - 14b)$ **9.** $2x(3x - 2)$ **10.** $4a(2a + 3)$ **11.** $b(b^2 + b + 1)$ **12.** $a(a^2 - a - 1)$ **13.** $ab(a + b)$ **14.** $xy(y - x)$ **15.** $3c(5a^2 - 1)$ **16.** $2y(6x^2 + 1)$ **17.** $5rs(r - 2s)$ **18.** $3ab(a - 3b)$ **19.** $t(1 + 2m)$ **20.** $x(3n - 1)$ **21.** $-6x(2x + 1)$ **22.** $(y + 2)(y - 1)$ **23.** $(a + 9)(a - 8)$ **24.** $(4c + 5d)(x - y)$ **25.** $2(x + 1)(x + 1)$ **26.** $2x(x - y)$ **27.** $(m - n)(m - n)$ **28.** $(b - 3c)(x^2 + y^2)$ **29.** $(k + 2)(k + t)$ **30.** $(x - 1)(x + y)$ **31.** $(d - f)(g + m)$ **32.** $(r - k)(s - t)$ **33.** $(3b + 2)(b + 4)$ **B** **34.** $4x(3x^3 - 2x + 5)$ **35.** $6y(3y^3 + 5y^2 - 7)$ **36.** $5rt(r - 2 + t)$ **37.** $3ac(6a^2c - 3a + 2c)$ **38.** $(x + 2)(3x - 2)$ **39.** $(2y + 3)(5y - 1)$ **40.** $x(x + a + b)$ **41.** $3a(a + 2x - 2y)$ **42.** $(m + 7a)(m + 2)$ **43.** $(2p + 1)(p + 2a)$ **44.** $(x^2 + 7)(x - 3)$ **45.** $(r^2 + s^2)(r - s)$

Page 244 Problems **1.** $A = r^2(\pi - 1)$ **2.** $A = r^2(16 - \pi)$ **3.** $A = S^2(4\pi - 1)$ **4.** $A = \pi(R^2 - r^2)$ **5.** $A = \pi(R^2 - 4r^2)$ **6.** $A = 2r^2(8 - \pi)$ **7.** $A = \frac{1}{2}r^2(8 + \pi)$ **8.** $A = \frac{1}{2}r^2(16 - \pi)$

Page 246 Written Exercises **A** **1.** 99 **2.** 2499 **3.** 396 **4.** 3596 **5.** 891 **6.** 4891 **7.** 1584 **8.** 6384 **9.** 2475 **10.** 8075 **11.** 1564 **12.** 9964 **B** **13.** 999,900 **14.** 249,600 **15.** $24\frac{3}{4}$ **16.** 1,209,100 **17.** 358,400 **18.** $80\frac{5}{9}$ **19.** 1,437,500 **20.** 86,400 **21.** $15\frac{15}{16}$

Pages 247–248 Written Exercises **A** **1.** $(x + 4)(x - 4)$ **2.** $(r + 3)(r - 3)$ **3.** $(x + 2y)(x - 2y)$ **4.** $(a + 2b)(a - 2b)$ **5.** $(R + r)(R - r)$ **6.** $(a + by^3)(a - by^3)$ **7.** $(4a + b^2)(4a - b^2)$ **8.** $(16x^2 + y)(16x^2 - y)$ **9.** $(2x^2 + z)(2x^2 - z)$ **10.** $(m + 4n^2)(m - 4n^2)$ **11.** $(14b + 11x)(14b - 11x)$ **12.** $(17x + 26y)(17x - 26y)$ **13.** $(2rt + 3)(2rt - 3)$ **14.** $(mn + 12)(mn - 12)$ **15.** $(1 + 3n)(1 - 3n)$ **16.** $(5m + 1)(5m - 1)$ **17.** $(.3a + 2)(.3a - 2)$ **18.** $(.2b + 7)(.2b - 7)$ **19.** $(c + .8)(c - .8)$ **20.** $(.9 + d)(.9 - d)$ **21.** $(x + \frac{1}{3})(x - \frac{1}{3})$ **22.** $(\frac{1}{4} + y)(\frac{1}{4} - y)$ **23.** $\left(a + \frac{b}{2}\right)\left(a - \frac{b}{2}\right)$ **24.** $\left(\frac{d}{6} + \frac{c}{5}\right)\left(\frac{d}{6} - \frac{c}{5}\right)$ **B** **25.** $x^4(5x + 1)(5x - 1)$ **26.** $s^2(4 + s)(4 - s)$ **27.** $a^2(1 + a)(1 - a)$ **28.** $c^3(1 + c)(1 - c)$ **29.** $xy(x + y^2)(x - y^2)$ **30.** $a^2b(a^2 + b)(a^2 - b)$ **31.** $2pq^2(pq + 6)(pq - 6)$ **32.** $3x^2y(7 + xy)(7 - xy)$ **33.** $2(a^2 + 4)(a + 2)(a - 2)$ **34.** $n(1 + n^2)(1 + n)(1 - n)$ **35.** $(1 + a^n)(1 - a^n)$ **36.** $(t + n^x)(t - n^x)$ **37.** $m^a(1 + m^{2a})(1 + m^a)(1 - m^a)$ **38.** $b^{2x}(b^{2x} + 1)(b^x + 1)(b^x - 1)$ **39.** $.02(3r^2 + 2s^3t)(3r^2 - 2s^3t)$ **C** **40.** $(a + 2b + x^2)(a + 2b - x^2)$ **41.** $(3a - 1 + y^3)(3a - 1 - y^3)$ **42.** $x(x - 2)$ **43.** $x(x + 6)$ **44.** a. $(n + 1)^2 - n^2 = 2n + 1 = n + (n + 1)$ b. $(2n + 3)^2 - (2n + 1)^2 = 8n + 8 = 2[(2n + 3) + (2n + 1)]$

Pages 252–253 Written Exercises **A** **1.** $(x + b)^2$ **2.** $(g - h)^2$ **3.** $(a - 6)^2$ **4.** $(y + 8)^2$ **5.** $(b + 7)^2$ **6.** $(n + 9)^2$ **7.** $(2a - b)^2$ **8.** $(5a - b)^2$ **9.** $(3x + 1)^2$ **10.** $(4x + 1)^2$ **11.** $(1 + n)^2$ **12.** $(1 + 2b)^2$ **13.** $(7x - 2)^2$ **14.** $(5x - 3)^2$ **15.** $(5a + 6b)^2$ **16.** $(12n + 5x)^2$ **17.** $(12x - 1)^2$ **18.** $(4x - 3y)^2$ **19.** $(6a + 5b)^2$ **20.** $(4r + 5t)^2$ **21.** $(r - 5)^2$ **22.** $(2m + n)^2$ **23.** $(11ab - 1)^2$ **24.** $3(9mn + 1)^2$ **25.** $3k(1 + 7k)^2$ **26.** $2n(2 + n)^2$ **27.** $(z - 2a^2)^2$ **28.** $(x^2 + y)^2$ **29.** $(y^3 - 4)^2$ **30.** $(n + 1)^2(n - 1)^2$ **B** **31.** $(x + 3)^2(x - 3)^2$ **32.** $(a^4 - 3)^2$ **33.** $(x - 2y + 3)(x - 2y - 3)$ **34.** $(a + 3b + 1)(a + 3b - 1)$ **35.** $(m + x - 1)(m - x + 1)$ **36.** $(n + y + 3)(n - y - 3)$ **C** **37.** 6 or -6 **38.** 5 or -5 **39.** 64

Pages 254–255 Written Exercises **A** **1.** $12a^2 + 25a + 12$ **2.** $27z^2 + 15z + 2$ **3.** $10b^2 - 38b + 36$ **4.** $14y^2 - 31y + 15$ **5.** $12c^2 + c - 6$ **6.** $42x^2 - 33x - 45$ **7.** $6w^2 + 23vw - 55v^2$ **8.** $28h^2 + hk - 84k^2$ **9.** $16x^2 + 34xy - 15y^2$ **10.** $48x^2 - 26xy - 35y^2$ **11.** $-x^2 - 4x + 21$ **12.** $-x^2 - x + 30$ **13.** $s^2 + 8s + 15$ **14.** $t^2 + 10t + 16$ **B** **15.** $-2y^2 + 19y - 24$ **16.** $-3y^2 - 19y - 20$ **17.** $6n^2 + 22n + 20$ **18.** $10m^2 + 19m + 6$ **19.** $-.48m^2 + 1.76m - 1.61$ **20.** $-6.51h^2 + 6.92h + 14.08$ **C** **21.** $\{17\}$ **22.** $\{16\}$ **23.** $\{\frac{2}{5}\}$ **24.** $\{\frac{3}{2}\}$ **25.** $\emptyset$ **26.** {directed numbers}

Pages 256–257 Written Exercises **A** **1.** $(n + 3)(n + 11)$ **2.** $(z + 3)(z + 9)$ **3.** $(x + 2)(x + 9)$ **4.** $(x + 8)(x + 2)$ **5.** $(h - 9)(h - 4)$ **6.** $(y - 7)(y - 8)$ **7.** $(x - 9)(x - 10)$ **8.** $(a - 45)(a - 2)$ **9.** $(m + 15)(m + 6)$ **10.** $(r + 30)(r + 3)$ **11.** $(33 - b)(1 - b)$ **12.** $(1 - k)(14 - k)$ **13.** $(3 + c)(14 + c)$ **14.** $(4 + s)(13 + s)$ **15.** $(x + 3)(x + 17)$ **16.** $(y + 51)(y + 1)$ **17.** $(x + 2y)(x + 12y)$ **18.** $(y + 24z)(y + 2z)$ **19.** $(m - 18n)(m - 4n)$ **20.** $(s - t)(s - 20t)$ **21.** $(b - 19c)(b - 4c)$ **22.** $(a - 3b)(a - 19b)$ **23.** $(z - 5b)(z - 24b)$ **24.** $(z - 15d)(z - 8d)$ **B** **25.** $21, 12, 9, -21, -12, -9$ **26.** $64, 24, 16, -64, -24, -16$ **27.** $2, -2$ **28.** $5, 4, -5, -4$ **29.** $10, 6, -10, -6$ **30.** $12, -12$ **31.** $4, 6$ **32.** $6, 10, 12$ **33.** $5, 8, 9$ **34.** $7 = 1 \cdot 7, (-1)(-7); 1 + 7 \neq 3, -1 + (-7) \neq 3$ **35.** $9 = 1 \cdot 9, 3 \cdot 3, (-1)(-9), (-3)(-3); 1 + 9 \neq 15, 3 + 3 \neq 15, -1 + (-9) \neq 15, -3 + (-3) \neq 15$ **36.** $1 = 1 \cdot 1, (-1)(-1); 1 + 1 \neq -4, -1 + (-1) \neq -4$ **37.** $3 = 1 \cdot 3, (-1)(-3); 1 + 3 \neq -3, -1 + (-3) \neq -3$ **38.** $4 = 1 \cdot 4, 2 \cdot 2, (-1)(-4), (-2)(-2); 1 + 4 \neq 0, 2 + 2 \neq 0, -1 + (-4) \neq 0, -2 + (-2) \neq 0$ **39.** $9 = 1 \cdot 9, 3 \cdot 3, (-1)(-9), (-3)(-3); 1 + 9 \neq 0, 3 + 3 \neq 0, -1 + (-9) \neq 0, -3 + (-3) \neq 0$

Pages 258–259 Written Exercises **A** **1.** $(x + 5)(x - 2)$ **2.** $(y + 7)(y - 2)$ **3.** $(y - 7)(y + 3)$ **4.** $(x - 5)(x + 3)$ **5.** $(h - 9)(h + 7)$ **6.** $(n + 8)(n - 7)$ **7.** $(k + 11)(k - 10)$ **8.** $(w - 12)(w + 8)$ **9.** $(r - 11)(r + 9)$ **10.** $(x - 15)(x + 2)$ **11.** $(u - 11)(u + 5)$ **12.** $(b + 12)(b - 11)$ **13.** $(x - 11)(x + 3)$ **14.** $(n - 12)(n + 4)$ **15.** $(x - 10)(x + 9)$ **16.** $(b - 8)(b + 3)$ **17.** $(z - 4t)(z + t)$ **18.** $(t - 5u)(t + 2u)$ **19.** $(y + 2z)(-y + z)$ **20.** $(x + 3y)(-x + 2y)$ **21.** $(a + 8)(a - 5)$ **22.** $(c + 12)(c - 3)$ **23.** $(d - 8)(d + 2)$

24. $(m - 6)(m + 5)$ **B** **25.** $-11, -4, -1, 11, 4, 1$ **26.** $-13, -5, 13, 5$ **27.** $-3, 0, 3$ **28.** $-15, -6, 0, 15, 6$ **29.** $-6, 6$ **30.** $-4, 4$ **31.** $-6, -14$ **32.** $-8, -18$ **33.** $-7, -16$ **C** **34.** $-7 = (1)(-7), (-1)(7); 1 + (-7) \neq 8, (-1) + 7 \neq 8$ **35.** $-4 = (1)(-4), (2)(-2), (-1)(4); 1 + (-4) \neq 5, 2 + (-2) \neq 5, (-1) + 4 \neq 5$ **36.** $-3 = (1)(-3), (-1)(3); 1 + (-3) \neq -4, (-1) + 3 \neq -4$ **37.** $-5 = (1)(-5), (-1)(5); 1 + (-5) \neq -6, (-1) + 5 \neq -6$ **38.** $-3 = (1)(-3), (-1)(3); 1 + (-3) \neq 0, (-1) + 3 \neq 0$ **39.** $-6 = (1)(-6), (2)(-3), (-1)(6), (-2)(3); 1 + (-6) \neq 0, 2 + (-3) \neq 0, (-1) + 6 \neq 0, (-2) + 3 \neq 0$

Page 260 Written Exercises **A** **1.** $(2y + 1)(y + 3)$ **2.** $(3x + 1)(x + 2)$ **3.** $(3n - 1)(n - 1)$ **4.** $(3x - 2)(x - 1)$ **5.** $(5x - 7)(x + 1)$ **6.** $(2x + 1)(x - 5)$ **7.** $(3a - 1)(a + 1)$ **8.** $(5a - 1)(a + 1)$ **9.** $(4x - 1)(2x - 3)$ **10.** $(3x - 1)(x + 7)$ **11.** $(7y - 3)(5y - 1)$ **12.** $(13y + 6)(y - 1)$ **B** **13.** $(6x + 7)(x + 3)$ **14.** $(14x + 5)(x + 2)$ **15.** $(7x - 11)(2x + 1)$ **16.** $(2x + 5y)(6x - y)$ **17.** $(4x - 3y)(6x + y)$ **18.** $(6n + 7s)(n - 9s)$ **19.** $(4m - 3)(2m + 5)$ **20.** $(5y + 2)(2y - 3)$ **C** **21.** $(3x + 4)$ and $(2x - 3)$ **22.** $(4n - 5)$ and $(n + 2)$ **23.** $[(30a - 14)$ and $(a + 3)]$ or $[(15a - 7)$ and $(2a + 6)]$ **24.** $[(14b + 10)$ and $(2b - 5)]$ or $[(7b + 5)$ and $(4b - 10)]$

Pages 262–263 Written Exercises **A** **1.** $3(x + 3)(x - 3)$ **2.** $14(y + 2)(y - 2)$ **3.** $2(w + 9)^2$ **4.** $8(z + 7)^2$ **5.** $(6x - 5)(x - 1)$ **6.** $(5r - 3)(r - 2)$ **7.** $(7n - 1)^2$ **8.** $(5x - 2)^2$ **9.** $-5a(x^2 + y^2)$ **10.** $-rt(t^2 + r^2)$ **11.** $(3x - 4)^2$ **12.** $(2x - 7)^2$ **13.** $(13a + 7b^2)(13a - 7b)^2$ **14.** $(7y^2 + 12z)(7y^2 - 12z)$ **15.** $(n + 2)(n - 20)$ **16.** $(n - 12)(n + 2)$ **17.** $(x - 7)(x + 12)$ **18.** $(x - 6)(x + 8)$ **19.** $(7 + m)(3 - m)$ **20.** $(161 + u)(1 - u)$ **21.** $(8 + s)(13 - s)$ **22.** $(12 + r)(15 - r)$ **23.** $5(1 - 6y)(1 + y)$ *or* $-5(6y - 1)(y + 1)$ **24.** $3(1 - 8y)(1 + y)$ *or* $-3(8y - 1)(y + 1)$ **25.** $(7n + 6)(4n + 9)$ **26.** $6(w - 19)(w + 3)$ **27.** $4(t - 15)(t + 8)$ **B** **28.** $a(z - 6w)^2$ **29.** $b(x - 7y)^2$ **30.** $3y(2y - 1)(y + 1)$ **31.** $2v(3v + 1)(v + 4)$ **32.** $-1(x - b)^2$ **33.** $-1(k - h)^2$ **34.** $-1(n - 1)^2$ **35.** $-1(2b - 1)^2$ **36.** $-1(7 + z)(2 + z)$ **37.** $-1(8 + k)(2 + k)$ **38.** $(7a + 2)(a - 1)(a + 1)$ **39.** $(4b + 1)(b - 1)(b + 2)$ **40.** $2(3x + 2)^2(x + 1)$ **41.** $3(7y - 1)^2(y - 1)$ **42.** $(x + 1)^2(x - 1)(1 + 4x)$ **43.** $(y + 2)(y - 2)(1 + 2y)(1 - y)$ **44.** $2(w^2 + 9)(w + 3)(w - 3)$ **45.** $3(m^2 + 25)(m + 5)(m - 5)$ **46.** $-1(4y - 7x)(5y - 2x)$ **47.** $-1(4m - 3n)(2m - 11n)$ **C** **48.** $(x^2 + 5)(x^2 - 12)$ **49.** $(n^2 + 15)(n^2 - 2)$ **50.** $(y + 2)(y - 2)(y + 1)(y - 1)$ **51.** $(m + 3)(m - 3)(m + 2)(m - 2)$ **52.** $(5a + b)(x + y)(x - y)$ **53.** $(m + 3)(m - 3)(r^2 + s^2)$ **54.** $(y^2 + 1)(y - 1)$ **55.** $(x + 1)^2(x - 1)$ **56.** -18 **57.** -10 **58.** 40 **59.** 4

Pages 263–264 Written Exercises **A** **1.** $\{2\}$ **2.** $\{8\}$ **3.** $\{-5\}$ **4.** $\{\frac{1}{8}\}$ **5.** $\{-\frac{7}{2}\}$ **6.** $\{-\frac{7}{3}\}$ **7.** $\{-12\}$ **8.** {directed nos.} **9.** {directed nos.} **10.** $\{\frac{1}{6}\}$ **11.** {any no. other than 0} **12.** {any no. other than 0} **B** **13.** $\{3, 5\}$ **14.** $\{-2, -7\}$ **15.** $\{-10, 7\}$ **16.** $\{8, -3\}$ **17.** $\{0, -\frac{1}{2}\}$ **18.** $\{0, -\frac{1}{3}\}$ **19.** $\{-\frac{1}{5}, 3\}$ **20.** $\{-\frac{1}{2}, -2\}$

Pages 266–267 Written Exercises **A** **1.** $\{-10, 9\}$ **2.** $\{-7, 2\}$ **3.** $\{5, -3\}$ **4.** $\{6, -3\}$ **5.** $\{-3, -5\}$ **6.** $\{-2, -9\}$ **7.** $\{3, 1\}$ **8.** $\{1, 5\}$ **9.** $\{-9, 12\}$ **10.** $\{11, -6\}$ **11.** $\{-2, 2\}$ **12.** $\{-16, 16\}$ **13.** $\{-\frac{5}{2}, \frac{5}{2}\}$ **14.** $\{-\frac{4}{3}, \frac{4}{3}\}$ **15.** $\{0, 24\}$ **16.** $\{0, 16\}$ **17.** $\{-2, -\frac{5}{2}\}$ **18.** $\{-2, -\frac{7}{3}\}$ **19.** $\{\frac{1}{2}, 5\}$ **20.** $\{0, -3\}$ **21.** $\{0, \frac{5}{2}\}$ **22.** $\{7, \frac{3}{2}\}$ **23.** $\{5, \frac{2}{3}\}$ **24.** $\{\frac{3}{5}, \frac{1}{2}\}$ **25.** $\{-8, 6\}$ **26.** $\{-7, 5\}$ **27.** $\{8, -3\}$ **28.** $\{9, -3\}$ **29.** $\{3\}$ **30.** $\{-4\}$ **31.** $\{0, 4\}$ **32.** $\{0, -\frac{1}{6}\}$ **B** **33.** $\{0, 10, -4\}$ **34.** $\{0, -14, 6\}$ **35.** $\{0, -\frac{1}{2}\}$ **36.** $\{0, \frac{2}{3}\}$ **37.** $\{-4, 4, -1, 1\}$ **38.** $\{-3, 3, -1, 1\}$ **C** **39.** $\{0, 1\}$ **40.** $\{-1, 1\}$ **41.** $\{0, -1, 1, -2, 2\}$ **42.** $\{0, -2, 2\}$ **43.** $y^2 - 4y + 3 = 0$ **44.** $z^2 + 3z + 2 = 0$ **45.** $w^3 + w^2 - 2w = 0$ **46.** $r^3 - 6r^2 + 8r = 0$ **47.** $p^3 - 3p^2 - p + 3 = 0$ **48.** $y^4 - 13y^2 + 36 = 0$

Pages 269–270 Problems **A** **1.** 6 ft; 11 ft **2.** 9 yd; 11 yd **3.** 20 sec **4.** 25 sec **5.** 26 **6.** 6 **7.** $-11, -9$ or $9, 11$ **8.** $-8; -6$ or $6; 8$ **9.** 9 M; 8 M **10.** 7 cm; 12 cm **11.** 2 sec **12.** 3 sec **13.** 3 sec **14.** 1 sec **B** **15.** 40 ft **16.** 10 ft **17.** a. at $t = 2$ sec or $t = 1$ sec; b. 3 sec **18.** a. at $t = 1$ sec or $t = 200$ sec; b. $3\frac{7}{20}$ min **20.** 16 yr **21.** 26 ft **22.** $n^2 + (n + 1)^2 = 2(n^2 + n) + 1$; since $2(n^2 + n)$ is even, then $2(n^2 + n) + 1$ is odd.

Page 273 Chapter Test **1.** 72 **2.** $7yz^2$ **3.** $16a(5a - b)$ **4.** $(x - 2)(x + 1)$ **5.** $(y^2 + 3)(y - 5)$ **6.** $81x^6$ **7.** $r^2s^2 - t^4$ **8.** $2(2m + 5)(2m - 5)$ **9.** $(x + 12)(x - 12)$ **10.** $9a^2 + 6ab + b^2$ **11.** $25k^2 - 20km + 4m^2$ **12.** $(5x + 9y)^2$ **13.** $x = 4$ *or* $\{4\}$ **14.** $(n + 14)(n + 3)$ **15.** $(r - 18s)(r - 5s)$ **16.** $(m + 9)(m - 4)$ **17.** $(k - 9)(k + 2)$ **18.** $(9n + 8)(4n + 7)$ **19.** $(3a + 4b)(a - 9b)$ **20.** $10(3t - 5)(t + 2)$ **21.** $\{2, -5\}$ **22.** $\{0, \frac{1}{2}\}$ **23.** $\{10, -9\}$ **24.** $\{-\frac{13}{7}, \frac{13}{7}\}$ **25.** 7 ft × 15 ft

Pages 274–276 Chapter Review **1.** 3, 2, 1 **2.** $-5bc^2$ **3.** 40 **4.** $(2x^3y^3)^1$ **5.** $5x^2(x^2 - 2x + 3)$ **6.** $-x(2x + 1)(x - 1)$ **7.** $n^2, 2$ **8.** $(y + 5)(y - 3)$ **9.** $(r + s)(r + 3)$ **10.** $3x(2x - 1)$ **11.** $(t - 3u)(2r + 5s)$ **12.** 256 **13.** $64x^2$ **14.** $49a^4b^6$ **15.** $25n^2 - x^2$ **16.** $9y^2 - \frac{1}{9}$ **17.** $(2x + 5)(2x - 5)$ **18.** $8(x + 2y)(x - 2y)$ **19.** $d(2d^2 - 1)$ **20.** $(x^2 + 9)(x + 3)(x - 3)$ **21.** $a^2 + 2ab + b^2$ **22.** $a^2 - 2ab + b^2$ **23.** $4x^2 - 4x + 1$ **24.** $25h^2 + 30hk + 9k^2$ **25.** a. Not trinomial square; b. Not trinomial square; c. Trinomial square **26.** $(4x - 1)^2$ **27.** $(3z + 5a)^2$ **28.** quadratic **29.** linear **30.** constant **31.** $8x^2 + 59x + 21$ **32.** $5n^2 - 64n + 48$ **33.** $x = 1$ *or* $\{1\}$ **34.** 12, 8 **35.** pos., neg. **36.** 6, 2 **37.** $(n + 11)(n + 4)$ **38.** $(x - 21)(x - 2)$ **39.** $-42, 1$ **40.** opposite **41.** 7, -6 **42.** $(x + 6)(x - 7)$ **43.** $(x + 5)(x - 7)$ **44.** $(x + 7)(x - 5)$ **45.** $b \in \{7, -7, 5, -5\}$ **46.** $b \in \{43, -43, 17, -17, 23, -23, 13, -13\}$ **47.** $b \in \{4, -4, 59, -59, 17, -17, 28, -28, 7, -7, 11, -11\}$ **48.** $(4y - 3)(5y - 1)$ **49.** $(2z + 1)(13z - 6)$ **50.** difference, squares **51.** perfect square, not a perfect square **52.** $x = -2$ *or* $\{-2\}$ **53.** $y = -2$ or $\frac{5}{2}$; *or* $\{-2, \frac{5}{2}\}$ **54.** $x^2 - 5x - 50 = 0$ **55.** $x = -14$ or $x = 5$; *or* $\{-14, 5\}$ **56.** $x = 12$ or $x = 7$; $\{12, 7\}$ **57.** $x = 0$, or $x = 10$, or $x = -3$; *or* $\{0, 10, -3\}$ **58.** 3

Chapter 8. Working with Fractions

Page 282 Written Exercises **A** **1.** $\frac{5}{y}$; $y \neq 0$ **2.** $\frac{-7}{z}$; $z \neq 0$ **3.** $\frac{17}{100}$; no exclusions **4.** $\frac{9}{10}$; no exclusions **5.** $\frac{x}{5}$; no exclusions **6.** $\frac{3y}{1}$; no exclusions **7.** $\frac{b-1}{b}$; b $\neq 0$ **8.** $\frac{a}{a-1}$; $a \neq 1$ **9.** $\frac{k}{6k-12}$; $k \neq 2$ **10.** $\frac{f}{3f+21}$; $f \neq -7$ **11.** $\frac{12x^2+4}{3}$; no exclusions **12.** $\frac{3y^2-2}{1}$; no exclusions **13.** $\frac{g-2}{14g+7}$; $g \neq -\frac{1}{2}$ **14.** $\frac{h-5}{33h-11}$; $h \neq \frac{1}{3}$ **15.** $\frac{1}{x(x-1)}$; $x \neq 0$ and $x \neq 1$ **16.** $\frac{1}{y(y+4)}$; $y \neq 0$ and $y \neq -4$
B **17.** $\{6, 2\}$ **18.** $\{-7, -2\}$ **19.** $\{\frac{1}{5}, -3\}$ **20.** $\{-\frac{2}{7}, 1\}$ **21.** $\{-2, 2\}$ **22.** $\{-3, 3\}$ **23.** $\emptyset$ **24.** $\emptyset$ **25.** $\{-4, 4\}$
C **26.** $a \neq 0$ and $a \neq b$ **27.** $c \neq -d$ **28.** $r \neq s$

Pages 284–285 Written Exercises **A** **1.** h; $h \neq 0, k \neq 0$ **2.** $\frac{t}{6}$; $s \neq 0, t \neq 0$ **3.** $-\frac{3}{4mn}$; $m \neq 0, n \neq 0$ **4.** $-\frac{5z^2}{9xy^3}$; $x \neq 0, y \neq 0$ **5.** $\frac{3}{4}$; $a \neq -b$ **6.** $\frac{c-d}{c+d}$; $c \neq -d$ **7.** $\frac{4}{x-y}$; $|x| \neq |y|$ **8.** $\frac{r-5}{3}$; $r \neq -5$ **9.** $\frac{2a+3b}{2ab}$; $a \neq 0, b \neq 0$ **10.** $\frac{5r+3b}{5rs}$; $r \neq 0, s \neq 0$ **11.** $-x - 4$; $x \neq 4$ **12.** $-1 - a$; $a \neq 1$ **13.** $\frac{m+2}{m-2}$; $m \neq 2$ **14.** $\frac{x+4}{x-4}$; $x \neq 4$ **15.** $\frac{3}{a+1}$; $a \neq -1$ **16.** $\frac{5}{x+2}$; $x \neq -2$ **17.** $s(r + 1)$; $r \neq 1$ **18.** $2(d - 1)$; $d \neq -1$ **19.** $\frac{x^2+1}{x+1}$; $x \neq -1$ **20.** $\frac{x-4}{x^2-4}$; $|x| \neq 2$ **21.** $\frac{3(a-3)}{a+1}$; $a \neq -3$ and $a \neq -1$ **22.** $\frac{4(a+1)}{4-a}$; $|a| \neq 4$ **23.** $\frac{c-3}{-c-6}$; $|c| \neq 6$ **24.** $\frac{3(d+3)}{d-8}$; $d \neq 8$ and $d \neq 3$ **B** **25.** $\frac{c+2d}{c-2d}$; $|c| \neq 2d$ **26.** $\frac{3a+b}{3a-b}$; $|b| \neq 3a$ **27.** $\frac{r}{r-1}$; $r \neq 1$ and $r \neq 3$ **28.** $\frac{t}{t-1}$; $t \neq 1$ and $t \neq 7$ **29.** $\frac{a-4}{a+1}$; $a \neq -1$ **30.** $\frac{b+6}{b-1}$; $b \neq 1$ **31.** $\frac{z-5}{z+4}$; $z \neq 3$ and $z \neq -4$ **32.** $\frac{m-4}{m-3}$; $m \neq 3$ and $m \neq -2$ **33.** $\frac{x-2}{-x-2}$; $|x| \neq 2$
C **40.** $\frac{3n+1}{3n-1}$; $n \neq \frac{1}{3}$ and $n \neq \frac{5}{2}$ **41.** $\frac{2n+3}{2n-3}$; $n \neq \frac{1}{4}$ and $n \neq \frac{3}{2}$ **42.** $\frac{-m+2}{2(m+2)}$; $m \neq 0$ and $m \neq -2$ and $m \neq -1$ **43.** $\frac{5(t-2)}{4(1-2t)}$; $t \neq 0$ and $|t| \neq \frac{1}{2}$ **44.** $\frac{(x+5)^2}{2(x^2+25)}$; $x \neq 0$ **45.** $\frac{(p+2)^2}{5(p^2+4)}$; $p \neq 0$ **46.** $\frac{2z^2(z+5x)}{z-3x}$; $z \neq -4x$ and $z \neq 3x$ **47.** $\frac{2n(m-4n)}{3m(m-3n)}$; $m \neq 0$ and $m \neq 3n$ and $m \neq -2n$

Pages 287–288 Written Exercises **A** **1.** $\frac{2}{3}$ **2.** $\frac{1}{2}$ **3.** $\frac{9}{5}$ **4.** $\frac{1}{4}$ **5.** $\frac{1}{4}$ **6.** $\frac{3}{5}$ **7.** $\frac{29}{10}$ **8.** $\frac{3}{10}$ **9.** $\frac{2}{1}$ **10.** $\frac{35}{23}$ **11.** $\frac{4}{5}$ **12.** $\frac{2}{1}$ **13.** $\frac{3}{2}$ **14.** $\frac{2}{3}$ **15.** $\frac{3}{4}$ **16.** $\frac{5}{2}$ **17.** $\frac{1}{1}$ **18.** $\frac{7}{1}$ **19.** $\frac{2}{3}$ **20.** $\frac{2}{3}$ **B** **21.** $\frac{2}{9}$ **22.** $\frac{2}{7}$ **23.** $\frac{4}{1}$ **24.** $\frac{1}{1}$ **25.** $\frac{1}{1}$ **26.** $\frac{-2}{1}$

Pages 288–289 Problems **A** **1.** $93\frac{1}{3}$ **2.** -15 **3.** 10, 25 **4.** 6 lb **5.** 64 in **6.** 2 **7.** copper, $4\frac{1}{2}$ lb; silver, $40\frac{1}{2}$ lb **8.** Ross, \$11,000; Morgan, \$13,200 **9.** \$1.05 for 21 lb **10.** second worker **11.** 12 cm; 18 cm **12.** 9 cm; 12 cm **B** **13.** 959,850 **14.** 33,880 mi **15.** (For prob. concerning sulfuric acid) 10 lb hydrogen, 160 lb sulfur, 320 lb oxygen **16.** Sand, 50 lb; silt, 225 lb; clay, 225 lb. **17.** \$560 **18.** \$3.00

Pages 290–291 Written Exercises **A** **1.** 17.5 **2.** .05 **3.** 4 **4.** 900 **5.** 4.5 **6.** 72 **7.** 24 **8.** 57 **9.** 3 **10.** .36 **11.** 1.27 **12.** .3 **13.** 80 **14.** 68 **15.** 42 **16.** 6.5 **17.** 195 **18.** 42 **19.** 216 **20.** 1700 **21.** 75% **22.** 60% **23.** 400% **24.** 250% **25.** .5% *or* $\frac{1}{2}$% **26.** 1.25% *or* $1\frac{1}{4}$% **27.** 1200% **28.** 2000%

Page 291 Problems **A** **1.** 1470 **2.** 1.25 lb **3.** .4 qt **4.** 151.2 lb **5.** \$580 **6.** \$950 **7.** \$53 **8.** \$12.50 **9.** \$700 **10.** \$60 **11.** 5% **12.** 24.5% *or* $24\frac{1}{2}$% **B** **13.** 20% **14.** 15% **15.** \$16.50 **16.** 5 lb **17.** \$600 **18.** \$3.05

Pages 293–294 Written Exercises **A** **1.** $\frac{15}{196}$ **2.** $-\frac{21}{80}$ **3.** $\frac{1}{6}$ **4.** $\frac{2}{5}$ **5.** $\frac{1}{2}$ **6.** $\frac{1}{5}$ **7.** $\frac{5c^5}{12}$ **8.** $\frac{7x^4}{16}$ **9.** $\frac{1}{a}$ **10.** $\frac{1}{r}$ **11.** $\frac{14}{5}$ **12.** $\frac{4}{9}$ **13.** $\frac{3cd}{4ab}$ **14.** $\frac{5rs}{7tu}$ **15.** $\frac{t^2 - v^2}{t^2 - 4v^2}$ **16.** $\frac{9r^2 - w^2}{r^2 - w^2}$ **17.** $3a - 3b$ **18.** $\frac{1}{1 + x}$ **19.** $\frac{2 + y}{2}$ **20.** $\frac{m - n}{2}$ **21.** 4 **B** **22.** $\frac{r + s}{m - n}$ **23.** $\frac{a - 7}{b + 5}$ **24.** $\frac{y - 1}{3x}$ **25.** $\frac{b + 1}{b^2}$ **26.** $\frac{3}{k}$ **27.** 3 **28.** 1 **29.** 1 **30.** $\frac{2s - 2t}{s + t}$ **31.** $\frac{3x + 3}{x - 1}$ **32.** $\frac{1}{2}$ **33.** 1 **34.** 1 **35.** $\frac{c + 1}{-c}$ *or* $-\frac{c + 1}{c}$ **C** **36.** -1 **37.** $-\frac{4}{5}$ **38.** $\frac{8(x + 5)}{15(x - 4)}$ **39.** $-a^2$ **40.** $-n(n + 3)$ *or* $-n^2 - 3n$ **41.** $\frac{k}{-5(2k + 3)}$ *or* $-\frac{k}{5(2k + 3)}$ **42.** $\frac{x}{x + 5}$ **43.** 1 **44.** $\frac{x^4(x - 2)}{2}$ **45.** $\frac{2a^2(3a^2 - b^2)}{3a - b}$

Pages 295–296 Written Exercises **A** **1.** $\frac{6}{11}$ **2.** $\frac{14}{11}$ **3.** b **4.** c **5.** $\frac{4c}{5b}$ **6.** $\frac{1}{12ar}$ **7.** $\frac{5}{6}$ **8.** $\frac{11}{20}$ **9.** $\frac{1}{2}$ **10.** 3 **11.** $\frac{1}{22x}$ **12.** $\frac{a}{14}$ **13.** $\frac{m + n}{mn}$ **14.** $\frac{a + b}{ab}$ **15.** $\frac{x}{x - 2}$ **16.** $\frac{m}{m + 3}$ **17.** $\frac{1}{(x - 4)^2}$ **18.** $\frac{5}{6}$ **19.** $\frac{ad}{bc}$ **20.** $\frac{m + n}{27mn}$ **21.** $\frac{50(a - b)}{ab}$ **22.** $\frac{x}{a}$ **23.** 1 **B** **24.** $\frac{a^2}{(1 - 2a)(3 - a)}$ **25.** $\frac{3(r^2 - s^2)}{r^2}$ **26.** $x - y$ **27.** $\frac{x - 1}{x + 1}$ **28.** $-\frac{5c^2}{(c + d)^2}$ *or* $\frac{5c^2}{-1(c + d)^2}$ **29.** $u - t$

Pages 296–297 Written Exercises **A** **1.** $\frac{y}{10}$ **2.** $\frac{b}{8}$ **3.** $\frac{c^2}{ab}$ **4.** $\frac{2t^3}{s^3}$ **5.** $\frac{5}{x}$ **6.** $\frac{3}{a + 2}$ **7.** $\frac{4(9m + n)}{n(3m - n)(3m + n)}$ **8.** $2d(2c - d)$ **9.** $\frac{b}{a}$ **10.** $\frac{r}{s}$ **11.** $\frac{1}{m}$ **12.** $\frac{1}{x}$ **13.** $\frac{a}{ab + 3}$ **14.** $\frac{1}{st + 2}$ **B** **15.** $\frac{10}{x + 10}$ **16.** $\frac{a - 8}{8}$ **17.** ab **18.** $\frac{xy^2(x + z)}{z(x + y)}$ **19.** $\frac{1}{d}$ **20.** vw

Pages 298–299 Written Exercises **A** **1.** $-\frac{11}{15}$ *or* $\frac{-11}{15}$ **2.** $-\frac{1}{2}$ *or* $\frac{-1}{2}$ **3.** $\frac{3}{z}$ **4.** $\frac{3}{a}$ **5.** $\frac{2a - 1}{r}$ **6.** $\frac{3b - 1}{3t}$ **7.** 1 **8.** 1 **9.** 3 **10.** 1 **11.** $a - b$ **12.** $z + 4$ **13.** 0 **14.** 0 **B** **15.** $x + y$ **16.** $m - n$ **17.** $\frac{1}{x - 4}$ **18.** $\frac{1}{x + 1}$ **19.** $\frac{1 - a - a^2}{a^2 + 3a - 4}$ **20.** $\frac{8 - b - b^2}{b^2 - 16}$

Pages 301–302 Written Exercises **A** **1.** $\frac{8b - a}{2a}$ **2.** $\frac{x + 6a}{3x}$ **3.** $\frac{3 - 2c}{2c}$ **4.** $-\frac{5a + 12}{5a}$ **5.** $\frac{-a + 17}{10}$ **6.** $\frac{9n - 20}{12}$ **7.** $\frac{8h - 3}{12}$ **8.** $-\frac{13}{18}$ *or* $\frac{-13}{18}$ **9.** $\frac{x^2 - 2x + 1}{x^3}$ **10.** $\frac{3 - 2a + a^2}{a^3}$ **11.** $\frac{c + a^2 - 2a}{abc}$ **12.** $\frac{4t - 2s + st}{rst}$ **13.** $\frac{ax + bx - cx}{abc}$ **14.** $\frac{ax - ay + az}{xyz}$ **15.** $\frac{3x + 2y - xy}{x^2y^2}$ **16.** $\frac{4s - 2r + r^2}{r^2s^2}$ **17.** $\frac{2a - b}{a^2 - b^2}$ **18.** $\frac{c + 2d}{c^2 - d^2}$ **19.** $\frac{7}{24}$

20. $\frac{11k}{24}$ **21.** $\frac{-13+3x}{12}$ **22.** $\frac{4+3x}{18}$ **23.** $\frac{-3x+10}{2x-4}$ **24.** $\frac{a+2}{2a+6}$ **B 25.** $\frac{1}{x-2}$ **26.** $\frac{1}{m-2}$ **27.** $\frac{1}{2x-10}$ **28.** $\frac{3}{2x-18}$ **29.** $\frac{4b^2-a^2}{4a^2b^2}$ **30.** $\frac{6n^2-3m^2}{4m^2n^2}$ **31.** 0 **32.** 0 **33.** $\frac{2y^2+3y-2}{9-y^2}$ **34.** $\frac{z^2+5z+13}{z^2-25}$ **C 35.** $\frac{4a^2+31a-54}{25(a+2)^2}$ **36.** $\frac{x^2+3x}{(x+1)(x-1)^2}$ **37.** $\frac{-10a-7b}{(2a+b)(a+2b)}$ **38.** $\frac{-x^2-5xy}{(x+y)^2(x-y)}$

Pages 303–304 Written Exercises A 1. $\frac{y^2+y+1}{y+1}$ **2.** $\frac{ab+b^2+a}{a+b}$ **3.** $\frac{2x}{x+y}$ **4.** $\frac{4a-2b}{a-b}$ **5.** $\frac{a^2}{a-1}$ **6.** $\frac{x^2}{x+2}$ **7.** $\frac{x+1}{x-1}$ **8.** $\frac{x+2}{x-2}$ **9.** $\frac{x^2+2xy+y^2}{xy}$ **10.** $\frac{a^2+2ab+b^2}{ab}$ **11.** $\frac{s^2}{s-2}$ **12.** $\frac{x^2}{x-4}$ **13.** $88\frac{4}{15}$ **14.** $199\frac{4}{13}$ **15.** $4x^2+\frac{2}{x}$ **16.** $\frac{4}{y}-y$ **17.** $2-\frac{1}{3a^3}$ **18.** $\frac{3}{2c^4}-2c^4$ **19.** $1+\frac{3}{ab}$ **20.** $\frac{2}{rs}-1$ **21.** $x+4+\frac{2}{x+1}$ **22.** $a+6+\frac{2}{a+1}$ **23.** $6x+5+\frac{-2}{2x-3}$ **24.** $3a+2+\frac{-21}{4a+2}$

Page 305 Written Exercises A 1. $\frac{2}{3}$ **2.** $\frac{4}{3}$ **3.** $\frac{b}{a}$ **4.** $\frac{x}{y}$ **5.** $\frac{2ab^2}{3c^2}$ **6.** $\frac{6r^2}{s^3}$ **7.** $\frac{3(k-m)}{k+m}$ **8.** $\frac{x+2}{2(x-2)}$ **9.** $\frac{x+4}{x}$ **10.** $a(a-5)$ **11.** $-\frac{20}{57}$ **12.** $-\frac{23}{26}$ **13.** $\frac{m+2}{2}$ **14.** $\frac{a}{x+b}$ **15.** $\frac{m-n}{m+n}$ **16.** $\frac{t+p}{t-p}$ **B 17.** $\frac{2(a+b)}{a-b}$ **18.** $\frac{x+3y}{3(x-y)}$ **19.** $-\frac{p^2}{q^2}$ **20.** $\frac{s^3+rs^2}{r^3-r^2s+2rs^2}$ *or* $\frac{-rs^2-s^3}{r^2s-2rs^2-r^3}$ *or* $-\frac{rs^2+s^3}{r^2s-2rs^2-r^3}$ **21.** $\frac{a^2+3}{2a^2+2a}$ **22.** $\frac{-6b+18}{b^2+8b+3}$ **C 23.** $\frac{10}{7}$ **24.** $\frac{a^3+a^2+2a+1}{a^3+2a}$ **25.** $\frac{2b^2+b+2}{b^2+b+1}$ **26.** $\{-1\}$ **27.** $\{-3\}$

Pages 307–308 Written Exercises A 1. $\{1\}$ **2.** $\{3\}$ **3.** $y > 6$ **4.** $x < 10$ **5.** $\{15\}$ **6.** $\{42\}$ **7.** $c \geq -10$ **8.** $c \leq -6$ **9.** $\{\frac{3}{2}\}$ **10.** $\{-1\}$ **11.** $n > -9$ **12.** $x < -8$ **13.** $\{54\}$ **14.** $\{24\}$ **B 15.** $\{9\}$ **16.** $\{2\}$ **17.** $n \geq -7$ **18.** $n \geq -5$ **19.** $\{9\}$ **20.** $\{-12\}$ **21.** $\{-7\}$ **22.** $\{-23\}$ **23.** $\{12{,}000\}$ **24.** $\{800\}$ **C 25.** $\{-\frac{1}{2}\}$ **26.** $\{-\frac{2}{3}\}$ **27.** $\{-3\}$ **28.** $\{-\frac{1}{2}\}$ **29.** {directed nos.} **30.** $\{-\frac{3}{2}\}$

Pages 309–310 Problems A 1. \$40 **2.** \$90 **3.** \$1000 **4.** \$1200 **5.** \$1344 **6.** \$840 **7.** \$400 **8.** \$2700 **9.** \$3000 at 7%; \$6000 at 3% **10.** \$2500 at 9%; \$17,500 at 3% **11.** Mr. Paxton, \$3500; wife, \$1500 **12.** Sally, \$250; Tom, \$350 **13.** \$5000 at 4%; \$5000 at 6% **14.** \$2800 at 5%; \$2000 at $3\frac{1}{2}$% **B 15.** \$22,000 **16.** \$12,000 **17.** \$800 at 4%; \$1300 at 5% **18.** \$4500 at $3\frac{1}{2}$%; \$1500 at 6% **19.** \$6000 at 5%; \$14,000 at 4% **20.** \$9900

Page 311 Problems A 1. 1 qt **2.** 24 oz **3.** 32 lb **4.** 20 lb **5.** $33\frac{1}{3}$ lb **6.** 100 lb **7.** 1 pt **8.** 4 qt **B 9.** 160 g **10.** 2 lb **11.** 76 oz **12.** 15 oz **13.** 80 oz 50% acid; 220 oz 80% acid **14.** $10\frac{2}{3}$ qt

Pages 313–314 Written Exercises A 1. $\{4\}$ **2.** $\{-4\}$ **3.** $\{5, -2\}$ **4.** $\{4, -3\}$ **5.** $\{8, -2\}$ **6.** $\{9, -4\}$ **7.** $\{2\}$ **8.** $\{3\}$ **9.** $\{-2\}$ **10.** $\{\frac{4}{13}\}$ **11.** $\{-1\}$ **12.** $\{-2\}$ **13.** $\{\frac{1}{3}\}$ **14.** $\{-5\}$ **B 15.** $\{6, -4\}$ **16.** $\{12, -1\}$ **17.** $\emptyset$ **18.** $\emptyset$ **19.** $\{\frac{2}{5}, -1\}$ **20.** $\{9\}$ **C 21.** $\{-2\}$ **22.** $\{0\}$ **23.** $\{0, 6\}$ **24.** $\{\frac{9}{2}, -2\}$ **25.** $\emptyset$ **26.** $\emptyset$

Pages 315–316 Problems A 1. $1\frac{7}{8}$ hr **2.** $1\frac{1}{3}$ hr **3.** 15 hr **4.** 6 min **5.** $1\frac{1}{11}$ hr **6.** 15 hr **7.** $2\frac{1}{4}$ hr **8.** 24 min **B 9.** 4 hr **10.** 120 min or 2 hr **11.** 6 hr **12.** $4\frac{1}{3}$ hr **13.** $2\frac{2}{11}$ hr **14.** 24 days **15.** 20 min **16.** 36 hr

Pages 318–319 Problems A 1. 12 mph **2.** 20 mph **3.** 2 mph **4.** 20 mph **5.** 3 mph **6.** 2 mph **7.** 3 mph **8.** 1 mph **B 9.** 16,500 ft/sec **10.** 4 ohms **11.** 15 mph **12.** 3 mph **13.** 200 km/sec **14.** 55

Pages 322–323 Chapter Test 1. $\{-4, 4\}$ **2.** $\frac{3r-5s}{r+2s}$ **3.** $\frac{z+3}{z+5}$ **4.** $\frac{1}{6-t}$ *or* $\frac{-1}{t-6}$ *or* $-\frac{1}{t-6}$ **5.** $\frac{2}{3}$ *or* 2:3 **6.** 6 M, 10 M **7.** 12% **8.** $\frac{5p^2}{3q^2t^2}$ **9.** $\frac{a-5}{4a}$ **10.** $\frac{m-n}{m}$ **11.** $\frac{1}{(y+5)^2}$ **12.** $\frac{2(a-b)}{a+b}$ **13.** 1 **14.** 3 **15.** $\frac{2m^2-3p}{6m^2p}$

16. $\frac{2h^2 + 10h + 13}{(h + 2)(h + 3)}$ **17.** $\frac{4x^2 + 2x}{(x + 1)(x - 1)}$ **18.** $\frac{2w^2 + 2wv}{2w + v}$ **19.** $x + 2 + \frac{5}{4x}$ **20.** a. $\frac{1}{yt}$ b. $\frac{2}{x + a}$ c. $\frac{k}{5k + n}$
21. $x \geq -8$ **22.** $700 **23.** 5 lb **24.** $\{5\}$ **25.** 40 min **26.** 10 mph

Pages 323–326 Chapter Review **1.** zero **2.** $y = \pm 3$ **3.** $x = 2$ **4.** $x = 0$ **5.** $a = b$ **6.** $\frac{2x}{3y}$ **7.** $\frac{1}{a + b}$
8. $\frac{4x + 3y}{4xy}$ **9.** $a - 2$ **10.** -1 **11.** $1 - x$ **12.** $-2 - y$; 2, $-\frac{3}{2}$ **13.** same unit **14.** 3:1 **15.** 1:6 **16.** 8:9
17. 17 **18.** 75 **19.** 150 **20.** $2240 **21.** 48.2% **22.** $\frac{xu}{tv}$ **23.** $\frac{10a^3x^2}{21b^2t^2}$ **24.** $\frac{25}{m^2 - n^2}$ **25.** $\frac{r + t}{t}$
26. $\frac{6z^2 + 5z - 6}{6z^2 - 5z - 6}$ **27.** $\frac{2a + 2}{5a - 5}$ **28.** $\frac{3}{s^2}$ **29.** $\frac{a + 2}{2}$ **30.** $\frac{x + 5}{x - 6}$ **31.** $\frac{k(r - 3k)(r + 3k)}{9r(3r + k)}$ **32.** $\frac{3s}{t}$
33. 3 **34.** $\frac{b - 1}{b + 1}$ **35.** $\frac{6b}{(2a + b)(a + 2b)}$ **36.** $\frac{r + s}{r - s}$ **37.** $\frac{x - 1}{12}$ **38.** $\frac{2n - 5}{n - 5}$ **39.** $\frac{a + 2}{a - 2}$
40. $\frac{y^2 + 3y + 4}{(y + 2)(y + 4)}$ **41.** $\frac{1}{a}$ **42.** $\frac{x^2 + y^2}{x + y}$ **43.** $m - 2 + \frac{1}{4m}$ **44.** $w + 3 + \frac{1}{w + 2}$ **45.** $\frac{x}{n}$ **46.** $\frac{3h - a}{h}$
47. $\{75\}$ **48.** {all numbers ≥ 54} **49.** $1500 **50.** $4500 **51.** $1700 at 5%, $3400 at 6% **52.** $4000 at $3\frac{1}{2}$%, $3000 at 7% **53.** 100 **54.** 3 **55.** 3, 3 **56.** 4 **57.** $100 - x$ **58.** 400 mil **59.** $\{12\}$ **60.** $\{-3\}$ **61.** $\frac{1}{40}$, $\frac{1}{40}$ **62.** $\frac{1}{x}$
63. $\frac{3}{5}$, $\frac{24}{x}$ **64.** 45 min **65.** $s + 4$, $s - 4$ **66.** 175 mph

Pages 326–328 Cum. Review 1–8 **1.** multiplication **2.** $-3 < y < 3$ **3.** $\{-4\}$ **4.** $a < b$ **5.** sometimes
6. $-2a + k$ **7.** $10r^2$ **8.** $\{3, -4\}$ **9.** $a < 0$ **10.** $t = 0$ **11.** $t \neq 3$ **12.** $t < 0$ **13.** $t \neq -1$ **14.** F **15.** F **16.** T
17. F **18.** F **19.** $(y + 2)(2 - y)^2$ **20.** $\frac{27(z - 3)}{z + 3}$ **21.** 3:7 **22.** $x^2 + 2x + 4$ **23.** $\frac{49x^2y}{121a^3b}$ **24.** 1, -1 **25.** $\{3\}$
26. $\{-4\}$ **27.** $\{0, 4\}$ **28.** {all directed nos. except 0 and -1} **29.** $7k^2(k - 3)(k + 3)$ **30.** $(12 - h)(5 + h)$ or $-1(h - 12)(h + 5)$ **31.** $(4c - 5d)(3r - 2s)$ **32.** $zt(2z - 5t)^2$ **33.** $(7n - 7k - 5)(7n - 7k + 5)$
34. $\frac{1}{6}(3x + 1)(2x - 1)$ **35.** $\frac{2(y^2 + y - 1}{y^2 - 4}$ **36.** $\frac{5x^2 - 66x + 189}{(x - 3)^2(x + 3)^2}$ *or* $\frac{(5x - 21)(x - 9)}{(x - 3)^2(x + 3)^2}$ **37.** $\frac{2a - 3b}{a - b}$ **38.** $\frac{40t^4}{9q^5s^2}$
39. v **40.** $240 **41.** $7\frac{1}{2}$ cm **42.** 8 lb; 12 lb **43.** $\frac{3}{5}$ **44.** 300 mph **45.** 2 hr

Chapter 9. Graphs

Pages 336–337 Written Exercises **A** **1.** $a = -1, b = 4$ **2.** $a = -3, b = -1$ **3.** $a = \frac{2}{3}, b = 3$ or $b = -3$ **4.** $a = -3, b = 0$ or $b = -1$ **5.** $a = -2$ or $a = 1, b = -6$ or $b = 6$ **6.** $a = 1$ or $a = -3, b = 0$
7. $\{(-6, -3), (0, 3), (6, 9)\}$ **8.** $\{(-6, 10), (0, 4), (6, -2)\}$ **9.** $\{(-6, 38), (0, 8), (6, -22)\}$ **10.** $\{(-6, -11), (0, 7), (6, 25)\}$ **11.** $\{(-6, -24), (0, -10), (6, 4)\}$ **12.** $\{(-6, 15), (0, 6), (6, -3)\}$ **13.** $\{(2, 2)\}$ **14.** $\{(-1, -2)\}$ **15.** $\{(5, 0)\}$ **16.** $\{(3, 1)\}$ **B** **17.** $\{(-4, 3)\}$ **18.** $\{(2, -3)\}$ **19.** $\{(0, -1), (0, 1), (1, 1)\}$ **20.** $\{(-1, -3), (0, -3), (0, 0)\}$ **21.** $\{(-1, 0), (-1, 1), (-1, 2), (3, 1), (3, 2), (4, 2)\}$ **22.** $\{(-2, 0), (3, 1), (3, 0)\}$ **C** **23.** $\{(-3, 1), (0, 4)\}$ **24.** $\{(0, -3), (4, -2)\}$ **25.** $\{(1, 2)\}$ **26.** $\{(-1, 7), (3, 3)\}$

Page 342 Written Exercises **B** **13.** a. $(6, 0)$; b. $(0, -10)$ **14.** a. $(7, 0)$; b. $(0, 4)$ **15.** a. $(5, 0)$; b. $(0, -12)$ **16.** a. $(-18, 0)$; b. $(0, 4)$ **17.** a. $(0, 0)$; b. $(0, 0)$ **18.** a. $(0, 0)$; b. $(0, 0)$

Pages 345–346 Written Exercises **B** **13.** $a = -1$ **14.** $a = -1$ **15.** $a = 7$ **16.** $a = -5$ **17.** $a = 0$ **18.** $a = 3$

Page 348 Written Exercises **1.** $y = 3x + 1$ **2.** $y = -2x + 5$ **3.** $4y = -x - 8$ **4.** $3y = x - 9$ **5.** $7y = -4x$ **6.** $y = 7$ **7.** $y = -10$ **8.** $y = -x + 2$ **9.** $y = x - 1$ **10.** $y = 7x$

Page 350 Written Exercises A 1. $y = 3x + 7$ **2.** $y = 5x + 5$ **3.** $y = -2x + 12$ **4.** $y = \frac{2}{7}x + \frac{10}{7}$ *or* $7y = 2x + 10$ **5.** $y = \frac{7}{2}x + 2$ *or* $2y = 7x + 4$ **6.** $y = -\frac{8}{3}x$ *or* $3y = -8x$ **7.** $y = -\frac{3}{8}x$ *or* $8y = -3x$ **8.** $y = -5$ **9.** $y = 3$ **10.** $y = x + 1$ **11.** $y = x + 3$ **12.** $x = 0$ **13.** $x = 0$ **14.** $y = -2x$ **15.** $y = -\frac{3}{5}x$ *or* $5y = -3x$ **16.** $y = \frac{5}{2}x + 6$ *or* $2y = 5x + 12$ **17.** $y = -\frac{3}{5}x - 2$ *or* $5y = -3x - 10$ **18.** $y = -\frac{1}{2}x - 1$ *or* $2y = -x - 2$ **B 19.** $a = -7$ **20.** $a = -2$ **21.** $a = 11$ **22.** $a = 7$ **23.** $y = -x + 7$ **24.** $y = x + 3$ **25.** $y = -\frac{1}{2}x - 2$ *or* $2y = -x - 4$ **26.** $y = \frac{1}{3}x$ *or* $3y = x$ **27.** $(7, 0)$ **28.** $(7, 0)$ **29.** $(3, 3)$ **30.** $(-1, 3)$

Pages 360–361 Chapter Test 1. $c = 11$ **2.** $a = 3$, $b = 1$ **3.** $\{(-1, 0), (-1, 1), (0, 0), (0, 1), (1, 0)\}$ **4.** a. $(-6, 9)$; b. $(3, 5)$; c. $(0, 2)$; d. $(-3, -2)$; e. $(5, -6)$ **7.** $(2, 7)$, yes; $(7, 2)$, no **8.** slope $= 2$ **10.** $a = 2$ **11.** slope $= \frac{5}{3}$; y-intercept $= -8$ **12.** $y = 10x - 4$ **13.** $y = -\frac{7}{4}x + 14$ *or* $4y = -7x + 56$ **14.** $y = -2x + 11$

Pages 361–364 Chapter Review 1. ordered pair **2.** $a = c$, $b = d$ **3.** -23 **4.** $\{(-1, -8), (1, -3)\}$ **5.** number lines, origin **6.** abscissa, ordinate **7.** 3, x **8.** first, fourth quadrants **9.** third **14.** satisfy **15.** 1 **16.** constant, one **17.** straight line **19.** steepness **20.** ordinates, abscissas **21.** -5 **22.** zero **23.** no **24.** positive **25.** negative **26.** m, b **27.** ordinate, y **28.** -5; 4 **29.** $y = -7x + 1$ **30.** 11 **31.** $y = -4x + 2$ **32.** $y = \frac{1}{2}x$ *or* $2y = x$ **33.** $y = 2x - 2$ **34.** half-planes; $>$; $<$ **35.** $=$; below, broken **36.** y **38.** Missing values of y: $-4, -1, -1, -4, -9$ **40.** $(0, 2)$ **41.** $(0, 3)$ **42.** plotted **44.** parts, total **45.** central

Chapter 10. Sentences in Two Variables

Page 372 Written Exercises A 1. $\{(45, 29)\}$ **2.** $\{(75, 47)\}$ **3.** $\{(17, 14)\}$ **4.** $\{(13, 21)\}$ **5.** $\{(3, -1)\}$ **6.** $\{(1, -2)\}$ **7.** $\{(6, -\frac{1}{2})\}$ **8.** $\{(0, -2)\}$ **9.** $\{(2, -3)\}$ **10.** $\{(6, 0)\}$ **11.** $\{(-1, -1)\}$ **12.** $\{(-1, \frac{2}{3})\}$ **13.** $\{(5, 17)\}$ **14.** $\{(27, 7)\}$ **15.** $\{(-6, 4)\}$ **16.** $\{(3, -9)\}$ **17.** $\{(18, 13)\}$ **18.** $\{(17, 21)\}$ **B 19.** $\{(6, 2)\}$ **20.** $\{(2, -4)\}$ **21.** $\{(-10, -6)\}$ **22.** $\{(4, -3)\}$ **23.** $\{(2, 12)\}$ **24.** $\{(1, 5)\}$ **25.** $\{(\frac{1}{2}, \frac{1}{3})\}$ **26.** $\{(-\frac{1}{5}, -\frac{1}{3})\}$

Page 374 Problems A 1. 42 ft; 30 ft **2.** 179 A; 97 A **3.** 12, 3 **4.** 15 yr; 11 yr **5.** 13 lb; 28 lb **6.** 15¢; 50¢ **7.** 17 doz; 11 doz **8.** 3 doz; 2 doz **9.** \$6.80; \$4.35 **10.** 25; 35

Pages 376–377 Written Exercises A 1. $\{(98, 25)\}$ **2.** $\{(82, 30)\}$ **3.** $\{(\frac{1}{2}, \frac{1}{3})\}$ **4.** $\{(0, -2)\}$ **5.** $\{(17, 11)\}$ **6.** $\{(9, 0)\}$ **7.** $\{(4, -\frac{1}{2})\}$ **8.** $\{(-4, -6)\}$ **9.** $\{(\frac{1}{3}, \frac{1}{4})\}$ **10.** $\{(-2, 2)\}$ **11.** $\{(-3, -7)\}$ **12.** $\{(2, 3)\}$ **B 13.** $\{(\frac{1}{2}, -1)\}$ **14.** $\{(-\frac{3}{2}, \frac{1}{3})\}$ **15.** $\{(-8, 3)\}$ **16.** $\{(-3, -\frac{1}{2})\}$ **17.** $\{(1, -2)\}$ **18.** $\{(-\frac{1}{3}, 0)\}$ **19.** $\{(2\text{b}, -3a)\}$ **20.** $\{(-3s, -5r)\}$

Page 377 Problems 1. \$70 **2.** 22, 8 **3.** 32 yr **4.** \$6500 **5.** 100 M; 250 M **6.** 24 in, 10 in **7.** 8¢ **8.** 67¢, 52¢ **9.** 65¢; 15¢ **10.** \$1.35; \$1.50 **11.** \$150; \$450 **12.** \$800; \$1300

Page 378 Written Exercises A 1. $\{(3, 1)\}$ **2.** $\{(1, 5)\}$ **3.** $\{(6, 5)\}$ **4.** $\{(9, 7)\}$ **5.** $\{(11, -1)\}$ **6.** $\{(3,.-2)\}$ **7.** $\{(7, 3)\}$ **8.** $\{(5, 2)\}$ **9.** $\{(7, 6)\}$ **10.** $\{(5, 4)\}$ **11.** $\{(\frac{33}{5}, \frac{22}{5})\}$ **12.** $\{(12, 9)\}$ **B 13.** $\{(6, 3)\}$ **14.** $\{(2, 10)\}$ **15.** $\{(18, 10)\}$ **16.** $\{(4, 2)\}$ **17.** $\{(5, 4)\}$ **18.** $\{(1, 7)\}$

Page 379 Problems A 1. 1, -2 **2.** 196 mi; 104 mi **3.** 5.5 blk.; 3.5 blk. **4.** 318 **5.** \$2000; \$1000 **6.** 400 cards/min; 800 cards/min **7.** $\frac{1}{8}$; $\frac{1}{12}$ **8.** \$450; \$750 **9.** \$750 **10.** $\frac{2}{1}$

Pages 382–383 Problems A 1. 58 **2.** 48 **3.** 36 **4.** 93 **5.** 25 **6.** 48 **7.** 53 **8.** 42 **9.** \$45 **10.** 52 **B 11.** 263 **12.** 937 **13.** 59 **C 15.** .45 **16.** .63 **17.** .49 **18.** .73

Pages 384–385 Problems A 1. 6 mph **2.** $17\frac{1}{2}$ mph **3.** 30 mph **4.** 330 mph **5.** $1\frac{1}{2}$ mph **6.** $5\frac{1}{2}$ mph **B 7.** 4 mph; 3 mph **8.** $8\frac{4}{7}$ mph; $4\frac{1}{2}$ mph **9.** 150 mph; 20 mph **10.** $s = 3c$

Pages 385–386 Problems A 1. 5 yr **2.** 4 yr **3.** 10 yr **4.** 15 yr; 20 yr **5.** 16 yr; 20 yr **6.** 6 yr **B 7.** 15 yr **8.** $\text{M} = \frac{4}{3}J$ **9.** 12 yr, 9 yr **10.** $\text{B} = 6b$

Page 387 Problems A 1. $\frac{5}{8}$ **2.** $\frac{4}{9}$ **3.** $\frac{5}{8}$ **4.** $\frac{4}{9}$ **5.** $\frac{9}{12}$ **6.** $\frac{8}{12}$ **B 7.** $\frac{24}{42}$ **8.** $\frac{3}{21}$ **9.** $\frac{39}{93}$ **10.** $\frac{214}{412}$ **C 11.** $\frac{12}{21}, \frac{24}{42}, \frac{36}{63}, \frac{48}{84}$ **12.** $\frac{102}{201}, \frac{204}{402}, \frac{306}{603}, \frac{408}{804}$

Pages 389–390 Chapter Test 2. $\{(\frac{45}{2}, \frac{29}{2})\}$ **3.** $\{(2, -1)\}$ **4.** $\{(28, 22)\}$ **5.** 20 **6.** $\{(41, 15)\}$ **7.** $\{(2, 3)\}$ **8.** $\{(-3, -5)\}$ **9.** $\{(9, -15)\}$ **10.** $\{(-12, 21)\}$ **12.** 78 **13.** 2 mph **14.** 35 yr; 15 yr **15.** $\frac{9}{12}$

Pages 390–393 Chapter Review **1.** same axes **2.** ordered pair; also, solution set **3.** root **4.** (6, 2) **5.** (−6, −4) **6.** (3, 5) **7.** $\{(5, 1)\}$ **8.** $\{(-30, -100)\}$ **9.** $\{(4, 3)\}$ **10.** equals **11.** $y + 5 = 2x;\ x + y = 100$ **12.** $l = 5w;\ 2l + 2w = 72$ **13.** $5\frac{1}{2}$ ft; $1\frac{1}{2}$ ft **14.** $\{(5, 7)\}$ **15.** $\{(\frac{6}{7}, \frac{1}{2})\}$ **16.** $\{(2, 3)\}$ **17.** $\{(8, -3)\}$ **18.** $\{(-2, -22)\}$ **19.** $\{(-4, 7)\}$ **20.** half-plane **21.** (0, 10) **22.** (0, −5) **23.** position (place) **24.** 3, 1 **25.** 97 **26.** 84 **27.** greater **28.** 205 **29.** 16 mph **30.** 20 mph **31.** 16 yr, 20 yr **32.** 13 yr, 18 yr **33.** 42 yr, 16 yr **34.** $\frac{10}{15}$ **35.** $\frac{4}{10}$

Chapter 11. The Real Numbers

Pages 399–400 Written Exercises A **1.** $\frac{8}{15} > \frac{7}{16}$ **2.** $\frac{-22}{45} < -\frac{12}{25}$ **3.** $\frac{19}{12} < \frac{56}{35}$ **4.** $-\frac{46}{49} > -\frac{53}{56}$ **5.** $\frac{200}{105} = \frac{240}{126}$ **6.** $\frac{138}{162} = \frac{184}{216}$ **7.** $-17\frac{1}{3} > -\frac{400}{12}$ **8.** $\frac{321}{15} > 21\frac{1}{5}$ **9.** $\{-\frac{4}{9}, \frac{1}{2}, \frac{16}{31}\}$ **10.** $\{-\frac{3}{7}, \frac{9}{14}, \frac{2}{3}\}$ **11.** $\{\frac{24}{21}, \frac{25}{20}, \frac{4}{3}\}$ **12.** $\{3, \frac{157}{50}, \frac{22}{7}, \frac{10}{3}\}$ **13.** $\{-\frac{11}{4}, -\frac{87}{32}, -\frac{16}{6}\}$ **14.** $\{-\frac{3}{2}, -\frac{17}{12}, -\frac{99}{70}, -\frac{7}{5}\}$ **15.** $\frac{23}{24}$ **16.** $\frac{19}{28}$ **17.** $\frac{11}{200}$ **18.** $-\frac{11}{2000}$ **19.** $-6\frac{27}{35}$ **20.** $4\frac{43}{48}$ **B** **21.** 1 **22.** $-\frac{5}{4}$ **24.** (a) $\frac{2p + q}{3}$; (b) $\frac{2p + q}{3}$ **27.** a. T; b. F; c. F; d. T

Page 403 Written Exercises A **1.** .08 **2.** .06 **3.** .21875 **4.** 1.125 **5.** $-.\overline{90}$ **6.** $-1.\overline{6}$ **7.** $-.\overline{285714}$ **8.** −.2125 **9.** 2.025 **10.** $.7\overline{428571}$ **11.** $-.2\overline{6}$ **12.** $-.\overline{380952}$ **13.** $\frac{333}{500}$ **14.** $\frac{2}{5}$ **15.** $\frac{76}{225}$ **16.** $\frac{167}{495}$ **17.** $\frac{3}{11}$ **18.** $\frac{101}{333}$ **19.** $-\frac{95}{37}$ **20.** $-\frac{541}{450}$ **B** **21.** $-.00\overline{27}$ **22.** $-.00\overline{33}$ **23.** $-.0\overline{9}$ **24.** $-.00\overline{16}$ **25.** $-.0000\overline{3}$ **26.** $-.0000\overline{09}$

Pages 405–407 Written Exercises A **1.** 22 **2.** 21 **3.** 36 **4.** 48 **5.** −34 **6.** −28 **7.** $-\frac{33}{5}$ **8.** $-\frac{22}{3}$ **9.** $\pm\frac{1}{18}$ **10.** $\pm\frac{1}{39}$ **11.** $\pm\frac{4}{35}$ **12.** $\pm\frac{7}{36}$ **13.** $\{6, -6\}$ **14.** $\{7, -7\}$ **15.** $\{\frac{1}{2}, -\frac{1}{2}\}$ **16.** $\{\frac{2}{3}, -\frac{2}{3}\}$ **17.** $\{3, -3\}$ **18.** $\{5, -5\}$ **B** **19.** 4 **20.** 31 **21.** −3 **22.** 4 **C** **25.** $\{-9\}$ **26.** $\emptyset$

Pages 410–411 Written Exercises A **1.** 1.9 **2.** 2.1 **3.** 3.2 **4.** 5.5 **5.** 42 **6.** 73 **7.** −17.8 **8.** −20.2 **9.** 156 **10.** 333 **11.** 1.41 **12.** 1.73 **13.** 7.09 **14.** 3.27 **15.** −18.87 **16.** −16.70 **17.** 5.20 **18.** 5.92 **B** **19.** $\{31.0, -31.0\}$ **20.** $\{20.1, -20.1\}$ **21.** $\{1.8, -1.8\}$ **22.** $\{3.5, -3.5\}$ **23.** $\{12.6, -12.6\}$ **24.** $\{11.2, -11.2\}$ **C** **25.** 4.58 **26.** 9.49 **27.** $\{1.1, -1.1\}$ **28.** $\{5.1, -5.1\}$ **29.** $\{1.8, -1.8\}$ **30.** $\{1.1, -1.1\}$

Page 411 Problems A **1.** 8.7 in **2.** 6.8 ft **3.** 5.29 cm **4.** 4.47 cm **5.** 2.0 M **6.** 2.0 M **7.** 26.1 M; 8.7 M **8.** 60.2 cm; 30.1 cm

Page 413 Written Exercises **1.** Yes **2.** Yes **3.** Yes **4.** Yes **5.** 25.00 in **6.** 40.00 M **7.** 6.25 mi **8.** 4.14 yd **9.** 40.00 M **10.** 22.92 ft

Pages 413–414 Problems A **1.** 15 ft **2.** 30 ft **3.** 94.34 ft **4.** 22.62 in **B** **5.** 6 in, 8 in, 10 in **6.** 3 ft, 4 ft, 5 ft **7.** 21 ft, 28 ft **8.** 15.86 mi **C** **9.** 17.57 ft **10.** 33.54 in

Pages 415–416 Written Exercises A **1.** 408 **2.** 1690 **3.** 6 **4.** $2\sqrt{42}$ **5.** 1 **6.** $3\sqrt{3}$ **7.** $\frac{1}{10}\sqrt{30}$ **8.** $\frac{1}{15}$ **9.** 6 **10.** 5 **11.** $-10a$ **12.** $-12b$ **13.** $3\sqrt{5}$ **14.** $\frac{1}{2}\sqrt{11}$ **15.** $9\sqrt{2}$ **16.** $6\sqrt{2}$ **17.** $28\sqrt{2}$ **18.** 5 **19.** $\frac{1}{7}$ **20.** $\frac{3}{2}\sqrt{3}$ **21.** $9\sqrt{3}$ **22.** $4\sqrt{5}$ **23.** $\frac{3\sqrt{5}}{5}$ **24.** a^2 **25.** $18\sqrt{6}$ **26.** $\sqrt{10}$ **27.** $\sqrt{33}$ **28.** $\sqrt{3}$ **29.** $\frac{\sqrt{10}}{3}$ **30.** $6c\sqrt{2}$ **31.** $2d^2\sqrt{10d}$ **32.** $9k\sqrt{10k}$ **B** **33.** $15ab$ **34.** $-14cd$ **35.** $x - 3\sqrt{x}$ **36.** $2\sqrt{y} - y$ **37.** −24 **38.** 140 **39.** $7y\sqrt{3}$ **40.** $6x\sqrt{2}$ **41.** $20a^3$ **42.** $18b^3$ **43.** $12\sqrt{3} + 6\sqrt{2}$ **44.** $50\sqrt{2} - 25\sqrt{6}$ **45.** $-\sqrt{42}$ **46.** $\sqrt{35}$ **47.** $-2\sqrt{15}$ **48.** $.647\sqrt{35}$ **49.** $2 + \sqrt{2}$ **50.** $25\sqrt{2}$ **51.** 7 **52.** −1 **53.** $7a + 1$ **54.** $1 - 4b$

Pages 416–417 Problems B **1.** ±.9, ±.3 **2.** ±5.2, ±3.9 **3.** 13.8 in, 9.2 in **4.** 11.8 in **5.** 894.4 sq cm **6.** .2 M **C** **7.** 3.2 **8.** 3.5 in **9.** Yes **11.** 1.7 **12.** 6.9

Pages 418–419 Written Exercises A **1.** $6\sqrt{5}$ **2.** $-9\sqrt{11}$ **3.** $6\sqrt{2} + 4\sqrt{3}$ **4.** $3\sqrt{2} - 2\sqrt{5}$ **5.** $-\sqrt{2}$ **6.** $3\sqrt{3}$ **7.** $\frac{1}{2}\sqrt{2}$ **8.** $\frac{4}{3}\sqrt{6}$ **9.** $-7\sqrt{2}$ **10.** $\sqrt{2}$ **11.** $\frac{19\sqrt{7}}{2}$ **12.** $\frac{17\sqrt{6}}{2}$ **13.** $\sqrt{10}$ **14.** $12\sqrt{15}$ **15.** 0 **16.** 0 **B** **17.** $5\sqrt{3}$ **18.** $-\frac{17\sqrt{10}}{2}$ **19.** $6\sqrt{6}$ **20.** $2\sqrt{10}$ **21.** $\frac{\sqrt{3}}{3}$ **22.** $\sqrt{15}$ **C** **23.** $(4 - 4x)\sqrt{3x}$ *or* $4(1 - x)\sqrt{3x}$ **24.** $3a^3\sqrt{ca}$ **25.** $\frac{5n}{24}$ **26.** $\frac{4t}{15}$ **27.** $\{\sqrt{3}\}$ **28.** $\{2\sqrt{2}\}$ **29.** $\{1\}$ **30.** $\{4\}$

Page 420 Written Exercises A **1.** 1 **2.** 11 **3.** −1 **4.** 1 **5.** $-3 - 4\sqrt{3}$ **6.** $46 - 15\sqrt{2}$ **7.** $8 + 2\sqrt{7}$ **8.** $35 - 10\sqrt{10}$ **9.** $51 - 10\sqrt{2}$ **10.** $67 + 12\sqrt{7}$ **11.** $12\sqrt{3} + 6\sqrt{2}$ **12.** $50\sqrt{2} - 25\sqrt{6}$ **B** **13.** $21 - 10\sqrt{3}$

14. $31 + 8\sqrt{7}$ **15.** $3 + 15\sqrt{2}$ **16.** $165 - 80\sqrt{3}$ **17.** $\dfrac{\sqrt{5}+1}{4}$ **18.** $\dfrac{\sqrt{7}-1}{6}$ **19.** $\dfrac{2-3\sqrt{2}}{-7}$ *or* $\dfrac{3\sqrt{2}-2}{7}$ **20.** $\dfrac{5\sqrt{6}+6}{19}$ **21.** $-1-\sqrt{5}$ **22.** $\dfrac{4\sqrt{6}-9}{3}$ **23.** $\dfrac{3\sqrt{2}+2}{7}$ **24.** $4\sqrt{3}-6$

Page 422 Written Exercises **A** **1.** $\{2\}$ **2.** $\{\frac{1}{27}\}$ **3.** $\{\frac{5}{4}\}$ **4.** $\{\frac{7}{16}\}$ **5.** $\{4\}$ **6.** $\{\frac{25}{4}\}$ **7.** $\{2\}$ **8.** $\{12\}$ **9.** $\{-1\}$ **10.** $\{85\}$ **11.** $\{6\}$ **12.** $\{-1\}$ **13.** $\{5\}$ **14.** $\{\frac{5}{2}\}$ **15.** $\emptyset$ **16.** $\emptyset$ **17.** $\{3\}$ **18.** $\{10\}$ **19.** $\{18\}$ **20.** $\{12\}$ **B** **21.** $\{7\}$ **22.** $\{1\}$ **23.** $\{20\}$ **24.** $\{\frac{16}{3}\}$ **25.** $\{2, -2\}$ **26.** $\{4, -4\}$ **27.** $E = \dfrac{mv^2}{2}$ **28.** $h = \dfrac{7v}{22r^2}$ **29.** $\{0\}$ **30.** $\{4\}$ **31.** $\{0\}$ **32.** $\{0\}$ **C** **33.** $\{\frac{3}{2}\}$ **34.** $\{-2\}$ **35.** $\{4, 2\}$ **36.** $\{6, 3\}$ **37.** $\{10\}$ **38.** $\{13\}$ **39.** $\emptyset$ **40.** $\emptyset$

Pages 422–423 Problems **A** **1.** 100 **2.** 225 **3.** 6 **4.** 11 **5.** $A = \dfrac{p^2}{16}$; 64 sq in **6.** $h = \dfrac{2d^2}{3}$; 54 ft **7.** $a = \sqrt{c^2 - b^2}$; 8 **8.** $f = 16s^2$; 576 ft **B** **9.** 86.4 horsepower **10.** 121 ft **11.** 92.5° **12.** 10% **13.** 8, 9 **14.** 30 ft

Pages 426–427 Chapter Test **1.** $\frac{8}{15}, \frac{9}{17}, \frac{7}{16}$ **2.** $\frac{79}{48}, \frac{470}{288}, \frac{117}{72}$ **3.** $\frac{29}{28}$ **4.** $-\frac{118}{63}$ **5.** T **6.** F **7.** $.\overline{714285}$ **8.** $-.8\overline{6}$ **9.** 12.375 **10.** $\frac{8}{11}$ **11.** $\frac{8}{9}$ **12.** $\frac{781}{333}$ **13.** 30 **14.** $-\frac{1}{24}$ **15.** $\{20, -20\}$ **16.** $\{11, -11\}$ **17.** 2.17 **18.** -2.87 **19.** 15 **20.** 20 **21.** 104 **22.** $\frac{7}{30}\sqrt{10}$ **23.** 252 **24.** $\dfrac{b\sqrt{2}}{2a}$ **25.** $4\sqrt{5}$ **26.** $-2\sqrt{2}$ **27.** $\frac{26}{3}\sqrt{3}$ **28.** $\frac{1}{4}\sqrt{6}$ **29.** 7 **30.** $\dfrac{3\sqrt{3}-2}{23}$ **31.** $21 + 8\sqrt{5}$ **32.** $\dfrac{9+4\sqrt{6}}{3}$ **33.** $\{0\}$ **34.** $\emptyset$

Pages 427–429 Chapter Review **1.** addition, multiplication **2.** $\frac{37}{40}$ **3.** $\frac{29}{48}$ **4.** infinite **5.** T **6.** F **7.** 2.3125 **8.** $-.91\overline{6}$ **9.** 1.09375 **10.** $\frac{7}{11}$ **11.** $\frac{22}{15}$ **12.** $\frac{11}{37}$ **13.** 8.4 **14.** 1.1 **15.** .7 **16.** 5.0 **17.** 5; radicand **18.** two **19.** 3; -3 **20.** 32 **21.** $\pm\frac{26}{5}$ **22.** $-\frac{1}{19}$ **23.** $\{8, -8\}$ **24.** $\{\frac{1}{2}, -\frac{1}{2}\}$ **25.** $\{10, 0\}$ **26.** y, v **27.** real, rational, irrational **28.** terminating, repeating **29.** 5.92 **30.** 3.10 **31.** $-.82$ **32.** 15.81 ft **33.** hypotenuse **34.** 17 cm **35.** 9 in **36.** Yes **37.** Yes **38.** 42 **39.** $3\sqrt{3}$ **40.** 5 **41.** $\dfrac{s}{3r^2|t|}$ **42.** $2\sqrt{6} - 9$ **43.** $1 - \sqrt{14}$ **44.** denominator **45.** rationalize, rational **46.** $6\sqrt{2}$ **47.** $-\frac{4}{5}\sqrt{5}$ **48.** $\frac{5}{4}\sqrt{2}$ **49.** $\dfrac{\sqrt{3}}{5}$ **50.** $4\sqrt{2} + 6$ **51.** 0 **52.** distributive **53.** 12.3 in **54.** $45 + 20\sqrt{5}$ **55.** 17 **56.** $\sqrt{6} - 2$ **57.** $2\sqrt{3} + \sqrt{2}$ **58.** equivalent **59.** $\{18\}$ **60.** $\{\frac{27}{4}\}$ **61.** $\{9\}$ **62.** $\emptyset$ **63.** $\emptyset$ **64.** $\{1\}$

Pages 429–431 Cum. Review **1.** $\dfrac{y^2}{x}$ **2.** 1 **3.** $a^2 - 2ab + b^2$ **4.** $\{3, -1\}$ **5.** x^2z^2 **6.** $(10{,}000 - x)$ dollars **7.** $r - c$ **8.** subset **9.** multiplicative inverse **10.** (a) **11.** (d) **12.** (b) **13.** (c) **14.** (e) **15.** $\{-3, 1\}$ **16.** $\{9\}$ **17.** $\{0, 3\}$ **18.** $\{(-1, -3)\}$ **19.** $r \geq -1$ **20.** $\{(2, -3)\}$ **22.** y increases **23.** $\frac{7}{12}$ **24.** .12 **25.** $\frac{2}{3}$ **26.** $5n^2\sqrt{2}$ **27.** $3\sqrt{3}$ **28.** 1 **29.** $\{4\}$ **30.** 9.5 in **31.** 15 **32.** Hard coal, \$24 per ton; soft coal, \$20 per ton **33.** $4\frac{1}{2}$ mph **34.** Ruth, 150 votes; Alan, 350 votes

Chapter 12. Functions and Variation

Pages 437–438 Written Exercises **A** **5.** Domain, $\{0, 1, 2\}$; range, $\{2, 3, 4\}$ **6.** Domain, $\{1, 2, 3\}$; range, $\{-1, 0, 1\}$ **7.** Domain, $\{-1, 3, 0\}$; range, $\{4, -2, 0\}$ **8.** Domain, $\{5, -2, 3\}$; range, $\{0, 3, 1, 6\}$ **9.** Domain, $\{-5, 4, 6, -7\}$; range, $\{4, -4\}$ **10.** Domain, $\{-6, -8, -9, -10\}$; range, $\{-1, -2\}$ **11.** $c = 32n$; gallons: 3, 12; cents: 128, 416 **12.** $A = 7h$; height: 7, 15; area: 63, 140 **13.** $F = 3.28$ M; meters: 6; feet: 3.28, 13.12 **14.** $P = 2.2$K; kilograms: 15; pounds: 2.2, 4.4 **15.** $y = x \cdot 0$ *or* $y = \dfrac{0}{x}$; x: any value except 0; y: 0 **16.** $y = |x|$; y: 1 **17.** $q = |p|$; q: 7 **18.** $y = 2x + 3$; y: 13 **19.** $t = n - 2$ **20.** $d = n - 3$ **21.** $t = d + 1$ **22.** $s = (n - 2)180°$

Pages 441–442 Written Exercises **A** **1.** a. $p = 2l + 6$ **2.** a. $d = 70t$ **3.** $c = 100m$ **4.** $f = 3.3m$ **5.** $k = .45p$ **6.** $d = 2.8p$ **7.** $l = 625d$ **8.** a. $d. = \dfrac{g}{100}$ **9.** $p = 62.4f$ **10.** $w = 3\frac{1}{2}p$ **11.** $d = 5y + 100$ **12.** $d = 7y + 200$ **13.** $d = .11m + 10$ **14.** $d = .12n + 25$

Pages 445–446 Written Exercises **A** **1.** 21 **2.** $13\frac{1}{2}$ **3.** $18\frac{2}{3}$ **4.** $8\frac{2}{3}$ **5.** 1.8 **6.** 12 **7.** $\{7\}$ **8.** $\{9\}$ **9.** $\{40\}$ **10.** $\{7\}$ **11.** $\{1\frac{5}{9}\}$ **12.** $\{17\frac{1}{6}\}$ **B** **13.** $\{6, -6\}$ **14.** $\{-\frac{1}{2}\}$ **15.** $\{-1\frac{2}{7}\}$ **16.** $y = 8$ **17.** $x = 21$ **18.** $b = 15$ **19.** $V = \frac{4\pi}{3}R^3$ **22.** 0 **23.** $\frac{5}{3}$ **24.** 2

Pages 446–447 Problems **A** **1.** 210 lb **2.** 35 bags **3.** 117 t **4.** $11\frac{3}{7}$ t **5.** 1200 g **6.** 250 volts **7.** 166.7 ft **8.** .081 ohms **9.** 1 in **10.** $\frac{1}{16}$ in = 1 in **B** **11.** limestone, 440 lb **12.** -19.4 g **13.** \$1600 **14.** 48 divisions **15.** 80 volts **16.** 97,969 sq mi **17.** 1225 men **18.** 1 ft per 100 ft; 10 ft

Pages 450–451 Written Exercises **A** **1.** 16 **2.** $\frac{1}{12}$ **3.** 32 **4.** .6 **5.** t is halved **6.** V is divided by 3 **7.** $t = 1$ **8.** $y = 7$ **B** **9.** $1\frac{1}{2}$ ft **10.** $F = 500$ **11.** T divided by 4 **12.** $R = 24$ **13.** $S = \frac{540}{L^3}$ **14.** $R = \frac{22}{T^2}$

Pages 451–452 Problems **A** **1.** $4\frac{1}{2}$ hr **2.** 3 days **3.** 5 ft **4.** 132 lb **5.** 20 cm **6.** \$30 **7.** 3% **8.** \$2,000 **9.** 45 in **10.** 252 rev. **11.** 15 cu ft **12.** 30 amp **13.** 300 rpm **14.** 150 rpm **B** **15.** 5.8 ft **16.** $\frac{1}{2}$ the distance **17.** 375 lb **18.** 27 ohms

Pages 454–455 Written Exercises **1.** $\frac{A}{h(a+b)} = k$ or $A = kh(a+b)$ **2.** $\frac{A}{lw} = k$ or $A = klw$ **3.** $\frac{V}{hB} = k$ or $V = khB$ **4.** $\frac{V}{hr^2} = k$ or $V = khr^2$ **5.** $\frac{nt}{w} = k$ or $n = \frac{kw}{t}$ **6.** $\frac{ts}{d} = k$ or $t = \frac{kd}{s}$ **7.** $\frac{g}{hr^2} = k$ or $g = khr^2$ **8.** $\frac{c}{md} = k$ or $c = kmd$ **9.** $\frac{fr}{s^2} = k$ or $f = \frac{ks^2}{r}$ **10.** $\frac{pd}{s^2} = k$ or $p = \frac{ks^2}{d}$

Pages 455–456 Problems **A** **1.** H made 12 times as large **2.** F is $\frac{1}{8}$ as large **3.** a. 45 b. $wz^2 = 45xy$ c. 315 **4.** a. $\frac{1}{3}$ b. $rtu^2 = \frac{s^3}{3}$ c. 5 **5.** 30 boys **6.** 2 lb **7.** 1680 lb **8.** 1500 lb **9.** 2.88¢ ≐ 3¢ **10.** 4 ohms **B** **11.** 24 mph **12.** 120 heat calories

Page 457 Chapter Test **1.** Domain = {nonnegative real no.}; range = {nonnegative real no.}; formula is $p = 62.4n$. **2.** Domain = $\{0, 1, 2, 3, 4, 5\}$; range = $\{-1, 1, 3, 5, 7, 9\}$; formula is $y = 2x - 1$. **3.** $\{(-3, -15), (-2, -8), (-1, -3), (0, 0), (1, 1), (2, 0)\}$($x$ an integer) **4.** No **5.** Domain = {real no. ≥ -3 and ≤ 3}; range = {real no. ≥ -3 and ≤ 9}; it is a function. **6.** $r = 2.4$ **7.** $k = 96$; 960 ft **8.** $V = 6$ **9.** $H = 4.8$ **10.** $\left(\frac{mh}{h-2}\right)$ men **11.** 24 cu in **12.** $n = 14$

Pages 458–459 Chapter Review **1.** relation **2.** graph, roster, rule **3.** domain; range **4.** open **5.** range and domain **6.** $\{(3, 0), (1, -2), (0, 0), (-1, 4), (-3, 18)\}$ **7.** Domain = $\{-2, -1, 0, 1, 2, 3\}$; range = $\{0, 1, 2, 3\}$; formula: $y = |x|$. **8.** relation **9.** only once; many **10.** relation **11.** function **12.** yes **13.** no **14.** function **15.** ordinate; constant function **16.** kx **17.** constant, proportionality **18.** $s_1 : s_2$ **19.** proportion **20.** $b, c; a, d$ **21.** product, extremes **22.** straight line, constant, proportionality, 0 **23.** $n = 16$ **24.** $L = 336$ **25.** 100 mi **26.** function **27.** $y = \frac{k}{x}$; $x \neq 0$ **28.** $s_2 : s_1$ **29.** hyperbola **30.** multiplicative inverse **31.** 9 ft **32.** $F = 270$ **33.** $y = \frac{9}{x}$, $x \neq 0$ **34.** directly, product **35.** $m = 48$ **36.** directly, inversely **37.** 6 hr

Chapter 13. Quadratic Equations and Inequalities

Pages 466–467 Written Exercises **A** **1.** $\{\frac{5}{3}, -\frac{5}{3}\}$ **2.** $\{\frac{3}{5}, -\frac{3}{5}\}$ **3.** $\{3\sqrt{3}, -3\sqrt{3}\}$ **4.** $\{5\sqrt{5}, -5\sqrt{5}\}$ **5.** $\{\frac{1}{3}, -\frac{1}{3}\}$ **6.** $\{\frac{1}{2}, -\frac{1}{2}\}$ **7.** $\{\frac{1}{10}, -\frac{1}{10}\}$ **8.** $\{\frac{1}{6}, -\frac{1}{6}\}$ **9.** $\{\frac{4}{3}\sqrt{3}, -\frac{4}{3}\sqrt{3}\}$ **10.** $\{\frac{11}{2}\sqrt{2}, -\frac{11}{2}\sqrt{2}\}$ **11.** $\{3, -1\}$ **12.** $\{2, -4\}$ **13.** $\{-1, -3\}$ **14.** $\{3, 1\}$ **15.** $\{-\frac{1}{2}, -\frac{11}{2}\}$ **16.** $\{\frac{22}{3}, \frac{8}{3}\}$ **17.** $\{\frac{19}{3}, -\frac{17}{3}\}$ **18.** $\{\frac{34}{7}, -\frac{36}{7}\}$ **19.** $\{\frac{5}{4}, -\frac{11}{4}\}$ **20.** $\{\frac{28}{5}, -\frac{22}{5}\}$ **21.** $\{-2, -8\}$ **22.** $\{8, 4\}$ **23.** $\{\frac{12}{11}, -\frac{10}{11}\}$ **24.** $\{\frac{8}{9}, -\frac{10}{9}\}$ **B** **25.** $\{6 + \sqrt{5}, 6 - \sqrt{5}\}$ **26.** $\{-6 + \sqrt{7}, -6 - \sqrt{7}\}$ **27.** $\{-1 + \sqrt{3}, -1 - \sqrt{3}\}$ **28.** $\{1 + \sqrt{2}, 1 - \sqrt{2}\}$ **29.** $\{0, 2, -2\}$ **30.** $\{0, 3, -3\}$ **31.** $\{0, 5, -5\}$ **32.** $\{0, 4, -4\}$ **33.** $\{0, \sqrt{5}, -\sqrt{5}\}$ **34.** $\{0, \sqrt{7}, -\sqrt{7}\}$ **35.** $\{0, -2\}$ **36.** $\{0, 3\}$ **C** **37.** $\{\sqrt{2}\}$ **38.** $\{-\sqrt{3}\}$ **39.** $\{3 + \sqrt{2}, -3 + \sqrt{2}\}$ **40.** $\{2 + \sqrt{3}, -2 + \sqrt{3}\}$ **41.** no **42.** no

Pages 468–469 Written Exercises A 1. correct **2.** correct **3.** correct **4.** incorrect **5.** incorrect **6.** correct **7.** incorrect **8.** incorrect **9.** correct **10.** incorrect **11.** correct **12.** incorrect **B 13.** $x^2 - 7x + 10 = 0$ **14.** $x^2 - x - 6 = 0$ **15.** $x^2 + 8x + 16 = 0$ **16.** $x^2 + 3x = 0$ **17.** $x^2 - 7 = 0$ **18.** $x^2 - 3 = 0$ **19.** $x^2 - 2x - 1 = 0$ **20.** $x^2 - 6x + 4 = 0$ **21.** $x^2 - 6\sqrt{5}\,x + 45 = 0$ **22.** $x^2 + \frac{7}{2}x + 3 = 0$ *or* $2x^2 + 7x + 6 = 0$

Pages 471–472 Written Exercises A 1. $\{-1 + 2\sqrt{2}, -1 - 2\sqrt{2}\}$; $\{1.8, -3.8\}$ **2.** $\{-2 + 3\sqrt{2}, -2 - 3\sqrt{2}\}$; $\{2.2, -6.2\}$ **3.** $\{2, -3\}$ **4.** $\{3, 1\}$ **5.** $\{4 + \sqrt{14}, 4 - \sqrt{14}\}$; $\{7.7, .3\}$ **6.** $\{-3 + \sqrt{5}, -3 - \sqrt{5}\}$; $\{-.8, -5.2\}$ **7.** $\left\{\frac{-7 + \sqrt{29}}{2}, \frac{-7 - \sqrt{29}}{2}\right\}$; $\{-.8, -6.2\}$ **8.** $\left\{\frac{5 + \sqrt{13}}{2}, \frac{5 - \sqrt{13}}{2}\right\}$; $\{4.3, .7\}$ **9.** $\{3, 0\}$ **10.** $\{0, -2\}$ **11.** $\{19, 1\}$ **12.** $\{23, 1\}$ **13.** $\left\{\frac{5 + 3\sqrt{5}}{2}, \frac{5 - 3\sqrt{5}}{2}\right\}$; $\{5.9, -.9\}$ **14.** $\left\{\frac{-3 + 3\sqrt{13}}{2}, \frac{-3 - 3\sqrt{13}}{2}\right\}$; $\{3.9, -6.9\}$ **15.** $\{-\frac{3}{2}\}$ **16.** $\{\frac{1}{3}\}$ **17.** $\left\{\frac{-7 + \sqrt{61}}{6}, \frac{-7 - \sqrt{61}}{6}\right\}$; $\{.1, -2.5\}$ **18.** $\left\{\frac{4 + \sqrt{21}}{5}, \frac{4 - \sqrt{21}}{5}\right\}$; $\{1.7, -.1\}$ **B 19.** $\{0, -\frac{2}{3}\}$ **20.** $\{1\frac{1}{2}, 0\}$ **21.** $\left\{\frac{-5 + 5\sqrt{17}}{4}, \frac{-5 - 5\sqrt{17}}{4}\right\}$; $\{3.9, -6.4\}$ **22.** $\left\{\frac{-1 + 2\sqrt{19}}{5}, \frac{-1 - 2\sqrt{19}}{5}\right\}$; $\{1.5, -1.9\}$ **23.** $\left\{\frac{3 + \sqrt{5}}{2}, \frac{3 - \sqrt{5}}{2}\right\}$; $\{2.6, .4\}$ **24.** $\left\{\frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}\right\}$; $\{1.6, -.6\}$ **25.** $\{2, 0\}$ **26.** $\{\frac{1}{3}, 0\}$ **27.** $\left\{\frac{9 + 3\sqrt{5}}{2}, \frac{9 - 3\sqrt{5}}{2}\right\}$; $\{7.9, 1.1\}$ **28.** $\left\{\frac{-1 + \sqrt{10}}{3}, \frac{-1 - \sqrt{10}}{3}\right\}$; $\{.7, -1.4\}$

Pages 472–473 Problems A 1. 14 in, 16 in **2.** 7 ft, 17 ft **3.** 15 ft × 12 ft, 17 ft × 30 ft **4.** 10 in, 15 in **5.** 30 mph, 40 mph **6.** 12¢ **7.** $12 **8.** 270 mph **9.** 5 **10.** 9

Page 475 Written Exercises A 1. $\left\{\frac{-5 + \sqrt{13}}{6}, \frac{-5 - \sqrt{13}}{6}\right\}$; $\{-.2, -1.4\}$ **2.** $\left\{\frac{-7 + \sqrt{17}}{8}, \frac{-7 - \sqrt{17}}{8}\right\}$; $\{-.4, -1.4\}$ **3.** $\left\{\frac{4 + \sqrt{10}}{2}, \frac{4 - \sqrt{10}}{2}\right\}$; $\{3.6, .4\}$ **4.** $\left\{\frac{3 + \sqrt{5}}{4}, \frac{3 - \sqrt{5}}{4}\right\}$; $\{1.3, .2\}$ **5.** $\{4, -\frac{3}{2}\}$ **6.** $\{\frac{7}{3}, -\frac{5}{2}\}$ **7.** $\{-2 + \sqrt{7}, -2 - \sqrt{7}\}$; $\{.7, -4.7\}$ **8.** $\{-3 + \sqrt{13}, -3 - \sqrt{13}\}$; $\{.6, -6.6\}$ **9.** $\{1 + \sqrt{2}, 1 - \sqrt{2}\}$; $\{2.4, -.4\}$ **10.** $\left\{\frac{-1 + 3\sqrt{5}}{2}, \frac{-1 - 3\sqrt{5}}{2}\right\}$; $\{2.9, -3.9\}$ **11.** $\{\frac{3}{5}, \frac{1}{4}\}$ **12.** $\{\frac{3}{2}, \frac{1}{5}\}$ **13.** $\{\frac{1}{3}, 0\}$ **14.** $\{\frac{1}{5}\sqrt{85}, -\frac{1}{5}\sqrt{85}\}$; $\{1.8, -1.8\}$ **B 15.** $(x - 1 - \sqrt{3})(x - 1 + \sqrt{3})$ **16.** $(x - 1 - \sqrt{5})(x - 1 + \sqrt{5})$ **17.** $(y + 3 - \sqrt{6})(y + 3 + \sqrt{6})$ **18.** $(y + 4 - \sqrt{3})(y + 4 + \sqrt{3})$ **19.** $x = \frac{2 \pm \sqrt{-4}}{2}$ **20.** $x = \frac{2 \pm \sqrt{-12}}{2}$ **21.** $y = \frac{-1 \pm \sqrt{-3}}{2}$ **22.** $y = \frac{-2 \pm \sqrt{-32}}{2}$

Pages 475–476 Problems A 1. 7 ft × 14 ft; 7 ft × 7 ft **2.** 2 in, 6 in **3.** 21.8 in sq **4.** 4.4 in sq **5.** $\frac{1}{2}$ ft, $\frac{2}{3}$ ft, $\frac{5}{6}$ ft **6.** 5 mph, 12 mph **B 7.** 50 mph **8.** 30 in; 41 in **9.** a. 5¢ or $2.20; b. two possible, one practical **10.** a. 300; b. one possible

Page 478 Written Exercises A Ex. 1, 2, 4, 5, 6, 10: Two different real roots. Ex. 3, 7, 8, 9: No real roots **B 11.** $(x - 1)(x - 1)$ **12.** $(x + 2 - 2\sqrt{2})(x + 2 + 2\sqrt{2})$ **13.** $2\left(x - 1 + \frac{\sqrt{2}}{2}\right)\left(x - 1 - \frac{\sqrt{2}}{2}\right)$ **14.** not factorable **15.** $(u + \frac{1}{2})(u - \frac{1}{4})$ **16.** not factorable **17.** $y(\frac{1}{4} - y)(\frac{1}{2} + \frac{3}{2}y)$ **18.** $\frac{1}{15}z(1 + 5z)(1 - 2z)$

Page 479 Problems A 1. (a) 3 sec; (b) 2 sec; (c) 3.4 sec **2.** 6 ft **3.** 5 **4.** 4 **5.** 7.0 mi/sec **6.** (a) 338.4 mi/min; (b) 46.3 mi/min; (c) 341.2 mi/min **7.** 23.3 mph **8.** 13, −11 **9.** 7.2 units **11.** 178.9 mi **12.** 24.8 mi

Page 482 Written Exercises C 19. $x \geq 7$ or $x \leq -5$ **20.** $x \geq 7$ or $x \leq 5$ **21.** $x \geq 0$ or $x \leq -5$ **22.** $x \geq 1$ or $x \leq 0$

Page 484 Written Exercises C 17. $x \leq -2$ or $x \geq 5$ **18.** $x \leq 0$ or $x \leq 3$

Page 486 Chapter Test 1. $\{\frac{1}{4}, -\frac{1}{4}\}$ **2.** $\{2, -1\}$ **3.** $\{\frac{1}{2}\}$ **4.** correct **5.** incorrect **6.** $c = .49$; $(x - .7)^2$ **7.** $(x - \frac{3}{2})^2 = \frac{45}{4}$ **8.** $\{8 + 2\sqrt{14}, 8 - 2\sqrt{14}\}$; $\{15.5, .5\}$ **9.** $\left\{\frac{-3 + \sqrt{2}}{2}, \frac{-3 - \sqrt{2}}{2}\right\}$; $\{-.8, -2.2\}$ **10.** 3 in **11.** two diff. **12.** a double **13.** none **14.** none

Pages 486–489 Chapter Review 1. $u = v$; $u = -v$ **2.** $\{2, -2\}$ **3.** $\{8, -8\}$ **4.** $\{4, -6\}$ **5.** opposite, coefficient, linear, constant **6.** incorrect **7.** correct **8.** $k = 100$; $(s - 10)^2$ **9.** $k = \frac{1}{4}$; $(y + \frac{1}{2})^2$ **10.** $k = .0625$;

$(z + .25)^2$ **11.** $\{-1 + 2\sqrt{6}, -1 - 2\sqrt{6}\}$; $\{3.9, -5.9\}$ **12.** $\{3 + \sqrt{21}, 3 - \sqrt{21}\}$; $\{7.6, -1.6\}$ **13.** $\left\{\frac{-5 + \sqrt{37}}{2}, \frac{-5 - \sqrt{37}}{2}\right\}$; $\{.5, -5.5\}$ **14.** $\{3 + 2\sqrt{5}, 3 - 2\sqrt{5}\}$; $\{7.5, -1.5\}$ **15.** 6 mph **16.** 40 mph, 50 mph **17.** 0 **18.** real **19.** $\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ **20.** $\left\{\frac{-11 + \sqrt{97}}{4}, \frac{-11 - \sqrt{97}}{4}\right\}$; $\{-.3, -5.2\}$ **21.** $\{9 + \sqrt{77}, 9 - \sqrt{77}\}$; $\{17.8, .2\}$ **22.** $\{5, -\frac{1}{3}\}$ **23.** $\left\{\frac{-4 + \sqrt{31}}{5}, \frac{-4 - \sqrt{31}}{5}\right\}$; $\{.3, -1.9\}$ **24.** \$10 **25.** 6 in sq **26.** $y = ax^2 + bx + c$ **28.** a. two diff.; b. a double; c. two diff.; d. none **29.** 1 sec **30.** 3.3 ft **31.** 36 **32.** $x - 2 > 0$; $x - 2 < 0$ **33.** $x - 1 \leq 0$; $x - 1 \geq 0$ **34.** no **35.** {real nos. > -1 or < -2} **36.** {real nos. ≥ -3 and ≤ 4} **37.** {real nos. ≥ 5 or ≤ -5} **38.** {real nos. > -1 and < 1} **39.** points for which $y > 0$ **40.** $(x + 1)(x - 2) < 0$, $x^2 - x < 2$

Chapter 14. Geometry and Trigonometry*

See page 587 for answers to Chapter 14 for the 1962 edition of this book.

Pages 496–497* **Written Exercises** **A** **1.** Axiom I **2.** Axiom II **3.** Axiom III **4.** Axiom II **5.** Axiom II **6.** Axiom IV **7.** Axioms IV and VI **8.** *R* **9.** *S* **10.** *S*, *W*, or *T* **11.** 6 **12.** 3 **13.** 5 **14.** 6 **15.** 6 **16.** 9

Pages 498–499* **Written Exercises** **A** **1.** $\angle PKM$ **2.** $\angle QKH$ **3.** Initial ray *KQ*, any two of the following: $\angle QKP$, $\angle QKL$, $\angle QKM$, $\angle QKR$, $\angle QKH$. Terminal ray *KQ*, any two of the following: $\angle PKQ$, $\angle LKQ$, $\angle MKQ$, $\angle RKQ$, $\angle HKQ$ **4.** Yes **5.** No **6.** Yes **7.** Straight **8.** Initial ray *LP*, terminal ray *LK* **9.** *KL* (or *KP*), *KM*, *KQ*, *KR*, *KH* **10.** A ray **11.** A point **12.** None of these **13.** A line **14.** A line segment **15.** None of these **16.** A line segment **17.** Two rays

Page 500* **Written Exercises** **A** **1.** $\triangle ABC$ is a right triangle. **2.** $\triangle JKL$ is a right triangle **3.** $\triangle DEF$ is not a right triangle **4.** $\triangle GHK$ is not a right triangle **5.** $\triangle PQR$ is a right triangle **6.** $\triangle XYZ$ is not a right triangle **7.** $AC = BC = \sqrt{2}$, or 1.414, approx. **8.** 90° **9.** 80° **10.** 90° **11.** Their sum does not equal 180° **12.** Yes

Page 506* **Written Exercises** **A** **1.** $\frac{1}{5}$ **2.** $\frac{2}{3}$ **3.** 1 **4.** 7 **5.** $-2\frac{1}{2}$ **6.** -1 **7.** $-\frac{2}{7}$ **8.** 0 **9.** 1 unit **10.** $\angle MOS$ **11.** *MK* **12.** $(1, -1)$

Pages 510–511* **Written Exercises** **A** **1.** $\frac{\sqrt{2}}{2}$ **2.** $\frac{\sqrt{2}}{2}$ **3.** $\frac{\sqrt{3}}{2}$ **4.** $-\frac{1}{2}$ **5.** 1 **6.** 0 **7.** -1 **8.** 0 **9.** (b) $\frac{3}{4}$ **10.** $\frac{5}{13}$ or $-\frac{5}{13}$ **11.** $\frac{12}{13}$ **12.** $\frac{5}{13}$ **13.** $\frac{12}{5}$ **14.** 1 **15.** 0 **16.** $\frac{3}{5}$ **17.** $\frac{4}{5}$ **18.** $\frac{3}{4}$ **20.** $\frac{1}{2}$ **21.** $\frac{1}{2}$ **22.** $\frac{\sqrt{3}}{2}$ **23.** $\frac{\sqrt{3}}{2}$

Page 512* **Written Exercises** **A** (In Ex. 1–8, the answers are given in the order: sine, cosine, tangent.) **1.** .0872, .9962, .0875 **2.** .0349, .9994, .0349 **3.** .4540, .8910, .5095 **4.** .5878, .8090, .7265 **5.** .7431, .6691, 1.1106 **6.** .8572, .5150, 1.6643 **7.** .0175, .9998, .0175 **8.** .8387, .5446, 1.5399 **9.** 33° **10.** 57° **11.** 29° **12.** 78° **13.** 30° **14.** 41°

Page 515* **Problems** **A** **1.** 30 ft **2.** 42 ft **3.** 10 ft **4.** 29 ft **5.** 37 ft **6.** 19 ft **7.** (a) 2403 ft (b) 689 ft **8.** 23 ft

Page 516* **Problems** **A** **1.** 386.2 ft **2.** 363.2 ft **3.** 100.7 ft **4.** 86.6 ft **5.** 321.7 ft **6.** 4813.5 ft **7.** 255.8 ft **8.** 3456.2 ft

Page 517* **Problems** **A** **1.** 35° **2.** 31° **3.** 7° **4** (a) 165.6 mi (b) 150.1 mi **5.** 28°, 62°, 90° **6.** 868.1 ft **B** **7.** 4.8 ft **8.** 129.8 ft

Pages 518–519* **Written Exercises** **A** **1.** 95° **2.** 9 ft **3.** 15 ft **4.** .56 meter **5.** 8.5 meters **6.** 9 in, 12 in **7.** $\frac{5}{12}$ **8.** $\frac{5}{12}$ **9.** $\frac{5}{12}$ **10.** $\frac{DE}{AE}$ and $\frac{BG}{AG}$.

Page 520* **Problems** **A** **1.** 33.75 ft **2.** 30 ft **3.** 17.5 ft **4.** 30.9 ft

Page 524* **Written Exercises** **A** **1.** No **2.** No **3.** Yes **4.** Yes **5.** Yes **6.** Yes **7.** Yes **8.** No **9.** $-\overrightarrow{MN}$ **10.** $-\overrightarrow{KL}$ **11.** $-\overrightarrow{NK}$ **12.** $-\overrightarrow{LM}$ **13.** $\overrightarrow{VS}$ **14.** $\overrightarrow{SV}$ **15.** $\overrightarrow{WR}$ **16.** $\overrightarrow{TW}$ **17.** $\overrightarrow{ST}$ **18.** $\overrightarrow{WS}$ **B** **19.** 122 lb at 35° **20.** 528 lb at 170° **21.** 100 kg at 307° **22.** Approx. 166 g at 245°

Pages 524–525* **Problems** **A** **1.** 141 lb at 45° **2.** Approx. 13 mph at 67° **3.** Approx. 151 mph at a bearing of 262° *or* at an angle of 188° **4.** Approx. 211 mph at a bearing of 276° **5.** Approx. 534 lb at an angle of 68° (or 112°) with the ground. **6.** Approx. 4326 lb at 324°

Page 526* **Problems** **A** **1.** 36 lb **2.** 17 lb **3.** 12 lb **4.** 130 lb **5.** 1128 lb **6.** 635 lb **7.** Force of the wind, 4285 lb; pull on the rope, 5230 lb **B** **8.** 177 lb

Page 528* **Chapter Test** **1.** *S* **2.** Ray **3.** None of these **4.** 71° **5.** d **6.** 2 **7.** 2.4 **8.** 12 ft **9.** 85 ft **10.** 162 lb **11.** Horizontal, 38 lb; vertical, 140 lb

Pages 529–530* **Chapter Review** **1.** Points **2.** two different, three **3.** *Q*, *P*, *R* **4.** $x \geq 5$ **5.** straight **6.** 70° **7.** 3 **8.** *OX* **9.** $\frac{2}{3}$ **10.** *NQ* **11.** *OM* **12.** 1 **13.** .8192 **14.** 72° **15.** 20 ft **16.** 62 ft **17.** Equal **18.** 40, 48 **19.** Yes **21.** 141 lb, 45° **22.** 5.4 mph at 22° east of north **24.** Horizontal component, 114.7 lb; vertical component, 163.8 lb

*The asterisks identify page references to the 1965 edition of the text.

Chapter 14. Geometry and Trigonometry

Pages 494–495 Written Exercises **1.** I **2.** II **3.** IV **4.** II **5.** II **6.** IV or III **7.** IV **8.** R **9.** S **10.** S, W, or T **11.** 6 **12.** 3 **13.** 5 **14.** 6 **15.** 6 **16.** 9

Pages 497–498 Written Exercises **17.** 80° **18.** 60° **19.** 90° **20.** 90° **21.** 55° **22.** 66° **23.** 20° **24.** 40°

Page 500 Written Exercises **1.** 95° **2.** 85° **3.** 9 ft **4.** 15 ft **5.** .26 M **6.** 8.5 M **7.** 9 in, 12 in **8.** 15 M, $22\frac{1}{2}$ M **9.** $\frac{5}{12}$ **10.** $\frac{5}{12}$ **11.** $\frac{5}{12}$ **12.** $\frac{BG}{AG}$, $\frac{DE}{AE}$ **13.** $\frac{AC}{AF}$, $\frac{AD}{AE}$ **14.** 12 in

Pages 500–501 Problems **1.** 30 ft **2.** $42\frac{6}{7}$ ft **3.** 30 ft **4.** $17\frac{1}{2}$ ft **5.** $30\frac{6}{7}$ ft **6.** 34 ft

Pages 503–504 Problems **1.** 30 ft **2.** 42 ft **3.** 101 ft **4.** 87 ft **5.** 322 ft **6.** 4813 ft **7.** 35° **8.** 31° **9.** 75 ft **10.** 320 ft

Pages 505–506 Written Exercises **1–6.** $\frac{4}{5}$ **7–14.** $\frac{3}{5}$ **15, 16.** $\frac{4}{5}$ **17.** $\frac{4}{3}$ **18.** $\frac{3}{4}$

Page 506 Problems **1.** 386.3 ft **2.** 363.2 ft **3.** 10.4 ft **4.** 29.0 ft **5.** 37.4 in **6.** 25.9 ft **7.** 255.8 ft **8.** 3403.3 ft or 3456.2 ft **9.** 23.5 ft **10.** 269.1 ft

Page 507 Written Exercises **1.** .0872, .9962, .0875 **2.** .0349, .9994, .0349 **3.** .9511, .3090, 3.0777 **4.** .4540, .8910, .5095 **5.** .5878, .8090, .7265 **6.** .8660, .5000, 1.7321 **7.** .7431, .6691, 1.1106 **8.** .8572, .5150, 1.6643 **9.** .9998, .0175, 57.2900 **10.** .0175, .9998, .0175 **11.** .8387, .5446, 1.5399 **12.** .7880, .6157, 1.2799 **13.** 33° **14.** 57° **15.** 29° **16.** 68° **17.** 78° **18.** 50° **19.** 30° **20.** 41° **21.** 53°

Pages 507–508 Problems **A** **1.** (a) 169.1 mi; (b) 61.6 mi **2.** (a) 10.6 ft; (b) 33.9 ft **3.** (a) 2403.3 ft; (b) 689.0 ft **4.** 128.6 ft **5.** 119.2 ft **6.** 1177.6 ft **7.** 7° **8.** (a) 165.6 mi; (b) 150.1 mi **9.** 28°; 52°; 90° **10.** 868.1 ft **B** **11.** 4.8 yd **12.** 129.8 ft

Page 511 Written Exercises **1.** No **2.** No **3.** Yes **4.** Yes **5.** Yes **6.** Yes **7.** Yes **8.** No **9.** $-\overrightarrow{MN}$ *or* $\overrightarrow{NM}$ **10.** $-\overrightarrow{KL}$ *or* $\overrightarrow{LK}$ **11.** $-\overrightarrow{NK}$ *or* $\overrightarrow{KN}$ **12.** $-\overrightarrow{LM}$ *or* $\overrightarrow{ML}$ **13.** $\overrightarrow{VS}$ **14.** $\overrightarrow{SV}$ **15.** $\overrightarrow{WR}$ **16.** $\overrightarrow{TW}$ **17.** $\overrightarrow{ST}$ **18.** $\overrightarrow{WS}$ **19.** (b) 122 lb at angle of 35° **20.** (b) 518 lb at angle of 170° **21.** (b) 100 kg at angle of 307° **22.** (b) 166 g at angle of 245°

Page 511–512 Problems **1.** 142 lb at an angle of 45° **2.** 13 mph at a bearing of 67° **3.** 151 mph at a bearing of 262° or an angle of 188° **4.** 211 mph at a bearing of 276° **5.** 534 lb at an angle of 68° with the ground **6.** 4326 lb at a bearing of 324°

Page 513 Written Exercises **1.** 106 lb horizontally; 106 lb vertically downward **2.** 105 lb horizontally; 182 lb vertically downward **3.** 376 mph east; 137 mph north **4.** 222 mph east; 611 mph north **5.** 18 mph east; 18 mph north **6.** 11 mph west; 11 mph south **7.** (14.7, 8.5) **8.** (4.2, 4.2) **9.** (10.0, .7) **10.** (.6, 8.0) **11.** (9, 9) **12.** (16, 12) **13.** (13, −2) **14.** (1, 9)

Page 514 Problems **1.** 36 lb **2.** 17 lb **3.** 12 lb **4.** 130 lb **5.** 1128 lb **6.** 308 lb **7.** 4284 lb; 5230 lb **8.** 88 lb **9.** 17 lb **10.** 93 lb **11.** 104 lb horizontally; 57 lb vertically **12.** 137 lb horizontally; 6 lb vertically

Pages 515–516 Chapter Test **1.** F **2.** T **3.** 115° **4.** a ray **5.** a line segment **6.** a point **7.** none of these **8.** 68 **9.** 240 ft **10.** .3 mi vertically; 2.0 mi horizontally **11.** 12 ft **12.** 161 *or* 162 lb **13.** 56° **14.** 38 lb horizontally, 140 lb vertically

Pages 516–518 Chapter Review **1.** two diff., three **2.** C, Q, P **3.** $x \geq 5$ **4.** 180° **5.** equal **6.** 40, 48 **7.** yes **8.** 16 **9.** 14 **10.** 2 : 5 **11.** opposite, adjacent **12.** $a:b$ **13.** $b:a$ **14.** 55 ft **15.** .6 in **16.** 1.1 in **17.** 2.0 in **18.** $a:c$ **19.** sin 55° **20.** .8192 **21.** .1736 **22.** .2309 **23.** 12° **24.** 72° **25.** 48° **26.** 56° **27.** 20 ft **28.** 62 ft **29.** 32° **31.** 141 lb; 45° **32.** 5.4 mph at a bearing of 22° **34.** 115 lb horizontally; 164 lb vertically

Chapter 15. Comprehensive Review and Tests

Properties of Numbers: Structure **1.** Distrib. Prop. **2.** Comm. Prop. for addition **3.** Meaning of subt. **4.** Prop. of opp. **5.** Mult. prop. of equality **6.** Theorem: a product is zero if, and only if, at least one of the factors is zero. **7.** Distrib. prop. **8.** Def. of an exponent **9.** Subt. prop. of equality **10.** Meaning of division **11.** Order prop. of numbers **12.** Closure prop. for addition **13.** Def. of subset **14.** Def. of opposite of zero; no other number has this prop. **15.** Def. of multiple **16.** Trans. prop. of inequality **17.** Def. of equivalent equations **18.** F **19.** T **20.** F **21.** F **22.** T **23.** F **24.** F **25.** (25, E) **26.** (26, B) *or* (26, C) **27.** (27, D) **28.** (28, A) **29.** (29, F) **30.** (30, C)

Algebraic Representation **1.** $\frac{a+n+p}{3}$ **2.** $p = 12n$ **3.** $c = \sqrt{n^2 + 25}$ **4.** $(12f - 12t - 3)$ in **5.** $\frac{25n}{c}$ **6.** $\frac{c - nt}{p}$ **7.** (a) $1.2m$ dollars; (b) $.9m$ dollars **8.** $12s \leq n < t$ **9.** $49\pi < A < 64\pi$ **10.** 9 **11.** −8

12. 16 **13.** -15 **14.** 5 **15.** -12 **16.** 59. **17.** 5.8 **18.** .1 **19.** 1. **20.** 1.0 **21.** .4 **22.** 7.5 (7.6 from table) **23.** 2.7 **24.** 1 : 2 **25.** 1 : 3 **26.** 7 : 13 **27.** 1 : 27 **28.** 1512.0 sq in **29.** 14 **30.** 7.7, 9.2

Fundamental Operations and Factoring **1.** $2(a-1)$ **2.** $6m^5n^2 - 10m^2n^3$ **3.** $6x^2 - x - 2$ **4.** $a^3 - 27b^3$ **5.** $2a - b$ **6.** $1 - 8a + 16a^2$ **7.** $x^2 - 2x + 4$ **8.** $-6x^2 + 2ax - a^2$ **9.** $5t^2 - 2t + 11 + \frac{-26}{t+3}$ **10.** $-10a + 8b + 2$ **11.** $18c + 33$ **12.** $6a^2 + 2ac - 3ab - bc$ **13.** $\frac{5(b+3)}{b}$ **14.** $\frac{3y}{20a^3x}$ **15.** $\frac{x+1}{x-1}$ **16.** $\frac{2}{a-7}$ **17.** $3c^2 + c$ **18.** $3x^2(1 - 7x)$ **19.** $(1 + 3a)(1 - 3a)$ **20.** $2\pi R(2R - H)$ **21.** $(y - 4)(y + 3)$ **22.** $(x - 5)^2$ **23.** $P(1 + RT)$ **24.** $(9x + 1)(4x - 1)$ **25.** $(d + .2)(d - .2)$ **26.** $\frac{1}{2}h(b_1 + b_2)$ **27.** $2(h^2 + 9)(h + 3)(h - 3)$ **28.** $(n + 3)(n - 3)(n + 4)(n - 4)$ **29.** $3r(6r - 5)(r + 6)$ **30.** $\frac{6a+5}{6(a+1)}$ **31.** $\frac{2h}{h^2-4}$ **32.** $\frac{3y^2 + 6xy - 2x^2}{x^2y^2}$ **33.** $\frac{6r^2 - 5rs - 10s^2}{12rs}$ **34.** $\frac{3 + x - x^2}{x^2 - 1}$ **35.** $\frac{-2}{t-1}$ *or* $\frac{2}{1-t}$ **36.** $\frac{2a-1}{a^2-b^2}$ **37.** $\frac{1}{2}$ **38.** $\frac{c(2c-1)}{6}$ **39.** $\frac{1-r}{1+r}$

Radicals **1.** $2\sqrt{7}$ **2.** $4|a|\sqrt{5}$ **3.** b^8 **4.** $2\sqrt{3}$ **5.** $\frac{1}{2}\sqrt{14}$ **6.** $\frac{2}{3}\sqrt{3}$ **7.** $6\sqrt{a^2 + b^2}$ **8.** $\frac{3}{2}\sqrt{2}$ **9.** $\frac{\sqrt{6}}{6}$ **10.** $5\sqrt{2}$ **11.** $1 + \sqrt{2}$ **12.** $\frac{\sqrt{2a}}{2}$ **13.** $\sqrt{2} - 1$ **14.** $\frac{3(3 + \sqrt{5})}{2}$ **15.** $2\sqrt{2}$ **16.** $-13\sqrt{2}$ **17.** $3 + 4\sqrt{3}$ **18.** $3\sqrt{2} - 14\sqrt{3}$ **19.** $-\frac{15\sqrt{6}}{4}$ **20.** 0 **21.** $3\sqrt{2} - 2\sqrt{3}$ **22.** $24\sqrt{3} - 24\sqrt{2}$ **23.** 15 **24.** $24\sqrt{6}$ **25.** 5.4 **26.** 20.6 **27.** 6.1 **28.** 2.9 **30.** $\{-\frac{2}{5}\}$ **31.** $\frac{46}{33}$ **32.** $\frac{26}{45}$

Equations **1.** $\{9\}$ **2.** $\{\frac{3}{4}\}$ **3.** $\{-2\}$ **4.** $\{10\}$ **5.** $\{\frac{2}{3}\}$ **6.** $\{-\frac{3}{2}\}$ **7.** $\{-\frac{3}{4}\}$ **8.** $\{\frac{1}{2}\}$ **9.** $\{0\}$ **10.** $\{8\}$ **11.** $\{2\}$ **12.** $\{6\}$ **13.** $\{0\}$ **14.** $\{6\}$ **15.** $\left\{\frac{3b}{a}\right\}$ **16.** $\{m - n\}$ **17.** $\left\{\frac{mn}{2m - 2n}\right\}$, where $m \neq n \neq 0$ **18.** $\{8\}$ **19.** $\{5, -5\}$ **20.** $\{0, 3\}$ **21.** $\{2.9, .1\}$ **22.** $\{32\}$ **23.** $\emptyset$ **24.** {real nos.} **25.** $\{-1\}$ **26.** $\{(-2, -3)\}$ **27.** $\{(-\frac{3}{2}, 3)\}$ **28.** $\{(7, 1)\}$ **29.** $\{(12, -7)\}$ **30.** $\{(15, 25)\}$ **32.** $\{-3.4, -1.8\}$ **34.** -2 **35.** $\frac{1}{2}$ **36.** $x = 3$ **37.** $y = -4$ **38.** $8x + 2y - 41 = 0$ **39.** $3x - 2y - 9 = 0$ **40.** $3x - y - 2 = 0$ **41.** $2x + 3y - 9 = 0$ **42.** $6x + y - 4 = 0$ **43.** $15x - 2y - 3 = 0$ **44.** -3

Functions and Variations **1.** $s = \frac{v^2}{2g}$ **2.** $F = \frac{9R + 128}{4}$ *or* $F = \frac{9}{4}R + 32$ **3.** $h = \frac{v^2}{2g}$ **4.** $B = \frac{2A}{h} - b$ **5.** $y = x - 1$ **6.** $y = 3x + 2$ **7.** $y = 2x - 1$ **8.** $c = 5n - 3$ **9.** $s = \frac{1}{2}t - 2$ *or* $2s = t - 4$ **10.** $y = -3x + 4$ **11.** Domain = {real numbers greater than 0 and less than or equal to $4\frac{2}{5}$} **12.** $S = 2wh + 2lh - 2ab - cd$ **13.** $179\frac{2}{3}$ **14.** $\frac{x_1}{y_1} = \frac{x_2}{y_2}$ **15.** $\frac{m_1}{m_2} = \frac{n_2^2}{n_1^2}$ **16.** $\frac{n_1}{n_2} = \frac{c_1d_2}{c_2d_1}$ **17.** $\frac{A_1}{h_1b_1} = \frac{A_2}{h_2b_2}$ **18.** $\frac{r_1}{r_2} = \frac{s_2}{s_1}$ **19.** $\frac{A_1}{B_1} = \frac{A_2}{B_2}$ **20.** $h = kt$ *or* $\frac{h}{t} = k$ **21.** $VP = k$ *or* $V = \frac{k}{P}$ **22.** $xy = k$ *or* $x = \frac{k}{y}$ **23.** $A = ks^2$ *or* $\frac{A}{s^2} = k$ **24.** 20 **25.** $\frac{5}{2}$ **26.** 12 **27.** $\frac{4}{9}$ **28.** $12\frac{4}{9}$ **29.** 980 rpm **30.** $626\frac{2}{23}$ rpm **31.** $2\frac{3}{16}$ hr **32.** a. $k = 1.15$; b. 180 naut. mi

Problems **1.** 76 **2.** 48, 60 **3.** 6, 8, 10 **4.** $\frac{2}{3}$, 2 **5.** 9, 29 **6.** $\frac{21}{28}$ **7.** 4 yr, 12 yr **8.** 3 ft **9.** 15° **10.** 40° **11.** 25 ft **12.** 135 **13.** 120 lb **14.** 39.2 g **15.** 18 ft 9 in, 13 ft 9 in **16.** 48 **17.** 75 ft **18.** 32 mph, 50 mph **19.** 9 mi **20.** $7\frac{1}{5}$ mi **21.** 6 mph, 9 mph **22.** \$600 at 5%, \$1000 at $3\frac{1}{2}$% **25.** \$1.39 for silk, \$.69 for cotton **26.** \$240 at 7%, \$760 at 2% **27.** 14 lb **28.** 32 quarters, 20 dimes **29.** 4 ml **30.** 10 pt **31.** 71 **32.** 7 hr 12 min **33.** 8 **34.** 52, 54, 56 **35.** 6.9 in **36.** 90 **38.** 35 **39.** 252 **40.** 126° **41.** Patio may be between 16 and 20 ft long **42.** More than 8 but at most 9 hr

Indirect Measurement: Vectors **1.** .7547 **2.** 3.7321 **3.** .8572 **4.** 56° **5.** 50° **6.** 77° **7.** 23° **8.** 78.8 **9.** 89.1 **10.** 39° **11.** a. 15 in; b. 75 sq in; c. 42 in **12.** 10° **13.** 480 ft **14.** 71° **15.** 25 in **16.** 854 ft **17.** 234 lb, 40° **18.** 16 lb, 48 lb **19.** 26 knots **20.** 13 mph at a bearing of 157°

True–False Test **1.** T **2.** T **3.** T **4.** F **5.** F **6.** F **7.** F **8.** F **9.** T **10.** F **11.** F **12.** F **13.** T **14.** T **15.** F **16.** F **17.** T **18.** T **19.** F **20.** T **21.** F **22.** T **23.** T **24.** T **25.** T **26.** F **27.** T **28.** T **29.** F **30.** T **31.** T **32.** F **33.** T **34.** F **35.** T **36.** F **37.** F **38.** F **39.** F **40.** T

Completion Test **1.** 12 **2.** 0 **3.** ksd^2 **4.** $\frac{d_2^2}{d_1^2}$ **5.** $m - 1, m + 1$ **6.** -1 **7.** 5 **8.** -1 **9.** 8 **10.** 5.3 **11.** -1 **12.** 11 **13.** 94% **14.** 24 **15.** $\frac{20 + rh}{4 + h}$ **16.** decreases **17.** increases **18.** 4 **19.** 48 **20.** 8 ft **21.** b **22.** -2 **23.** 2 **24.** 68.6 cm **25.** $\{2, 3\}$ **26.** 59 **27.** 45 **28.** 4 **29.** 10 **30.** 8 **31.** $\frac{25}{4}$ **32.** $x - 3$ **33.** $a = 0, b \neq 0$ **34.** $\frac{7}{11}$ **35.** function **36.** 4 **37.** $y = 2x - 1$ **38.** $\sqrt{3} : 1$ **39.** $\{10, 11, 12\}$ **40.** $3\sqrt{3}$

Date Due

EDUC OC 1 '74 — EDUC MR 22 '78
OCT 6 RETURN — RETURNED MAR 22'78
EDUC OC 16 '74
OCT 16 RETURN — Date Due APR 30'79
APR 23 RETURN
EDUC OC 27 '74 — EDUC NO 20 '79
OCT 26 RETURN — EDUC NO 27 '79
NOV 27 RETURN
NOV 13 RETURN — DUE EDUC MAY 22
EDUC FE 2 '75 — MAY 15 RETURN
EDUC MR 22 '75
APR 19 RETURN — DUE EDUC JUN 4'82
EDUC APR 28 '78
APR 14 RETURN — RETURN JUN 4 '82
EDUC AP 12 '77
APR 12 RETURN
DUE Educ OCT 30'77
DUE Educ NOV 6'77
Returned NOV 5-'77

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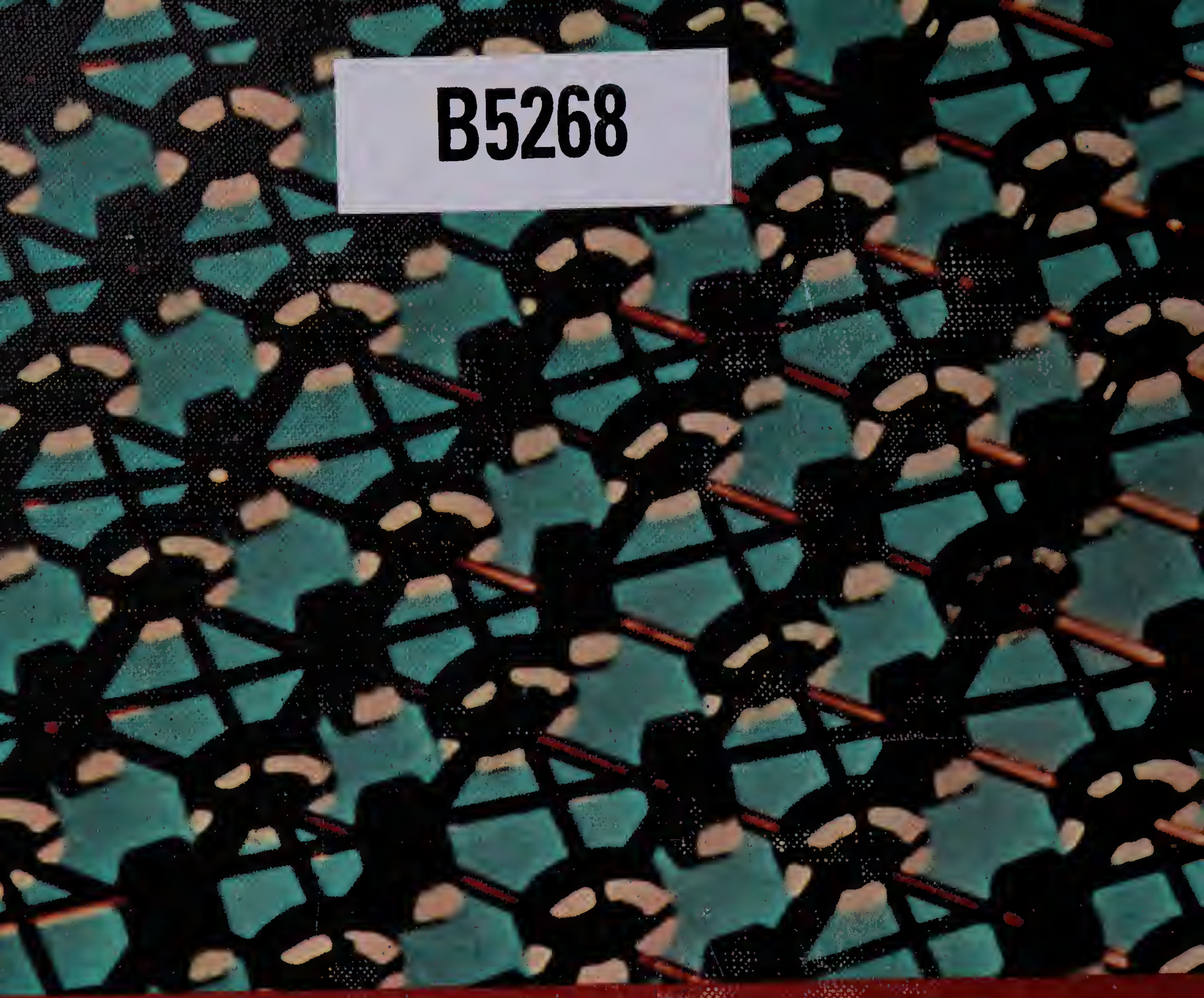

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